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Fractional Hermite-Hadamard type inequalities for functions whose mixed derivatives are co-ordinated $(\log, (s, m))$-convex

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Abstract

In this paper, we introduce the class of $(\log, (s, m))$-convexity on the co-ordinates, we establish a new identity involving the functions of two independent variables, and then we derive some fractional Hermite-Hadamard type integral inequalities for functions whose second derivatives are co-ordinated $(\log, (s, m))$-convex.

Keywords: co-ordinated $(log, (s, m))$-convex, co-ordinated convex, Hölder inequality
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1 Introduction

One of the most well-known inequalities in mathematics for convex functions is the so-called Hermite-Hadamard integral inequality, that can be stated as follows: for every convex function f on the finite interval $[a, b]$ we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

If the function f is concave, then (1.1) holds in the reverse direction (see [12]). In [4] Dragomir established the bidimensional analog of (1.1) given by

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{2}\left(\frac{1}{b-a}\int_a^b f(x, \tfrac{c+d}{2})dx + \frac{1}{d-c}\int_c^d f(\tfrac{a+b}{2}, y)dy\right) \leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \\ &\leq \frac{1}{4}\left(\frac{1}{b-a}\int_a^b f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx + \frac{1}{d-c}\int_c^d f(a,y)dy + \frac{1}{d-c}\int_c^d f(b,y)dy\right) \\ &\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned} \tag{1.2}$$

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The inequalities (1.1) and (1.2) has attracted many researchers, various generalizations, refinements, extensions and variants have been appeared in the literature, see [1, 2, 3, 6, 7, 8, 9, 10, 11, 14, 16] and references therein.

Sarikaya [13] gave the following fractional Hermite-Hadamard for co-ordinated convex functions.

Theorem 1.1. Let $f:\Delta\to\mathbb{R}$ be a partial differentiable mapping on $\Delta=[a,b]\times[c,d]\subset\mathbb{R}^2$. If $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ is a convex function on the co-ordinates on Δ, then one has the inequalities

$$\begin{aligned}&\left|\tfrac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^\alpha(d-c)^\beta}\left(J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right)-A\right|\\ &\leq \tfrac{(b-a)(d-c)}{(\alpha+1)(\beta+1)}\left(\frac{\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|}{4}\right),\end{aligned}$$

where

$$\begin{aligned}A &= \tfrac{\Gamma(\beta+1)}{4(d-c)^\beta}\left(J^\beta_{c^+}f(a,d)+J^\beta_{c^+}f(b,d)+J^\beta_{d^-}f(a,c)+J^\beta_{d^-}f(b,c)\right)\\ &\quad+\tfrac{\Gamma(\alpha+1)}{4(b-a)^\alpha}\left(J^\alpha_{a^+}f(b,c)+J^\alpha_{a^+}f(b,d)+J^\alpha_{b^-}f(a,c)+J^\alpha_{b^-}f(a,d)\right). \qquad (1.3)\end{aligned}$$

Theorem 1.2. Let $f:\Delta\to\mathbb{R}$ be a partial differentiable mapping on $\Delta=[a,b]\times[c,d]\subset\mathbb{R}^2$. If $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ is a convex function on the co-ordinates on Δ, where $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then one has the inequalities

$$\begin{aligned}&\left|\tfrac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^\alpha(d-c)^\beta}\left(J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right)-A\right|\\ &\leq \tfrac{(b-a)(d-c)}{((\alpha p+1)(\beta p+1))^{\frac{1}{p}}}\left(\frac{\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q}{4}\right)^{\frac{1}{q}},\end{aligned}$$

where A is defined by (1.3).

Motivated by the above results, in this paper, we introduce the concept of $(\log,(s,m))$ convexity on the co-ordinates, we also establish a new fractional identity involving functions of two independent variables, and we derive some fractional Hermite-Hadamard type integral inequalities for functions whose second derivatives are in this class of functions.

2 Preliminaries

In this section, we recall some definitions and lemmas that's well known in the literature, and assume that $\Delta:=[a,b]\times[c,d]$ and $\Delta_0=[0,b]\times[c,d]$ are two bidimensional interval in $\mathbb{R}^2$ with $a<b$ and $c<d$.

Definition 2.1. [15] A function $f:\Delta_0\to(0,+\infty)$ is said to be co-ordinated $(log,(\alpha,m))$-convex on Δ_0, if the following inequality

$$f(tx+(1-t)u,\lambda y+m(1-\lambda)v)\leq\left[\lambda^\alpha f(x,y)+m\lambda^\alpha f(x,v)\right]^t\left[\lambda^\alpha f(u,y)+m(1-\lambda^\alpha)f(u,v)\right]^{1-t}$$

holds for all $t,\lambda\in[0,1],\alpha,m\in(0,1]$ and $(x,u),(y,v)\in\Delta_0$.

Definition 2.2. [5] Let $f\in L_1[a,b]$. The Riemann-Liouville integrals $J^\alpha_{a^+}f$ and $J^\alpha_{b^-}f$ of order $\alpha>0$ with $a\geq 0$ are defined by

$$\begin{aligned}J^\alpha_{a^+}f(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)dt, \qquad x>a\\ J^\alpha_{b^-}f(x) &= \frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)dt, \qquad b>x\end{aligned}$$

respectively, where $\Gamma(\alpha)=\int\limits_0^\infty e^{-t}t^{\alpha-1}dt$, is the Gamma function and $J^0_{a+}f(x)=J^0_{b-}f(x)=f(x)$.

Definition 2.3. [5] Let $f\in L([a,b]\times[c,d])$. The Riemann-Liouville integrals $J^{\alpha,\beta}_{a^+,c^+}$, $J^{\alpha,\beta}_{a^+,d^-}$, $J^{\alpha,\beta}_{b^-,c^+}$, and $J^{\alpha,\beta}_{b^-,d^-}$of order $\alpha,\beta>0$ with $a,c\geq 0$, $a<b$ and $<d$ are defined by

$$J^{\alpha,\beta}_{a^+,c^+}f\left(b,d\right)=\frac{1}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}\int\limits_a^b\int\limits_c^d\left(b-x\right)^{\alpha-1}\left(d-y\right)^{\beta-1}f\left(x,y\right)dydx, \tag{2.1}$$

$$J^{\alpha,\beta}_{a^+,d^-}f\left(b,c\right)=\frac{1}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}\int\limits_a^b\int\limits_c^d\left(b-x\right)^{\alpha-1}\left(y-c\right)^{\beta-1}f\left(x,y\right)dydx, \tag{2.2}$$

$$J^{\alpha,\beta}_{b^-,c^+}f\left(a,d\right)=\frac{1}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}\int\limits_a^b\int\limits_c^d\left(x-a\right)^{\alpha-1}\left(d-y\right)^{\beta-1}f\left(x,y\right)dydx, \tag{2.3}$$

$$J^{\alpha,\beta}_{b^-,d^-}f\left(a,c\right)=\frac{1}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}\int\limits_a^b\int\limits_c^d\left(x-a\right)^{\alpha-1}\left(y-c\right)^{\beta-1}f\left(x,y\right)dydx, \tag{2.4}$$

where Γ is the Gamma function, and

$$J^{0,0}_{a^+,c^+}f\left(b,d\right)=J^{0,0}_{a^+,d^-}f\left(b,c\right)=J^{0,0}_{b^-,c^+}f\left(a,d\right)=J^{0,0}_{b^-,d^-}f\left(a,c\right)=f\left(x,y\right).$$

Definition 2.4. [13] Let $f\in L([a,b]\times[c,d])$. The Riemann-Liouville integrals $J^{\alpha}_{b^-}f\left(a,c\right)$, $J^{\alpha}_{a^+}f\left(b,c\right)$, $J^{\beta}_{d^-}f\left(a,c\right)$, and $J^{\alpha}_{c^+}f\left(a,d\right)$ of order $\alpha,\beta>0$ with $a,c\geq 0$, $a<b$, and $c<d$ are defined by

$$J^{\alpha}_{b^-}f\left(a,c\right)=\frac{1}{\Gamma\left(\alpha\right)}\int\limits_a^b\left(x-a\right)^{\alpha-1}f\left(x,c\right)dx, \tag{2.5}$$

$$J^{\alpha}_{a^+}f\left(b,c\right)=\frac{1}{\Gamma\left(\alpha\right)}\int\limits_a^b\left(b-x\right)^{\alpha-1}f\left(x,c\right)dx, \tag{2.6}$$

$$J^{\beta}_{d^-}f\left(a,c\right)=\frac{1}{\Gamma\left(\beta\right)}\int\limits_c^d\left(y-c\right)^{\beta-1}f\left(a,y\right)dy, \tag{2.7}$$

$$J^{\alpha}_{c^+}f\left(a,d\right)=\frac{1}{\Gamma\left(\beta\right)}\int\limits_c^d\left(d-y\right)^{\beta-1}f\left(a,y\right)dy, \tag{2.8}$$

where Γ is the Gamma function.

3 Main results

In what follows, we assume that $\Delta=[a,b]\times[c,d]$ with $a<b$, $0<c<d$ and $\Delta_0=[a,b]\times\left[0,\frac{d}{m}\right]$ where $m\in(0,1]$.

Definition 3.1. A function $f:\Delta_0\to(0,+\infty)$ is said to be co-ordinated $(log,(s,m))$-convex on Δ_0 if the following inequality

$$f\left(tx+\left(1-t\right)u,\lambda y+m\left(1-\lambda\right)v\right)\leq\left[\lambda^s f(x,y)+m\left(1-\lambda\right)^s f(x,v)\right]^t\left[\lambda^s f(u,y)+m((1-\lambda)^s f(u,v)\right]^{1-t}$$

holds for all $t,\lambda\in[0,1]$, $s,m\in(0,1]$ and $(x,u),(y,v)\in\Delta_0$.

Lemma 3.2. Let $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a differentiable mapping on $\Delta=[a,b]\times[c,d]$ with $a<b$ and $c<d$. If $\frac{\partial^2 f}{\partial t\partial\lambda}\in L(\Delta)$, then the following fractional equality holds

$$\begin{aligned}F(f,a,b,c,b,\alpha,\beta,A,J) &= \tfrac{(b-a)(d-c)}{4}\left(\int_0^1\int_0^1 kh\tfrac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\,dtd\lambda\right.\\ &\left.-\int_0^1\int_0^1((1-t)^\alpha-t^\alpha)((1-\lambda)^\beta-\lambda^\beta)\times\tfrac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\,dtd\lambda\right),\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}F(f,a,b,c,b,\alpha,\beta,A,J) &= f\left(\tfrac{a+b}{2},\tfrac{c+d}{2}\right)-\frac{f\left(a,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},d\right)}{2}+A\\ &-\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta}\times\left(J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right),\end{aligned} \tag{3.2}$$

$$k=\begin{cases}1 & \text{if}\quad 0\le t<\frac12,\\ -1 & \text{if}\quad \frac12\le t<1,\end{cases} \tag{3.3}$$

$$h=\begin{cases}1 & \text{if}\quad 0\le \lambda<\frac12,\\ -1 & \text{if}\quad \frac12\le \lambda<1,\end{cases} \tag{3.4}$$

and

$$\begin{aligned}A &= \tfrac{\Gamma(\beta+1)}{4(d-c)^\beta}\left(J^\beta_{d^-}f(a,c)+J^\beta_{d^-}f(b,c)+J^\alpha_{c^+}f(a,d)+J^\alpha_{c^+}f(b,d)\right)\\ &+\tfrac{\Gamma(\alpha+1)}{4(b-a)^\alpha}\left(J^\alpha_{b^-}f(a,c)+J^\alpha_{b^-}f(a,d)+J^\alpha_{a^+}f(b,c)+J^\alpha_{a^+}f(b,d)\right).\end{aligned} \tag{3.5}$$

Proof . Let

$$I=\tfrac{(b-a)(d-c)}{4}(I_1-I_2), \tag{3.6}$$

where

$$I_1=\int_0^1\int_0^1 kh\tfrac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\,dtd\lambda,$$

and

$$I_2=\int_0^1\int_0^1((1-t)^\alpha-t^\alpha)((1-\lambda)^\beta-\lambda^\beta)\times\tfrac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\,dtd\lambda.$$

Clearly, we have

$$\begin{aligned}I_1 &= \int_0^{\frac12}\int_0^{\frac12}\tfrac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\,dtd\lambda-\int_0^{\frac12}\int_{\frac12}^1\tfrac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\,dtd\lambda\\ &-\int_{\frac12}^1\int_0^{\frac12}\tfrac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\,dtd\lambda+\int_{\frac12}^1\int_{\frac12}^1\tfrac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\,dtd\lambda\\ &= \tfrac{1}{(b-a)(d-c)}\Big(f\left(\tfrac{a+b}{2},\tfrac{c+d}{2}\right)-f\left(b,\tfrac{c+d}{2}\right)-f\left(\tfrac{a+b}{2},d\right)+f(b,d)-f\left(a,\tfrac{c+d}{2}\right)+f\left(\tfrac{a+b}{2},\tfrac{c+d}{2}\right)+f(a,d)\\ &-f\left(\tfrac{a+b}{2},d\right)-f\left(\tfrac{a+b}{2},c\right)+f(b,c)+f\left(\tfrac{a+b}{2},\tfrac{c+d}{2}\right)-f\left(b,\tfrac{c+d}{2}\right)\\ &+f(a,c)-f\left(\tfrac{a+b}{2},c\right)-f\left(a,\tfrac{c+d}{2}\right)+f\left(\tfrac{a+b}{2},\tfrac{c+d}{2}\right)\Big)\\ &= \tfrac{4}{(b-a)(d-c)}\left(\left(f\left(\tfrac{a+b}{2},\tfrac{c+d}{2}\right)\right)+\tfrac{f(b,d)+f(a,d)+f(b,c)+f(a,c)}{4}-\frac{f\left(a,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},d\right)}{2}\right)\end{aligned} \tag{3.7}$$

Now, by integration by parts, I_2 gives

$$I_2 = \int_0^1 \left((1-\lambda)^\beta - \lambda^\beta\right) \times \left(\int_0^1 ((1-t)^\alpha - t^\alpha) \tfrac{\partial^2 f}{\partial t \partial \lambda} (ta + (1-t)b, \lambda c + (1-\lambda)d)\, dt\right) d\lambda$$

$$= \tfrac{1}{(b-a)(d-c)} (f(a,c) + f(a,d) + f(b,c) + f(b,d)) - \tfrac{\beta}{(b-a)(d-c)} \left(\int_0^1 (1-\lambda)^{\beta-1} f(a, \lambda c + (1-\lambda)d)\, d\lambda\right.$$

$$+ \int_0^1 \lambda^{\beta-1} f(a, \lambda c + (1-\lambda)d)\, d\lambda + \int_0^1 \lambda^{\beta-1} f(b, \lambda c + (1-\lambda)d)\, d\lambda + \left.\int_0^1 (1-\lambda)^{\beta-1} f(b, \lambda c + (1-\lambda)d)\, d\lambda\right)$$

$$- \tfrac{\alpha}{(b-a)(d-c)} \left(\int_0^1 (1-t)^{\alpha-1} f(ta + (1-t)b, c)\, dt + \int_0^1 t^{\alpha-1} f(ta + (1-t)b, c)\, dt + \int_0^1 t^{\alpha-1} f(ta + (1-t)b, d)\, dt\right.$$

$$+ \left.\int_0^1 (1-t)^{\alpha-1} f(ta + (1-t)b, d)\, dt\right) + \tfrac{\alpha\beta}{(b-a)(d-c)} \left(\int_0^1\int_0^1 t^{\alpha-1}\lambda^{\beta-1} f(ta + (1-t)b, \lambda c + (1-\lambda)d)\, d\lambda dt\right.$$

$$+ \int_0^1\int_0^1 (1-t)^{\alpha-1} \lambda^{\beta-1} f(ta + (1-t)b, \lambda c + (1-\lambda)d)\, d\lambda dt$$

$$+ \int_0^1\int_0^1 t^{\alpha-1} (1-\lambda)^{\beta-1} f(ta + (1-t)b, \lambda c + (1-\lambda)d)\, d\lambda dt$$

$$+ \left.\int_0^1\int_0^1 (1-t)^{\alpha-1} (1-\lambda)^{\beta-1} f(ta + (1-t)b, \lambda c + (1-\lambda)d)\, d\lambda dt\right).$$

Substituting (3.7) and (3.8) in (3.6), and putting $x = ta + (1-t)b$ and $y = \lambda c + (1-\lambda)d$, we get

$$I = f\left(\tfrac{a+b}{2}, \tfrac{c+d}{2}\right) - \frac{f\left(a, \frac{c+d}{2}\right) + f\left(\frac{a+b}{2}, c\right) + f\left(b, \frac{c+d}{2}\right) + f\left(\frac{a+b}{2}, d\right)}{2}$$

$$+ \tfrac{\beta}{4(d-c)^\beta} \left(\int_c^d (y-c)^{\beta-1} f(a,y)\, dy + \int_c^d (y-c)^{\beta-1} f(b,t)\, dy + \int_c^d (d-y)^{\beta-1} f(a,y)\, dy + \int_c^d (d-y)^{\beta-1} f(b,y)\, dy\right)$$

$$+ \tfrac{\alpha}{4(b-a)^\alpha} \left(\int_0^1 (x-a)^{\alpha-1} f(x,c)\, dx + \int_a^b (x-a)^{\alpha-1} f(x,d)\, dx + \int_a^b (b-x)^{\alpha-1} f(x,c)\, dx + \int_a^b (b-x)^{\alpha-1} f(x,d)\, dx\right)$$

$$- \tfrac{\alpha\beta}{4(b-a)^\alpha (d-c)^\beta} \left(\int_a^b\int_c^d (b-x)^{\alpha-1} (d-y)^{\beta-1} f(x,y)\, dy dx + \int_a^b\int_c^d (x-a)^{\alpha-1} (d-y)^{\beta-1} f(x,y)\, dy dx\right.$$

$$+ \left.\int_a^b\int_c^d (b-x)^{\alpha-1} (y-c)^{\beta-1} f(x,y)\, dy dx + \int_a^b\int_c^d (x-a)^{\alpha-1} (y-c)^{\beta-1} f(x,y)\, dy dx\right). \quad (3.8)$$

Using (2.1)-(2.8) in (3.9), we obtain the desired result. □

Theorem 3.3. Let $f : \Delta \to \mathbb{R}$ be a partially differentiable function on Δ such that $\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right| \in L(\Delta_0)$. If $\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|$ is co-ordinated $(\log,(s,m))$-convex on Δ_0 for some fixed $s,m\in(0,1]$, then the following fractional inequality holds

$$\begin{aligned}
&\left|F\left(f,a,b,c,b,\alpha,\beta,A,J\right)\right|\\
\leq\ &\tfrac{(b-a)(d-c)}{4}\left(\left(\tfrac{1}{2(s+1)}+\tfrac{1}{\alpha+1}\left(B\left(s+1,\beta+1\right)+\tfrac{1}{\beta+s+1}\right)\right)\right.\\
&\left.\times\left(\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|+m\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|+\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|+m\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|\right)\right),
\end{aligned}$$

where F is defined as in (3.2) and $B\left(.,.\right)$ is the beta function.

Proof . From Lemma 3.2, properties of modulus, and $(log,(s,m))$-convexity on the co-ordinates of $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|$, we have

$$\begin{aligned}
&\left|F\left(f,a,b,c,b,\alpha,\beta,A,J\right)\right|\\
\leq\ &\tfrac{(b-a)(d-c)}{4}\left(\int_0^1\int_0^1\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(ta+(1-t)\,b,\lambda c+(1-\lambda)\,d\right)\right|dtd\lambda+\int_0^1\int_0^1\left((1-t)^\alpha+t^\alpha\right)\left((1-\lambda)^\beta+\lambda^\beta\right)\right.\\
&\left.\times\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(ta+(1-t)\,b,\lambda c+(1-\lambda)\,d\right)\right|dtd\lambda\right)\\
\leq\ &\tfrac{(b-a)(d-c)}{4}\left(\int_0^1\int_0^1\left[\lambda^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|+m\,(1-\lambda)^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|\right]^t\right.\\
&\times\left[\lambda^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|+m\,(1-\lambda)^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|\right]^{1-t}dtd\lambda+\int_0^1\int_0^1\left((1-t)^\alpha+t^\alpha\right)\left((1-\lambda)^\beta+\lambda^\beta\right)\\
&\left.\times\left[\lambda^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|+m\,(1-\lambda)^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|\right]^t\times\left[\lambda^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|+m\,(1-\lambda)^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|\right]^{1-t}dtd\lambda\right)
\end{aligned}\tag{3.9}$$

Applying Young's inequality for (3.10) we get

$$\begin{aligned}
&\left|F\left(f,a,b,c,b,\alpha,\beta,A,J\right)\right|\leq\tfrac{(b-a)(d-c)}{4}\times\left(\int_0^1\int_0^1 t\left[\lambda^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|+m\,(1-\lambda)^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|\right]dtd\lambda\right.\\
&+\int_0^1\int_0^1(1-t)\left[\lambda^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|+m\,(1-\lambda)^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|\right]dtd\lambda\\
&+\int_0^1\int_0^1\left((1-t)^\alpha+t^\alpha\right)\left((1-\lambda)^\beta+\lambda^\beta\right)\times\left[t\lambda^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|+m\,(1-\lambda)^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|\right]dtd\lambda\\
&\left.+\int_0^1\int_0^1\left((1-t)^\alpha+t^\alpha\right)\left((1-\lambda)^\beta+\lambda^\beta\right)(1-t)\times\left[\lambda^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|+m\,(1-\lambda)^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|\right]^{1-t}dtd\lambda\right)\\
=\ &\tfrac{(b-a)(d-c)}{4}\left(\left(\int_0^1 tdt\right)\times\left(\int_0^1\left[\lambda^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|+m\,(1-\lambda)^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|\right]d\lambda\right)\right.\\
&+\left(\int_0^1(1-t)\,dt\right)\times\left(\int_0^1\left[\lambda^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|+m\,(1-\lambda)^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|\right]d\lambda\right)
\end{aligned}$$

$$+\left(\int_0^1 \left(t\left(1-t\right)^{\alpha}+t^{\alpha+1}\right)dt\right)$$
$$\times\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|\int_0^1 (\lambda^s(1-\lambda)^{\beta}+\lambda^{\beta+s})d\lambda+m\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|\int_0^1((1-\lambda)^{\beta+s}+\lambda^{\beta}\left(1-\lambda\right)^s)d\lambda\right)$$
$$+\left(\int_0^1 \left(\left(1-t\right)^{\alpha+1}+t^{\alpha}\left(1-t\right)\right)dt\right)$$
$$\times\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|\int_0^1 (\lambda^s(1-\lambda)^{\beta}+\lambda^{\beta+s})d\lambda+m\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|\int_0^1((1-\lambda)^{\beta+s}+\lambda^{\beta}\left(1-\lambda\right)^s)d\lambda\right)\Bigg)$$
$$=\frac{(b-a)(d-c)}{4}\left(\frac{1}{2(s+1)}\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|+m\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|\right)+\frac{1}{2(s+1)}\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|+m\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|\right)\right.$$
$$+\frac{1}{\alpha+1}\left(B\left(s+1,\beta+1\right)+\frac{1}{\beta+s+1}\right)\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|+m\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|\right)$$
$$\left.+\frac{1}{\alpha+1}\left(B\left(s+1,\beta+1\right)+\frac{1}{\beta+s+1}\right)\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|+m\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|\right)\right)$$
$$=\frac{(b-a)(d-c)}{4}\left(\frac{1}{2(s+1)}+\frac{1}{\alpha+1}\left(B\left(s+1,\beta+1\right)+\frac{1}{\beta+s+1}\right)\right)$$
$$\times\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|+m\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|+m\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|\right)\Bigg),$$

which is the desired result. □

Corollary 3.4. Under the conditions of Theorem 3.3,

1. If $m=1$, then

$$\left|F\left(f,a,b,c,b,\alpha,\beta,A,J\right)\right|$$
$$\leq\frac{(b-a)(d-c)}{4}\left(\left(\frac{1}{2(s+1)}+\frac{1}{\alpha+1}\left(B\left(s+1,\beta+1\right)+\frac{1}{\beta+s+1}\right)\right)\times\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right)\right).$$

2. If $s=1$, then

$$\left|F\left(f,a,b,c,b,\alpha,\beta,A,J\right)\right|$$
$$\leq\frac{(b-a)(d-c)}{4}\left(\left(\frac{1}{4}+\frac{1}{(\alpha+1)(\beta+1)}\right)\times\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|+m\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|+m\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|\right)\right).$$

3. If $m=s=1$, then

$$\left|F\left(f,a,b,c,b,\alpha,\beta,A,J\right)\right|\leq\frac{(b-a)(d-c)}{4}\left(\left(\frac{1}{4}+\frac{1}{(\alpha+1)(\beta+1)}\right)\times\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right)\right).$$

4. If $\alpha=\beta=1$, then

$$\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{f\left(a,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},d\right)}{2}+A-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\right|$$
$$\leq\frac{(b-a)(d-c)}{4(s+1)}\times\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|+m\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|+m\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|\right).$$

5. If $\alpha=\beta=s=1$, then

$$\left|f\left(\tfrac{a+b}{2},\tfrac{c+d}{2}\right)-\frac{f\left(a,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},d\right)}{2}+A-\tfrac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\right|$$
$$\leq \tfrac{(b-a)(d-c)}{8}\times\left(\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|+m\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|+\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|+m\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|\right).$$

6. If $\alpha=\beta=m=1$, then

$$\left|f\left(\tfrac{a+b}{2},\tfrac{c+d}{2}\right)-\frac{f\left(a,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},d\right)}{2}+A-\tfrac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\right|$$
$$\leq \tfrac{(b-a)(d-c)}{4(s+1)}\times\left(\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|+\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|+\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|+\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right).$$

7. If $\alpha=\beta=s=m=1$, then

$$\left|f\left(\tfrac{a+b}{2},\tfrac{c+d}{2}\right)-\frac{f\left(a,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},d\right)}{2}+A-\tfrac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\right|$$
$$\leq \tfrac{(b-a)(d-c)}{8}\times\left(\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|+\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|+\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|+\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right),$$

where

$$A=\tfrac{1}{2(b-a)}\int_a^b\left(f(x,c)+f(x,d)\right)dx+\tfrac{1}{2(d-c)}\int_c^d\left(f(a,y)+f(b,y)\right)dy.$$

Theorem 3.5. Let $f:\Delta\to\mathbb{R}$ be a partially differentiable function on Δ such that $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|\in L(\Delta_0)$. If $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$ is co-ordinated $(\log,(s,m))$-convex on Δ_0 where $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ for some fixed $s,m\in(0,1]$, then the following fractional inequality holds

$$|\digamma(f,a,b,c,b,\alpha,\beta,A,J)|\leq\frac{(b-a)(d-c)}{2^{4+\frac{2}{q}+\frac{s}{q}}(s+1)^{\frac{1}{q}}}\left(\left(\frac{\Upsilon_1+\left(2^{s+1}-1\right)m\Psi_{1,m}}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left(2^{s+1}-1\right)\Upsilon_1+m\Psi_{1,m}}{s+1}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{\Upsilon_2+\left(2^{s+1}-1\right)m\Psi_{2,m}}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left(2^{s+1}-1\right)\Upsilon_2+m\Psi_{2,m}}{s+1}\right)^{\frac{1}{q}}$$
$$\left.+\frac{16\times 2^{\frac{s}{q}+\frac{1}{q}}\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+m\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|^q\right)\right)^{\frac{1}{q}}}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\right),$$

where

$$\Upsilon_1=\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+3\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q, \tag{3.10}$$

$$\Psi_{1,m}=\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|^q+3\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|^q, \tag{3.11}$$

$$\Upsilon_2=3\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q, \tag{3.12}$$

$$\Psi_{2,m}=3\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|^q+\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|^q, \tag{3.13}$$

and $\digamma$ is defined as in (3.2).

Proof . From Lemma 3.2, properties of modulus, and Hölder inequality, we have

$$|F(f,a,b,c,b,\alpha,\beta,A,J)| \leq \frac{(b-a)(d-c)}{4} \times \left(\left(\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} dtd\lambda\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} \left|\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q dtd\lambda\right)^{\frac{1}{q}}\right.\right.$$

$$+\left(\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} dtd\lambda\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} \left|\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q dtd\lambda\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}} dtd\lambda\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}} \left|\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q dtd\lambda\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} dtd\lambda\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} \left|\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q dtd\lambda\right)^{\frac{1}{q}}\right)$$

$$+\left(\left(\int_0^1\int_0^1 (1-t)^{\alpha p}(1-\lambda)^{\beta p} dtd\lambda\right)^{\frac{1}{p}} + \left(\int_0^1\int_0^1 t^{\alpha p}(1-\lambda)^{\beta p} dtd\lambda\right)^{\frac{1}{p}} + \left(\int_0^1\int_0^1 (1-t)^{\alpha p}\lambda^{\beta p} dtd\lambda\right)^{\frac{1}{p}}\right.$$

$$\left.\left.+\left(\int_0^1\int_0^1 t^{\alpha p}\lambda^{\beta p} dtd\lambda\right)^{\frac{1}{p}}\right) \times \left(\int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q dtd\lambda\right)^{\frac{1}{q}}\right)$$

$$= \frac{(b-a)(d-c)}{4^{1+\frac{1}{p}}}\left(\left(\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} \left|\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q dtd\lambda\right)^{\frac{1}{q}} + \left(\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} \left|\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q dtd\lambda\right)^{\frac{1}{q}}\right.\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}} \left|\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q dtd\lambda\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} \left|\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q dtd\lambda\right)^{\frac{1}{q}}\right)$$

$$\left.+\frac{4^{1+\frac{1}{p}}}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}} \times \left(\int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q dtd\lambda\right)^{\frac{1}{q}}\right).$$

Using the $(\log,(s,m))$-convexity on the co-ordinates of $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$ and Young's inequality, we obtain

$$|F(f,a,b,c,b,\alpha,\beta,A,J)| \leq \frac{(b-a)(d-c)}{4^{1+\frac{1}{p}}}\left(\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} t\left[\lambda^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q + m(1-\lambda)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|^q\right]dtd\lambda\right.\right.$$

$$\left.+\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} (1-t)\left[\lambda^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q + m((1-\lambda)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|^q\right]dtd\lambda\right)^{\frac{1}{q}}$$

$$+\left(\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} t\left[\lambda^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q + m(1-\lambda)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|^q\right]dtd\lambda\right.$$

$$+\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1}(1-t)\left[\lambda^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+m((1-\lambda)^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|^q\right]dtd\lambda\Bigg)^{\frac{1}{q}}$$

$$+\Bigg(\int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}}t\left[\lambda^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|+m\left(1-\lambda\right)^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|\right]dtd\lambda$$

$$+\int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}}(1-t)\left[\lambda^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+m((1-\lambda)^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|^q\right]dtd\lambda\Bigg)^{\frac{1}{q}}$$

$$+\Bigg(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}t\left[\lambda^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+m\left(1-\lambda\right)^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|^q\right]dtd\lambda$$

$$+\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}(1-t)\left[\lambda^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+m((1-\lambda)^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|^q\right]dtd\lambda\Bigg)^{\frac{1}{q}}$$

$$+\frac{4^{1+\frac{1}{p}}}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\times\Bigg(\Bigg(\int_0^1\int_0^1 t\left[\lambda^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+m\left(1-\lambda\right)^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|^q\right]dtd\lambda$$

$$+\int_0^1\int_0^1(1-t)\left[\lambda^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+m((1-\lambda)^s\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|^q\right]dtd\lambda\Bigg)^{\frac{1}{q}}\Bigg)$$

$$=\frac{(b-a)(d-c)}{2^{4+\frac{2}{q}+\frac{s}{q}}(s+1)^{\frac{1}{q}}}\times\Bigg(\Bigg(\frac{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+3\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+\left(2^{s+1}-1\right)m\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|^q+3\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|^q\right)}{s+1}\Bigg)^{\frac{1}{q}}$$

$$+\left(\frac{\left(2^{s+1}-1\right)\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+3\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q\right)+m\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|^q+3\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|^q\right)}{s+1}\right)^{\frac{1}{q}}$$

$$+\left(\frac{3\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+\left(2^{s+1}-1\right)m\left(3\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|^q\right)}{s+1}\right)^{\frac{1}{q}}$$

$$+\left(\frac{\left(2^{s+1}-1\right)\left(3\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q\right)+m\left(3\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|^q\right)}{s+1}\right)^{\frac{1}{q}}$$

$$+\frac{16\times 2^{\frac{s}{q}+\frac{1}{q}}}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\times\left(\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+m\left(\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|^q+\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|^q\right)\right)^{\frac{1}{q}}\Bigg),$$

which is the desired result. □

Corollary 3.6. Under the conditions of Theorem 3.5,

1. If $m=1$, then

$$|F\left(f,a,b,c,b,\alpha,\beta,A,J\right)|\leq\frac{(b-a)(d-c)}{2^{4+\frac{2}{q}+\frac{s}{q}}(s+1)^{\frac{1}{q}}}\left(\left(\frac{\Upsilon_1+\left(2^{s+1}-1\right)\Psi_{1,1}}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left(2^{s+1}-1\right)\Upsilon_1+\Psi_{1,1}}{s+1}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{\Upsilon_2+\left(2^{s+1}-1\right)\Psi_{2,1}}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left(2^{s+1}-1\right)\Upsilon_2+\Psi_{2,1}}{s+1}\right)^{\frac{1}{q}}$$

$$\left.+\frac{16\times2^{\frac{s}{q}+\frac{1}{q}}\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right)^{\frac{1}{q}}}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\right).$$

2. If $s=1$, then

$$|F\left(f,a,b,c,b,\alpha,\beta,A,J\right)|\leq\frac{(b-a)(d-c)}{2^{4+\frac{4}{q}}}$$

$$\left(\left(\frac{\Upsilon_1+3m\Psi_{1,m}}{2}\right)^{\frac{1}{q}}+\left(\frac{3\Upsilon_1+m\Psi_{1,m}}{2}\right)^{\frac{1}{q}}+\left(\frac{\Upsilon_2+3m\Psi_{2,m}}{2}\right)^{\frac{1}{q}}+\left(\frac{3\Upsilon_2+m\Psi_{2,m}}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\frac{16\times2^{\frac{2}{q}}\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+m\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|^q\right)\right)^{\frac{1}{q}}}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\right).$$

3. If $m=s=1$, then

$$|F\left(f,a,b,c,b,\alpha,\beta,A,J\right)|\leq\frac{(b-a)(d-c)}{2^{4+\frac{4}{q}}}$$

$$\left(\left(\frac{\Upsilon_1+3\Psi_{1,1}}{2}\right)^{\frac{1}{q}}+\left(\frac{3\Upsilon_1+\Psi_{1,1}}{2}\right)^{\frac{1}{q}}+\left(\frac{\Upsilon_2+3\Psi_{2,1}}{2}\right)^{\frac{1}{q}}+\left(\frac{3\Upsilon_2+\Psi_{2,1}}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\frac{16\times2^{\frac{2}{q}}\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+m\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right)\right)^{\frac{1}{q}}}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\right).$$

4. If $\alpha=\beta=1$, then

$$\left|f\left(\tfrac{a+b}{2},\tfrac{c+d}{2}\right)-\frac{f\left(a,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},d\right)}{2}+A-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\right|$$

$$\leq\frac{(b-a)(d-c)}{2^{4+\frac{2}{q}+\frac{s}{q}}(s+1)^{\frac{1}{q}}}\left(\left(\frac{\Upsilon_1+\left(2^{s+1}-1\right)m\Psi_{1,m}}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left(2^{s+1}-1\right)\Upsilon_1+m\Psi_{1,m}}{s+1}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{\Upsilon_2+\left(2^{s+1}-1\right)m\Psi_{2,m}}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left(2^{s+1}-1\right)\Upsilon_2+m\Psi_{2,m}}{s+1}\right)^{\frac{1}{q}}$$

$$\left.+\frac{16\times2^{\frac{s}{q}+\frac{1}{q}}\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+m\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|^q\right)\right)^{\frac{1}{q}}}{(p+1)^{\frac{2}{p}}}\right).$$

5. If $\alpha=\beta=m=1$, then

$$\left|f\left(\tfrac{a+b}{2},\tfrac{c+d}{2}\right)-\frac{f\left(a,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},d\right)}{2}+A-\tfrac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\right|$$

$$\leq \frac{(b-a)(d-c)}{2^{4+\frac{2}{q}+\frac{s}{q}}(s+1)^{\frac{1}{q}}}\left(\left(\tfrac{\Upsilon_1+\left(2^{s+1}-1\right)\Psi_{1,1}}{s+1}\right)^{\frac{1}{q}}+\left(\tfrac{\left(2^{s+1}-1\right)\Upsilon_1+\Psi_{1,1}}{s+1}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{\Upsilon_2+\left(2^{s+1}-1\right)\Psi_{2,1}}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left(2^{s+1}-1\right)\Upsilon_2+\Psi_{2,1}}{s+1}\right)^{\frac{1}{q}}$$

$$\left.+\frac{16\times 2^{\frac{s}{q}+\frac{1}{q}}\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right)^{\frac{1}{q}}}{(p+1)^{\frac{2}{p}}}\right).$$

6. If $\alpha=\beta=s=1$, then

$$\left|f\left(\tfrac{a+b}{2},\tfrac{c+d}{2}\right)-\frac{f\left(a,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},d\right)}{2}+A-\tfrac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\right|$$

$$\leq \frac{(b-a)(d-c)}{2^{4+\frac{4}{q}}}\left(\left(\tfrac{\Upsilon_1+3m\Psi_{1,m}}{2}\right)^{\frac{1}{q}}+\left(\tfrac{3\Upsilon_1+m\Psi_{1,m}}{2}\right)^{\frac{1}{q}}+\left(\frac{\Upsilon_2+3m\Psi_{2,m}}{2}\right)^{\frac{1}{q}}+\left(\frac{3\Upsilon_2+m\Psi_{2,m}}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\frac{16\times 2^{\frac{2}{q}}\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+m\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,\frac{d}{m}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(b,\frac{d}{m}\right)\right|^q\right)\right)^{\frac{1}{q}}}{(p+1)^{\frac{2}{p}}}\right).$$

7. If $\alpha=\beta=m=s=1$, then

$$\left|f\left(\tfrac{a+b}{2},\tfrac{c+d}{2}\right)-\frac{f\left(a,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},d\right)}{2}+A-\tfrac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\right|$$

$$\leq \frac{(b-a)(d-c)}{2^{4+\frac{4}{q}}}\left(\left(\tfrac{\Upsilon_1+3\Psi_{1,1}}{2}\right)^{\frac{1}{q}}+\left(\tfrac{3\Upsilon_1+\Psi_{1,1}}{2}\right)^{\frac{1}{q}}+\left(\frac{\Upsilon_2+3\Psi_{2,1}}{2}\right)^{\frac{1}{q}}+\left(\frac{3\Upsilon_2+\Psi_{2,1}}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\frac{16\times 2^{\frac{2}{q}}\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right)^{\frac{1}{q}}}{(p+1)^{\frac{2}{p}}}\right),$$

where $\Upsilon_1,\Psi_{1,m},\Upsilon_2,\Psi_{2,m}$ are defined as in (3.11)-(3.14) respectively, and

$$A=\tfrac{1}{2(b-a)}\int_a^b\left(f(x,c)+f(x,d)\right)dx+\tfrac{1}{2(d-c)}\int_c^d\left(f(a,y)+f(b,y)\right)dy.$$

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