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The largest size of the arc of degree three in a projective plane of order sixteen

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Abstract

An $(n;3)$-arc K in projective plane $PG(2,q)$ of size n and degree three is a set of n points satisfies that every line meet it in less than or equal three points, also it is complete if it is not contained in $(n+1;3)$-arc. The goals of this paper are to construct the projectively inequivalent $(n;3)$-arcs in $PG(2,16)$, determined the largest complete arc in $PG(2,16)$, the stabilizer group of these arcs and we have identified the group with which its isomorph.

Keywords: Projective Plane, Complete Arc.
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1. Introduction

Let F_q, $q=p^t$ the Galois field of q elements, for some prime number p and with some integer t and $V(3,q)$ be the vector space of row vectors of length three with entries in F_q. Let $PG(2,q)$ be the corresponding projective plane. The points $[x_0,x_1,x_2]$of $PG(2,p)$ are the 1-dimensional subspaces of $V(3,q)$. Subspaces of dimension two of the form $V(aX_0+bX_1+cX_2)$ are called lines. The number of points and the number of lines in $PG(2,q)$ is q^2+q+1. There are $q+1$ points on every line and $q+1$ lines through every point. Many researches have been studies the subject of projective geometries for examples see[9, 8, 4, 3, 1, 10, 14],the tools of this paper Gap-Groups,Algorithms, programming a system for computational discrete algebra [6].

A full classification of $(n;3)$-arcs are given by Cook[5]. For $q=11$, a maximal $(n;3)$-arcs has been found by Marcugini[12]. For $q=13$ in [13] the maximal arc has been found. Many studies and improvements have been given to get a largest (n;r)-arc of size two, three, four, etc. For more

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detailed to the size of $(n;3)$-arcs see [14]. The main aim of this paper is to construct the full numbers of $(n;3)$-arcs in $PG(2,16)$ and then classify the arcs which of them is an equivalent or an inequivalent and determined the largest size of $(n;3)$-arc then determined the stabilizer group of each an equivalent arc.we depend on two techniques the first once is based on choosing the number of inequivalent classes that depends on choosing the number of inequivalent classes $\{l_0\ ,l_1,l_2,l_3\}$ of $i-secant$ distributions which represents the 0-secant, 1-secant, 2-secant and 3-secant in each procedure. The second technique is based on constructed the special arcs in $PG(2,q)$ to find the largest size of complete $(n;3)$-arcs [15]. So we obtained that the largest size of complete $(n;3)$-arc is equal to 27.

2. The projective $PG(2,16)$

In $PG(2,16)$ there are 273 points and lines, 17 points on each line and 17 lines passage through each point. Take $H(X)=X^2+X+\omega^2$ in $F_{16}[X]$, where

$$F_{16}=\left\{0,1,\omega,\omega^2,\omega^3,\omega^4,\omega^5,\omega^6,\omega^7,\omega^8,\omega^9,\omega^{10},\omega^{11},\omega^{12},\omega^{13},\omega^{14}\right\},$$

a polynomial is primitive in F_{16}. The companion matrix of H is

$$T=\begin{bmatrix}0&1&0\\0&0&1\\\omega^7&1&0\end{bmatrix}.$$

The points of $PG(2,16)$ are generated by T as follows: $P_i=P\,[1,0,0]\,T^i;\quad i=0,\ldots 272$.
To find the lines in $PG(2,16)$:Let l_0 contains of 17 points such that the third coordinate of it is equal to zero. Then the points $P_i=i$ and the lines l_i in $PG(2,16)$ can be represented by:

$$l_0=\{0,1,3,7,15,31,63,90,116,127,136,181,194,204,233,138,255\}.$$

Moreover, $l_i=l_0T^i;\quad i=0,\ldots 272$.

3. Some definitions and basic properties

Definition 3.1. *[2] An $(n;r)$-arc K in $PG(2,q)$ is a set of n points satisfies that every line meet it in less than or equal r points, that is $|K\cap l|\le r$, for all $l\in PG\,(2,16)$.*

Definition 3.2. *[7] (1) An $(n;r)$-arc K is a complete if it is not contained in a $(n+1;r)$-arc. (2) The largest values of n for which $(n;3)$-arc exists in $PG(2,q)$ is denoted by$m_r\,(2,q)$.*

Definition 3.3. *[10] If a line l of meet K in i points, that is $|K\cap l|=i$, then it is an i-secant of an (n,r)-arc K. Thus, for $i=0,1,2,3$, 0-secsnt is external, 1-secsnt is unisecant, 2-secsnt is bisecantand 3-secsnt is trisecant.*

Definition 3.4. *[9] The points out of arc K which passes through it $i2-bisecant$ of K is called a point of index i.The number of these points is denoted by c_i. So which represents the number of the points not on $2-bisecant$ of K.*

Notation . Let $\rho_i$be the number of$i-secant$of a (n;r)-arc K in $PG(2,q)$.

Definition 3.5. *[8] The secant distribution of a (n; r) -arc K is the ordered (r+1) tuple $(l_0,l_1,l_2,\ldots,l_r)$.*

Definition 3.6. *[11] Two (n;r)-arcsis said to be equivalent if they have the same $i-$ secant distribution*

Theorem 3.7. *[8] If K is a maximal (n;r)-arc in $PG(2,q)$, then the following holds:*
(1) $r=q+1$ and $K=PG(2,q)$;
(2) $r=q$ and $K=APG\,(2,q)=PG(2,q)\backslash l$ for some line l.
(3) $2\leq r<q$, $n\backslash q$, and the dual of the external lines of K forms a $((q+1-r)\,q/r;q/r)$-arc also maximal

Proof . See [8]. □

Corollary 3.8. *[8] An (n;r)-arc is a maximal if and only if every line in $PG(2,q)$ is ether n-secant or an external line.*

Corollary 3.9. *[8] If $2<r<q$ and n does not divided q, then $m_r\,(2,q)\leq(r-1)\,q+r-2$.*

Corollary 3.10. *[8] If K is an $(n;r)$-arc in $PG(2,q)$, that has an external line and if $(r,q)=1$, then $n\leq(r-1)\,q+1$.*

Proof . See [8]. □

Definition 3.11 (Stabilizer Group). *[1] To determine the automorphism stabilizer group of (n,r)-arc, K calculate every projective $\beta\in PG(2,16)$ that maps K to itself, that is $K\beta=K$ .This achieved by finding β that maps (5,3)-arc onto K,and then determining if $K\beta=K$.*

Lemma 3.12. *For a (n,r)-arc K ,the following equations hold:*

$$\sum_{i=0}^{r}\rho_i=q^2+q+1 \tag{3.1}$$

$$\sum_{i=0}^{r}i\rho_i=K(q+1)\,(1.2) \tag{3.2}$$

$$\sum_{i=0}^{r}\frac{1}{2}i\,(i-1)\,\rho_i\;=\frac{1}{2}K(K-1) \tag{3.3}$$

Proof . See [9]. □

Theorem 3.13. *[14] An (n,r) -arc K in $PG(2,q)$ is a complete if and only if $c_0=0$.*
we will study and compute the arcs K of degree 3 with different size.

4. The constructions and classifications of $(n;3)$-arc

The technique that has been used to construct and classify the set of $(n;3)$-arcs in $PG(2,16)$ is based on fixing a set of four points, that is $K=\{0,1,2,253\}$ be the set of the frame points in $PG(2,16)$ no three of them are collinear, where $P_0=(1,0,0)$, $P_1=(0,1,0)$, $P_2=(0,0,1)$, $P_{253}=(1,1,1)$.The (5,3)-arcs can be constructed by adding to A one point from $PG(2,16)/K_4$ which is 269 points which makes three a collinear as follows: $K_5=K\cup\{3\}$ Where, $P_3=(\omega^7,1,0)$. Then the K (5,3)-arc is the set $\{0,1,2,3,253\}$. The stabilizer of (5;3)-arc is Z_2.

The stabilizer group T has two elements $T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ of order one and $T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ of order two. To show that T isomorphic to $(Z_2, +_2)$. (1) define a mapping $f : (T, .) \to (Z_2, +_2)$ by $f(T_1) = [0]$ and $f(T_2) = [1]$. (2) Define T and Z_3 by the following tables:

$\cdot$	T_1	T_2
T_1	T_1	T_2
T_2	T_2	T_1

$+_2$	[0]	[1]
[0]	[0]	[1]
[1]	[1]	[0]

It is clear that γ is bijective function. To prove that γ is homomorphism, we find

$$f(T_1.T_1) = f(T_1) = [0] \text{ and } f(T_1) +_2 f(T_1) = [0];$$
$$f(T_1.T_2) = f(T_2) = [1] \text{ and } f(T_1) +_2 f(T_2) = [1];$$
$$f(T_2.T_2) = f(T_1) = [0] \text{ and } f(T_2) +_2 f(T_2) = [0].$$

So the stabilizer group with multiplicative of matrices isomorphic to $(Z_2, +_2)$.

In the same way constructed the $(n;3)$-arc for $n > 6$ by calculating the $\boldsymbol{c_0}$ of each $(n;3)$-arcs and adding them to each $(n;3)$-arcs to construct $(n+1;3)$-arcs Then finding the secant distribution $\{l_0, l_1, l_2, l_3\}$ for each $(n+1;3)$-arc, then choosing a $(n+1;3)$-arc from each set of the same secant distribution to get the inequivalent $(n+1;3)$-arcs.

The results of the construction and classifications of $(n;3)$-arcs $n \geq 6$ in $PG(2,16)$ are in the following subsections:

4.1. The constructions and classifications of (6;3)-arcs

To construct (6;3)-arc compute the c_0, which is 254, that is the points not on any bisecant of (5;3)-arc, then added them to the (5;3)-arc these points as follows:
{ 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54,55, 56, 57, 58, 59, 60, 61, 62, 64, 65, 66, 67, 68, 69, 70, 71,72,73,74,75,76, 77,78, 79,80,81,82,83,84,85, 86, 87, 88, 89, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104,105,106,107, 108,109, 110,111, 112, 113, 114, 115, 117, 118, 119, 120, 21, 122, 123, 124, 125, 126, 128, 129, 130, 131, 132, 133, 134, 135, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 193, 195, 196, 197, 198, 199, 200, 201, 202, 203, 205, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 216, 217, 218, 219, 220, 221, 222, 223, 224, 225, 226, 227, 228, 229, 230, 231, 232, 234, 235, 236, 237, 239, 240, 241, 242, 243, 244, 245, 246, 247, 248, 249, 250, 251, 252, 254, 256, 257, 258, 259, 260, 261, 262, 263, 264, 265, 266, 267, 268, 269, 270, 271, 272 }.

Then added one point separately of $\boldsymbol{c_0}$ to (5;3)-arc. Hence the number of (6;3)-arcs is 254. The secant distribution$\{l_0\ , l_1, l_2, l_3\}$ for each (6;3)-arc is calculated, and then the (6;3)-arcs separated into a number of sets of which have inequivalent secant distribution. Then choosing a (6;3)-arc from each set of the same secant distribution and we obtained an inequivalent $K_6^i\ ; i = 1, 2, 3$. The stabilizes of the three projectivelty inequivalent (6;3)-arc are I, Z_3, I respectively since .

(1) The Group of projectivities of $K_6^{(1)}$ and $K_6^{(3)}$have one projective, $T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ of order one.

(2) The Group of projectivities of $K_6^{(2)}$ has three projectiveties, $T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $T_2 = \begin{bmatrix} 0 & 0 & 1 \\ \omega^7 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ and $T_3 = \begin{bmatrix} \omega^8 & \omega^8 & \omega^8 \\ 1 & \omega^8 & 1 \\ \omega^2 & 0 & 0 \end{bmatrix}$,one matrix of order one and two matrix of order three.

The stabilizer group of $K_6^{(2)}$ isomorphic to $(Z_3, +_3)$. To show that Let $T = \{T_1, T_2, T_3\}$, $Z_3 = \{[0], [1], [2]\}$ and define $\gamma : (T, .) \to (Z_3, +_3)$ such that; $\gamma(T_1) = [0]$, $\gamma(T_2) = [1]$ and $\gamma(T_3) = [2]$.

(3) Define T and Z_3 by the following tables:

$\cdot$	T_1	T_2	T_3
T_1	T_1	T_2	T_3
T_2	T_2	T_3	T_1
T_3	T_3	T_1	T_1

$+_2$	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

It is clear that γ is bijective function. To prove that γ is homomorphism, we find
$\gamma(T_1.T_1) = \gamma(T_1) = [0]$ and $\gamma(T_1) +_3 \ \gamma(T_1) = [0]$;
$\gamma(T_1.T_2) = \gamma(T_2) = [1]$ and $\gamma(T_1) +_3 \ \gamma(T_2) = [1]$;
$\gamma(T_1.T_3) = \gamma(T_3) = [2]$ and $\gamma(T_1) +_3 \ \gamma(T_3) = [2]$;
$\gamma(T_2.T_2) = \gamma(T_3) = [2]$ and $\gamma(T_2) +_3 \ \gamma(T_2) = [2]$;
$\gamma(T_3.T_3) = \gamma(T_2) = [1]$ and $\gamma(T_3) +_3 \ \gamma(T_3) = [1]$;
$\gamma(T_2.T_3) = \gamma(T_1) = [0]$ and $\gamma(T_2) +_3 \ \gamma(T_3) = [0]$;
$\gamma(T_3.T_2) = \gamma(T_1) = [0]$ and $\gamma(T_3) +_3 \ \gamma(T_2) = [0]$. Hence γ is homomorphism implies that α is isomorphism.
The details of (6;3)-arc in the table1.

Table 1: Details of K_6 in $PG(2, 16)$.

$\boldsymbol{K_6^{(i)}; i = 1, 2, 3}$	$\{\boldsymbol{l_0}\ ,\boldsymbol{l_1}, \boldsymbol{l_2}, \boldsymbol{l_3}\}$	**Inequivalent arc**	**The No. of $\boldsymbol{c_0}$**	**Stabilize group**
$K_6^{(1)}$	$\{185, 75, 12, 1\}$	K$\cup\{4\}$	239	I
$K_6^{(2)}$	$\{184, 78, 9, 2\}$	K$\cup\{5\}$	225	Z_3
$K_6^{(3)}$	$\{183, 81, 6, 3\}$	K$\cup\{10\}$	253	I

4.2. The constructions and classifications of (7;3)-arcs

In the same above way, the number of (7;3)-arc that have been constructed is (717). Among of these arcs, there are (5) have the inequivalent classes of secant distribution and c_0 that shown in table 2.

Table 2: Distinct classes and inequivalent arcs with No. of c_0 of K_7.

$\boldsymbol{K_7^{(i)}}$;$\boldsymbol{i = 1, 2, 3,4,5}$	$\{\boldsymbol{l_0}$,$\boldsymbol{l_1}, \boldsymbol{l_2}, \boldsymbol{l_3}\}$	**Inequivalent Arcs**	**No.of $\boldsymbol{c_0}$**	Stabilize group
$K_7^{(1)}$	$\{170, 92, 6, 5\}$	K∪ $\{5, 32\}$	238	$Z_2 \times Z_2$
$K_7^{(2)}$	$\{171, 89, 9, 4\}$	K∪ $\{5, 4\}$	252	I
$K_7^{(3)}$	$\{172, 86, 12, 3\}$	K∪ $\{18, 4\}$	224	I
$K_7^{(4)}$	$\{173, 83, 15, 2\}$	K∪ $\{4, 11\}$	196	I
$K_7^{(5)}$	$\{174, 80, 18, 1\}$	K∪ $\{10, 12\}$	210	I

4.3. The constructions and classifications of (8;3)-arcs

Here the number of (8;3)-arcs that have been constructed is (1120). The number of distinct (8;3)-arcs is (7) arcs, having the inequivalent classes distributions and c_0 that shown in the table 3:

Table 3: Distinct classes and inequivalent arcs with No. of c_0 of K_8.

$\boldsymbol{K_8^{(i)}}$;$\boldsymbol{i = 1, \ldots ,7}$	$\{\boldsymbol{l_0}$,$\boldsymbol{l_1}, \boldsymbol{l_2}, \boldsymbol{l_3}\}$	**Inequivalent Arcs**	**No. of $\boldsymbol{c_0}$**	**Stabilizer group**
$K_8^{(1)}$	$\{158, 101, 7, 7\}$	K∪ $\{4, 5, 6\}$	223	I
$K_8^{(2)}$	$\{159, 98, 10, 6\}$	K∪ $\{4, 5, 36\}$	251	I
$K_8^{(3)}$	$\{160, 95, 13, 5\}$	K∪ $\{5, 32, 224\}$	209	Z_2
$K_8^{(4)}$	$\{161, 92, 16, 4\}$	K∪ $\{10, 12, 11\}$	170	I
$K_8^{(5)}$	$\{162, 89, 19, 3\}$	K∪ $\{4, 11, 13\}$	197	I
$K_8^{(6)}$	$\{163, 86, 22, 2\}$	K∪ $\{18, 4, 9\}$	183	I
$K_8^{(7)}$	$\{164, 83, 25, 1\}$	K∪ $\{4, 11, 21\}$	224	I

4.4. The constructions and classifications of (9;3)-arcs

Here, by the same above way, the number of (9;3)-arcs that have been constructed is (1457). Among these arcs, there are (9) distinct arcs where each one has the class of secant distribution and c_0 that shown in table 4.

Table 4: Distinct classes and inequivalent arcs with No. of c_0 of K_9.

$\boldsymbol{K_9^{(i)};i=1,\ldots,9}$	$\{\boldsymbol{l_0},\boldsymbol{l_1},\boldsymbol{l_2},\boldsymbol{l_3}\}$	**Inequivalent Arcs**	**No.of $\boldsymbol{c_0}$**	**Stabilize group**
$K_9^{(1)}$	$\{147,108,9,9,\}$	K$\cup\{10,12,11,24\}$	237	I
$K_9^{(2)}$	$\{148,105,12,7\}$	K$\cup\{4,5,36,6\}$	159	I
$K_9^{(3)}$	$\{149,102,15,7\}$	K$\cup\{5,32,224,80\}$	169	I
$K_9^{(4)}$	$\{150,99,18,6\}$	K$\cup\{10,12,11,19\}$	250	I
$K_9^{(5)}$	$\{151,96,21,5\}$	K$\cup\{4,11,13,26\}$	222	I
$K_9^{(6)}$	$\{152,93,24,4\}$	K$\cup\{18,4,9,12\}$	195	I
$K_9^{(7)}$	$\{153,90,27,3\}$	K$\cup\{4,11,13,25\}$	210	I
$K_9^{(8)}$	$\{154,87,30,2\}$	K$\cup\{4,11,21,5\}$	183	I
$K_9^{(9)}$	$\{155,84,33,1\}$	A$\cup\{5,32,224,43\}$	146	I

4.5. The constructions and classifications of(10;3)-arcs

The number of (10;3)-arcs that have been made up is (1771). Here there are (11) (10;3)-arcs. The related information of (10;3)-arcs shown in table 5.

Table 5: Distinct classes and inequivalent arcs with No. of c_0 of K_{10}.

$\boldsymbol{K_9^{(i)};i=1,\ldots,11}$	$\{\boldsymbol{l_0},\boldsymbol{l_1},\boldsymbol{l_2},\boldsymbol{l_3}\}$	**Inequivalent Arcs**	**No.of $\boldsymbol{c_0}$**	**Stabilizer group**
$K_{10}^{(1)}$	$\{137,113,12,11\}$	$K_9^{(9)}\cup\{102\}$	123	I
$K_{10}^{(2)}$	$\{138,110,15,10\}$	$K_9^{(2)}\cup\{21\}$	136	I
$K_{10}^{(3)}$	$\{139,107,18,9\}$	$K_9^{(8)}\cup\{40\}$	147	I
$K_{10}^{(4)}$	$\{140,104,21,8\}$	$K_9^{(7)}\cup\{23\}$	172	I
$K_{10}^{(5)}$	$\{141,101,24,7\}$	$K_{19}^{(3)}\cup\{6\}$	156	I
$K_{10}^{(6)}$	$\{142,98,27,6\}$	$K_9^{(1)}\cup\{14\}$	196	I
$K_{10}^{(7)}$	$\{143,95,30,5\}$	$K_9^{(6)}\cup\{11\}$	183	I
$K_{10}^{(8)}$	$\{144,92,33,4\}$	$K_9^{(4)}\cup\{5\}$	208	I
$K_{10}^{(9)}$	$\{145,89,36,3\}$	$K_{10}^{(5)}\cup\{24\}$	221	I
$K_{10}^{(10)}$	$\{146,86,39,2\}$	$K_9^{(1)}\cup\{48\}$	236	I
$K_{10}^{(11)}$	$\{147,83,42,1\}$	$K_9^{(4)}\cup\{71\}$	249	I

4.6. The constructions and classifications of (11;3)-arcs

The number of (11;3)-arcs that have been established is (2027) and the number of distinct arcs is (13) that have the inequivalent classes of secant distribution and c_0 shown in the table 6.

Table 6: Distinct classes and inequivalent arcs with No. of c_0 of K_{11}.

$K_{11}^{(i)}$;$i = 1, \ldots$,13	$\{l_0, l_1, l_2, l_3\}$	**Inequivalent arcs**	**No. of c_0**	**S.G**
$K_{11}^{(1)}$	$\{127, 119, 13, 14\}$	$K_{10}^{(1)} \cup \{126\}$	94	I
$K_{11}^{(2)}$	$\{128, 116, 16, 13\}$	$K_{10}^{(2)} \cup \{37\}$	105	I
$K_{11}^{(3)}$	$\{129, 113, 19, 12\}$	$K_{10}^{(3)} \cup \{158\}$	116	I
$K_{11}^{(4)}$	$\{130, 110, 22, 11\}$	$K_{10}^{(4)} \cup \{124\}$	126	I
$K_{11}^{(5)}$	$\{131, 107, 25, 10\}$	$K_{10}^{(5)} \cup \{10\}$	136	I
$K_{11}^{(6)}$	$\{132, 104, 28, 9\}$	$K_{10}^{(6)} \cup \{137\}$	149	I
$K_{11}^{(7)}$	$\{133, 101, 31, 8\}$	$K_{10}^{(7)} \cup \{19\}$	158	I
$K_{11}^{(8)}$	$\{134, 95, 34, 7\}$	$K_{10}^{(8)} \cup \{, 6\}$	170	I
$K_{11}^{(9)}$	$\{135, 95, 37, 6\}$	$K_{10}^{(7)} \cup \{104\}$	182	I
$K_{11}^{(10)}$	$\{136, 92, 40, 5\}$	$K_{10}^{(10)} \cup \{4\}$	194	I
$K_{11}^{(11)}$	$\{137, 89, 43, 4\}$	$K_{10}^{(10)} \cup \{8\}$	208	I
$K_{11}^{(12)}$	$\{138, 86, 46, 3\}$	$K_{10}^{(9)} \cup \{39\}$	220	I
$K_{11}^{(13)}$	$\{139, 83, 49, 2\}$	$K_{10}^{(11)} \cup \{29\}$	234	I

4.7. The constructions and classifications of(12;3)-arc

The number of all (12;3)-arcs is (2092). Among of these arcs there are (15). The related information of these shown in table 7.

4.8. The constructions and classifications of(13;3)-arcs

Here the number of (13;3)-arcs that has been constructed is (2118) and the number of distinct arc is (17).The related information are shown in table 8.

4.9. The constructions and classifications of(14;3)-arcs

In this process, the number of (14;3)-arc established is (2132) .also there are (18) distinct (14;3)-arc, The related information are shown in table 9.

4.10. The constructions and classifications of(15;3)-arc

The number of (15;3)-arc is (1838) and there are (18) (15;3)-arcs, having the following an inequivalent classes of secant distribution and c_0 that shown in table 10.

4.11. The constructions and classifications of(16;3)-arc

The number of (16;3)-arcs in $PG(2, 16)$ is (1481) and the number of distinct (16;3)-arcs is (20),having the following classes of secant distribution and c_0 that shown in table11

4.12. The constructions and classifications of (17;3)-arc

The number of (17;3)-arc is (1206) and there are (22) distinct (17 ;3)-arcs ,having the following information are shown in table12.

Table 7: Distinct classes and inequivalent arcs with No. of c_0 of K_{12}.

$K_{12}^{(i)}$;$i = 1,\ldots,15$	$\{l_0, l_1, l_2, l_3\}$	Inequivalent arcs	No.of c_0	S.G
$K_{12}^{(1)}$	$\{118, 123, 15, 17\}$	$K_{11}^{(1)} \cup \{57\}$	72	I
$K_{12}^{(2)}$	$\{119, 120, 18, 16\}$	$K_{11}^{(2)} \cup \{19\}$	80	I
$K_{12}^{(3)}$	$\{120, 117, 21, 15\}$	$K_{11}^{(3)} \cup \{126\}$	89	I
$K_{12}^{(4)}$	$\{121, 114, 24, 14\}$	$K_{11}^{(4)} \cup \{6\}$	99	I
$K_{12}^{(5)}$	$\{122, 111, 27, 13\}$	$K_{11}^{(5)} \cup \{22\}$	107	I
$K_{12}^{(6)}$	$\{123, 108, 30, 12\}$	$K_{11}^{(6)} \cup \{27\}$	118	I
$K_{12}^{(7)}$	$\{124, 105, 33, 11\}$	$K_{11}^{(7)} \cup \{20\}$	128	I
$K_{12}^{(8)}$	$\{125, 102, 36, 10, \}$	$K_{11}^{(8)} \cup \{21\}$	139	I
$K_{12}^{(9)}$	$\{126, 99, 39, 9\}$	$K_{11}^{(9)} \cup \{35\}$	149	I
$K_{12}^{(10)}$	$\{127, 96, 42, 8\}$	$K_{11}^{(10)} \cup \{17\}$	160	I
$K_{12}^{(11)}$	$\{128, 93, 45, 7\}$	$K_{11}^{(11)} \cup \{13\}$	172	I
$K_{12}^{(12)}$	$\{129, 90, 48, 6\}$	$K_{11}^{(12)} \cup \{9\}$	183	I
$K_{12}^{(13)}$	$\{130, 87, 51, 5\}$	$K_{11}^{(13)} \cup \{17\}$	195	I
$K_{12}^{(14)}$	$\{131, 84, 54, 4\}$	$K_{11}^{(12)} \cup \{77\}$	207	I
$K_{12}^{(15)}$	$\{132, 81, 57, 3\}$	$K_{11}^{(13)} \cup \{38\}$	221	I

4.13. The constructions and classifications of (18;3)-arc

The number of (18;3)-arcs is (952) and there are (21) distinct (18;3)-arcs. The related information are shown in table 13. .

4.14. The constructions and classifications of (19;3)-arc

In this process, the number of (19;3)-arc is (604) and there are (21) distinct (19;3)-arcs. The details of these arcs are shown in table 14.

4.15. The constructions and classifications of (20;3)-arc

The number of all (19;3)-arcs in $PG(2,16)$ is (389) and there are (20) distinct (20;3)-arcs. The details of these arcs are shown in table 15.

4.16. The constructions and classifications of (21;3)-arc

Hence the number of all (21;3)-arcs that have been constructed is (194) and there are (19) distinct (21;3)-arcs. The related information are shown in table16.

4.17. The constructions and classifications of (22;3)-arc

The number of all (22;3)-arc established is (102) and there are (16) distinct (22;3)-arcs. The details of these arcs are shown in table 17.

Table 8: Distinct classes and inequivalent arcs with No. of c_0 of K_{13}.

$K_{13}^{(i)}$; $i = 1,\ldots,17$	$\{l_0, l_1, l_2, l_3\}$	Inequivalent arcs	No. of c_0	S.G
$K_{13}^{(1)}$	$\{110, 125, 18, 20\}$	$K_{12}^{(1)} \cup \{53\}$	55	I
$K_{13}^{(2)}$	$\{111, 122, 21, 19\}$	$K_{12}^{(1)} \cup \{10\}$	66	I
$K_{13}^{(3)}$	$\{112, 119, 24, 18\}$	$K_{12}^{(2)} \cup \{46\}$	67	I
$K_{13}^{(4)}$	$\{113, 116, 27, 17\}$	$K_{12}^{(3)} \cup \{10\}$	78	I
$K_{13}^{(5)}$	$\{114, 113, 30, 16\}$	$K_{12}^{(4)} \cup \{24\}$	83	I
$K_{13}^{(6)}$	$\{115, 110, 33, 15\}$	$K_{12}^{(5)} \cup \{12\}$	91	I
$K_{13}^{(7)}$	$\{116, 107, 36, 14\}$	$K_{12}^{(6)} \cup \{19\}$	103	I
$K_{13}^{(8)}$	$\{117, 104, 39, 13\}$	$K_{12}^{(7)} \cup \{13\}$	110	I
$K_{13}^{(9)}$	$\{118, 101, 42, 12\}$	$K_{12}^{(8)} \cup \{16\}$	119	I
$K_{13}^{(10)}$	$\{119, 98, 45, 11\}$	$K_{12}^{(9)} \cup \{13\}$	129	I
$K_{13}^{(11)}$	$\{120, 95, 48, 10\}$	$K_{12}^{(10)} \cup \{19\}$	141	I
$K_{13}^{(12)}$	$\{121, 92, 51, 9\}$	$K_{12}^{(11)} \cup \{19\}$	150	I
$K_{13}^{(13)}$	$\{122, 89, 54, 8\}$	$K_{12}^{(12)} \cup \{53\}$	193	I
$K_{13}^{(14)}$	$\{123, 86, 57, 7\}$	$K_{12}^{(13)} \cup \{30\}$	161	I
$K_{13}^{(15)}$	$\{124, 83, 60, 6\}$	$K_{12}^{(14)} \cup \{33\}$	182	I
$K_{13}^{(16)}$	$\{125, 80, 63, 5\}$	$K_{12}^{(15)} \cup \{23\}$	196	I
$K_{13}^{(17)}$	$\{126, 77, 66, 4\}$	$K_{12}^{(15)} \cup \{121\}$	208	I

4.18. The constructions and classifications of (23;3)-arc

The number of (23;3)-arcs in $PG(2,16)$ is (39) and the number of distinct (23;3)-arcs is (12). The related information of these arcs are shown in table 18.

5. The largest size of arc of degree three in $PG(2,16)$

In this section we used the theorem 5.1 to compute the $(n;3)$-arcs for some $n = 24$ in order to determinate the size of largest $(n;3)$-arc in $PG(2,16)$.

Theorem 5.1. *[15] Let $q \equiv 1(mod\ 3)$. Let $\omega \in F_q$, $\omega \neq 1$ such that $\omega^3 = 1$. Let $c \in F_q$. Consider the sets S_1 and $S_2(c)$ of points with the following coordinates:*

$$S_1 = \{\ [1,1,1],\ [1,1,\omega],\ [1,1,\omega^2],\ [1,\omega,1],\ [1,\omega,\omega],[1,\omega,\omega^2],[1,\omega^2,1],[1,\omega^2,\omega],[1,\omega^2,\omega^2]\ \}$$

$$S_2(c) = \{\ [1,0,c],\ [\omega,0,c],\ [\omega^2,0,c],\ [0,c,1],\ [0,c,\omega],[0,c,\omega^2],[c,1,0],[c,\omega,0],[c,\omega^2,0]\ \}$$

Then $S_1 \cup S_2(c)$ is an (18;3)-arc if and only if $c \neq 0$ and $c^3 \neq \pm 1$

The values ω=,ω^5, ω^{10} and the values of c as follows:

$$c = \omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^7, \omega^9, \omega^{12}$$

Table 9: Distinct classes and inequivalent arcs with No. of c_0 of K_{14}.

$K_{14}^{(i)}$;$i = 1, \ldots, 18$	$\{l_0, l_1, l_2, l_3\}$	Inequivalent arcs	No. of c_0	S.G
$K_{14}^{(1)}$	$\{103, 15, 22, 23\}$	$K_{13}^{(1)} \cup \{19\}$	41	I
$K_{14}^{(2)}$	$\{104, 122, 25, 22\}$	$K_{13}^{(2)} \cup \{107\}$	43	I
$K_{14}^{(3)}$	$\{105, 119, 28, 21\}$	$s \cup \{50\}$	50	I
$K_{14}^{(4)}$	$\{106, 116, 31, 20\}$	$K_{13}^{(4)} \cup \{49\}$	58	I
$K_{14}^{(5)}$	$\{107, 113, 34, 19\}$	$K_{13}^{(5)} \cup \{43\}$	61	I
$K_{14}^{(6)}$	$\{108, 110, 37, 18\}$	$K_{13}^{(6)} \cup \{24\}$	71	I
$K_{14}^{(7)}$	$\{108, 107, 40, 17\}$	$K_{13}^{(7)} \cup \{34\}$	82	I
$K_{13}^{(8)}$	$\{110, 104, 43, 16\}$	$K_{13}^{(8)} \cup \{36\}$	84	I
$K_{14}^{(9)}$	$\{111, 101, 46, 15\}$	$K_{13}^{(9)} \cup \{28\}$	92	I
$K_{14}^{(10)}$	$\{112, 98, 49, 14\}$	$K_{13}^{(10)} \cup \{22\}$	105	I
$K_{14}^{(11)}$	$\{113, 95, 52, 13\}$	$K_{13}^{(11)} \cup \{23\}$	113	I
$K_{14}^{(12)}$	$\{114, 92, 55, 12\}$	$K_{13}^{(12)} \cup \{14\}$	112	I
$K_{14}^{(13)}$	$\{115, 89, 58, 11\}$	$K_{13}^{(13)} \cup \{20\}$	119	I
$K_{14}^{(14)}$	$\{116, 86, 61, 10\}$	$K_{13}^{(14)} \cup \{48\}$	140	I
$K_{14}^{(15)}$	$\{117, 83, 64, 9\}$	$K_{13}^{(15)} \cup \{22\}$	145	I
$K_{14}^{(16)}$	$\{118, 80, 67, 8\}$	$K_{13}^{(16)} \cup \{33\}$	161	I
$K_{14}^{(17)}$	$\{119, 77, 67, 8\}$	$K_{13}^{(17)} \cup \{22\}$	170	I
$K_{14}^{(18)}$	$\{120, 74, 73, 6\}$	$K_{13}^{(17)} \cup \{28\}$	184	I

5.1. The constructions and classifications of (24;3)-arc

To construct (24;3)-arcs we choose the case whenω=ω^5, c =ω^2. So the points of the arc $S_1 \cup S_2(c)$ as follows:

$$\left\{ \begin{array}{c} [1,1,1]\,,\ [\omega^{10},\omega^{10},1]\,,[\omega^5,\omega^5,1]\,,\ [1,\omega^5,1]\,,\ [\omega^{10},1,1]\,,[\omega^5,\omega^{10},1]\,,[1,\omega^{10},1]\,,[\omega^{10},\omega^5,1]\,, \\ [\omega^5,1,1]\,,[\omega^{13},0,1]\,,[\omega^{14},0,1]\,,[\omega,0,1]\,,[0,\omega^2,1]\,,[1,\omega,1]\,,[0,1,1]\,,[\omega^2,1,0]\,,[\omega,1,0]\,,[1,1,0] \end{array} \right\}.$$

Then the points of index zero calculated and we obtained that is $c_0 = 15$. So $S_1 \cup S_2\,(c) \cup \{37, 44, 47, 58, 58, 201\}$ is (24;3)-arc which is obtained by adding first point of c_0 of each $(n;3)$-arc, for some $n = 18, \ldots 23$. The (24;4) is complete since $c_0 = 0$.

5.2. The constructions and classifications of (25;3)-arc

To construct (25;3)-arcs we choose the case whenω=ω^5, c =ω^7. So the points of the arc $S_1 \cup S_2(c)$ as follows:

$$\left\{ \begin{array}{c} [1,1,1]\,,\ [\omega^{10},\omega^{10},1]\,,[\omega^5,\omega^5,1]\,,\ [1,\omega^5,1]\,,\ [\omega^{10},1,1]\,,[\omega^5,\omega^{10},1]\,,[1,\omega^{10},1]\,,[\omega^{10},\omega^5,1]\,,[\omega^5,1,1]\,, \\ [\omega^8,0,1]\,,[\omega^9,0,1]\,,[\omega^{10},0,1]\,,[0,\omega^7,1]\,,[1,\omega^6,1]\,,[0,\omega^5,1]\,,[\omega^7,1,0]\,,[\omega^6,1,0]\,,[\omega^5,1,0] \end{array} \right\}$$

Table 10: Distinct classes and inequivalent arcs with No. of c_0 of K_{15}.

$K_{15}^{(i)}; i=1,\dots,17$	$\{l_0, l_1, l_2, l_3\}$	**Inequivalent arcs**	**No.of c_0**	**S.G**
$K_{15}^{(1)}$	$\{97,123,27,26\}$	$K_{14}^{(1)} \cup \{11\}$	30	I
$K_{15}^{(2)}$	$\{98,120,30,25\}$	$K_{14}^{(2)} \cup \{48\}$	34	I
$K_{15}^{(3)}$	$\{99,117,33,24\}$	$K_{14}^{(3)} \cup \{45\}$	36	I
$K_{15}^{(4)}$	$\{100,114,36,24\}$	$K_{14}^{(4)} \cup \{28\}$	45	I
$K_{15}^{(5)}$	$\{101,111,39,22\}$	$K_{14}^{(5)} \cup \{26\}$	53	I
$K_{15}^{(6)}$	$\{102,108,42,21\}$	$K_{14}^{(6)} \cup \{119\}$	53	I
$K_{15}^{(7)}$	$\{103,105,45,20\}$	$K_{14}^{(7)} \cup \{29\}$	63	I
$K_{15}^{(8)}$	$\{104,102,48,19\}$	$K_{13}^{(8)} \cup \{30\}$	67	I
$K_{15}^{(9)}$	$\{105,99,51,18\}$	$K_{14}^{(9)} \cup \{37\}$	70	I
$K_{15}^{(10)}$	$\{106,96,54,17\}$	$K_{14}^{(10)} \cup \{24\}$	87	I
$K_{15}^{(11)}$	$\{107,93,57,16\}$	$K_{14}^{(11)} \cup \{22\}$	87	I
$K_{15}^{(12)}$	$\{108,90,60,15\}$	$K_{14}^{(12)} \cup \{35\}$	98	I
$K_{15}^{(13)}$	$\{109,87,63,14\}$	$K_{14}^{(13)} \cup \{57\}$	102	I
$K_{15}^{(14)}$	$\{110,84,66,13\}$	$K_{14}^{(14)} \cup \{35\}$	109	I
$K_{15}^{(15)}$	$\{111,81,69,12\}$	$K_{14}^{(15)} \cup \{12\}$	123	I
$K_{15}^{(16)}$	$\{112,78,72,11\}$	$K_{14}^{(16)} \cup \{28\}$	134	I
$K_{15}^{(17)}$	$\{113,75,75,10\}$	$K_{14}^{(17)} \cup \{23\}$	140	I
$K_{15}^{(18)}$	$\{114,72,78,9\}$	$K_{14}^{(18)} \cup \{45\}$	150	I

Then the points of index zero calculated and we obtained that $c_0 = 21$. So $S_1 \cup S_2(c) \cup \{22,46,55,59,76,228,186\}$ is (25;3)-arc which is obtained by adding first point of c_0 of $(n;3)$-arc, for some $n = 18,\dots 24$. The (25;4) is complete since $c_0 = 0$.

5.3. The constructions and classifications of (26;3)-arc

In ordered to establish (26;3)-arcs we choose the case when $\omega{=}\omega^5, c=\omega$. So the points of the arc $S_1 \cup S_2(c)$ as follows:

$$\left\{ \begin{array}{c} [1,1,1],\ [\omega^{10},\omega^{10},1],[\omega^5,\omega^5,1],\ [1,\omega^5,1],\ [\omega^{10},1,1],[\omega^5,\omega^{10},1],[1,\omega^{10},1],[\omega^{10},\omega^5,1],[\omega^5,1,1], \\ [\omega^{14},0,1],[1,0,1],[\omega,0,1],[0,\omega,1],[0,1,1],[1,\omega^{14},0],[\omega,1,0],[1,1,0],[\omega^{14},1,0] \end{array} \right\}.$$

Then the points of index zero calculated and we obtained that $c_0 = 30$. Hence $S_1 \cup S_2(c) \cup \{5,13,20,23,27,83,192,224\}$ is (26;3)-arc which is obtained by adding first point of c_0 of $(n;3)$-arc, for some $n = 18,\dots 25$. The (26;4) is complete since $c_0 = 0$.

5.4. The constructions and classifications of (27;3)-arc

To establish (27;3)-arcs we choose the case whenω=ω^5, $c=\omega^6$. So the points of the arc $S_1 \cup S_2(c)$ as follows:

$$\left\{ \begin{array}{c} [1,1,1]\,,\ [\omega^{10},\omega^{10},1]\,,[\omega^5,\omega^5,1]\,,\ [1,\omega^5,1]\,,\ [\omega^{10},1,1]\,,[\omega^5,\omega^{10},1]\,,[1,\omega^{10},1]\,,[\omega^{10},\omega^5,1]\,,[\omega^5,1,1]\,, \\ [\omega^9,0,1]\,,[\omega^{10},0,1]\,,[\omega^{11},0,1]\,,[0,\omega^6,1]\,,[0,\omega^5,1]\,,[0,\omega^4,1]\,,[\omega^6,1,0]\,,[\omega^5,1,0]\,,[\omega^4,1,0] \end{array} \right\}.$$

Then the points of index zero calculated and we obtained that $c_0 = 30$. So $S_1 \cup S_2(c) \cup \{18, 52, 54, 76, 77, 121, 209, 251, 261\}$ is (27;3)-arc which is obtained by adding first point of c_0 of each $(n;3)$-arc, for some $n = 18, \ldots 26$. The (27;4) is complete since $c_0 = 0$.

Remark 5.2. *When we tried the rest of the possibilities for the values of* ω, c *we got the same conclusion for that the largest size of* $(n;3)$*-arc in* $PG(2,16)$ *is 27.*

6. Conclusion

In this paper we study the number of full $(n;3)$-arcs, which have the inequivalent secant distributions and the largest size of $(n;3)$-arcs with property of completeness, from the previous calculation the following results obtained:

1- The number of $(n;3)$-arcs in $PG(2,16)$ is classified.
2- The number of inequivalent $(n;3)$-arcs in $PG(2,16)$ is classified.
3- The largest complete $(n;3)$-arcs of size $n = 27$ is established.
4- The stabilizer group of these arcs are computed.

References

[1] N.A.M. Al-Seraji, *The group action on the finite projective planes of orders 53,61,64*, J. Discrete Math. Sci. Crypt. 23(8) (2020) 1573–1582.
[2] N.A.M. Al-Seraji, A. Bahkeet and Z. Sadiq, *Study of Orbits on the finite projective plane*, J. Interdiscip. Math. 23(6) (2020) 1187–1195.
[3] S. Ball and J.W.P. Hirschfeld, *Bounds on (n; r)-Arcs and Their Application to Linear Codes*, Department of Math., University of Sussex, Brighton BNI 9RF. UK., 2005
[4] D. Barlotti, S. Marvugini and F. Pambianco, *The non-exstence of some NMDS codes and the extremal size of complete(r;3)-arcs in* $PG(2,16)$, Des. Codes Cryptogr. 72 (2014) 129–134.
[5] G.R. Cook, *Arcs in a Finite Projective Plane*, PH.D., Thesis, University of Sussex, United Kingdom, 2011.
[6] Gap Group, *GAP. 2021*, Reference manual URL, `http://www.gap-system.org`.
[7] J.W.P. Hircshfield, *Finite Projective Space of Three Dimensions*, Oxford University Press, Oxford, 1985.
[8] J.W.P. Hirschfeld, *Projective Geometries Over Finite Feilds*, 2nd Edition, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1998.
[9] J.W.P. Hirschfeld and J.A. Thas, *General Galois Gemetries*, Oxford University Press, Oxford, 1991.
[10] J.W.P. Hirschfeld and J.A. Thas, *Hermitian varieties*, Gen.l Galois Geom. (2016) 57–97.
[11] J.W.P. Hirschfield and J.F. Voloch, *Group-arcs of prime order on cubic curves*, Finite Geom. Combin. 191(2015) 177–185.
[12] S. Marcugini, A. Milani and F. Pambianco, *Maximal (n;3)-arcs in PG(2,13)*, Discrete Math. 228 (1999) 421–426.
[13] S. Marcugini, A. Milani and F. Pambianco, *Classification of the (n; 3)-arcs in PG(2,13)*, J. Discrete Math. 294 (2005) 139–145.
[14] E.V.D. Pichanick and J.W.P. Hirschfeld, *Bounded for arcs of arbitrary degree in finite Desarguesian Planes*, J. Combin. Designs 24(4) (2016) 184–196.
[15] H. Sticker, *Classification of Arcs in small Desarguesian Projective Panes*, University Gent, 2012.

Table 11: Distinct classes and inequivalent arcs with No. of c_0 of K_{16}.

$\boldsymbol{K_{16}^{(i)}; i = 1, \ldots, 20}$	$\{\boldsymbol{l_0}, \boldsymbol{l_1}, \boldsymbol{l_2}, \boldsymbol{l_3}\}$	**Inequivalent arcs**	**No.of $\boldsymbol{c_0}$**	**S.G**
$K_{16}^{(1)}$	$\{90, 125, 27, 3\}$	$K_{15}^{(1)} \cup \{164\}$	14	I
$K_{16}^{(2)}$	$\{91, 122, 30, 30\}$	$K_{15}^{(2)} \cup \{265\}$	19	I
$K_{16}^{(3)}$	$\{92, 119, 33, 29\}$	$K_{15}^{(3)} \cup \{53\}$	19	I
$K_{16}^{(4)}$	$\{93, 116, 36, 28\}$	$K_{15}^{(4)} \cup \{72\}$	25	I
$K_{16}^{(5)}$	$\{94, 113, 39, 27\}$	$K_{15}^{(5)} \cup \{59\}$	32	I
$K_{16}^{(6)}$	$\{95, 110, 42, 26\}$	$K_{15}^{(6)} \cup \{45\}$	32	I
$K_{16}^{(7)}$	$\{96, 107, 45, 25\}$	$K_{15}^{(7)} \cup \{55\}$	38	I
$K_{16}^{(8)}$	$\{97, 104, 48, 24\}$	$K_{15}^{(8)} \cup \{41\}$	45	I
$K_{16}^{(9)}$	$\{98, 101, 51, 23\}$	$K_{15}^{(9)} \cup \{43\}$	47	I
$K_{16}^{(10)}$	$\{99, 98, 54, 22\}$	$K_{15}^{(10)} \cup \{37\}$	59	I
$K_{16}^{(11)}$	$\{100, 95, 57, 21\}$	$K_{15}^{(11)} \cup \{43\}$	57	I
$K_{16}^{(12)}$	$\{101, 92, 60, 20\}$	$K_{15}^{(12)} \cup \{28\}$	59	I
$K_{16}^{(13)}$	$\{102, 89, 63, 19\}$	$K_{15}^{(13)} \cup \{4\}$	71	I
$K_{16}^{(14)}$	$\{103, 86, 66, 18\}$	$K_{15}^{(14)} \cup \{10\}$	74	I
$K_{16}^{(15)}$	$\{104, 93, 69, 17\}$	$K_{15}^{(15)} \cup \{6\}$	85	I
$K_{15}^{(16)}$	$\{105, 80, 72, 16\}$	$K_{15}^{(16)} \cup \{16\}$	91	I
$K_{16}^{(17)}$	$\{106, 77, 75, 15\}$	$K_{15}^{(17)} \cup \{17\}$	97	I
$K_{16}^{(18)}$	$\{107, 74, 78, 14\}$	$K_{15}^{(18)} \cup \{14\}$	104	I
$K_{16}^{(19)}$	$\{108, 71, 81, 13\}$	$K_{15}^{(18)} \cup \{28\}$	114	I
$K_{16}^{(20)}$	$\{109, 68, 84, 12\}$	$K_{15}^{(17)} \cup \{23\}$	124	I

Table 12: Distinct classes and inequivalent arcs with No. of c_0 of K_{17}.

$K_{17}^{(i)}; i = 1, \ldots, 22$	$\{l_0\ , l_1, l_2, l_3\}$	**Inequivalent arcs**	**No. of c_0**	**S.G**
$K_{17}^{(1)}$	$\{84, 125, 28, 36\}$	$K_{16}^{(1)} \cup \{123\}$	6	I
$K_{17}^{(2)}$	$\{85, 122, 31, 35\}$	$K_{16}^{(2)} \cup \{215\}$	9	I
$K_{17}^{(3)}$	$\{86, 119, 34, 34\}$	$K_{16}^{(3)} \cup \{248]$	8	I
$K_{17}^{(4)}$	$\{87, 116, 37, 33\}$	$K_{16}^{(4)} \cup \{56\}$	14	I
$K_{17}^{(5)}$	$\{88, 113, 40, 32\}$	$K_{16}^{(5)} \cup \{65\}$	19	I
$K_{17}^{(6)}$	$\{89, 110, 43, 31\}$	$K_{16}^{(6)} \cup \{53\}$	6	I
$K_{17}^{(7)}$	$\{90, 107, 46, 30\}$	$K_{16}^{(7)} \cup \{80\}$	22	I
$K_{17}^{(8)}$	$\{91, 104, 49, 29\}$	$K_{16}^{(8)} \cup \{71\}$	25	I
$K_{17}^{(9)}$	$\{92, 101, 52, 28\}$	$K_{16}^{(9)} \cup \{48\}$	29	I
$K_{17}^{(10)}$	$\{93, 98, 55, 27\}$	$K_{16}^{(10)} \cup \{62\}$	39	I
$K_{17}^{(11)}$	$\{94, 95, 58, 25\}$	$K_{16}^{(11)} \cup \{38\}$	35	I
$K_{17}^{(12)}$	$\{95, 92, 61, 25\}$	$K_{16}^{(12)} \cup \{33\}$	41	I
$K_{17}^{(13)}$	$\{96, 89, 64, 24\}$	$K_{16}^{(13)} \cup \{14\}$	51	I
$K_{17}^{(14)}$	$\{97, 86, 67, 23\}$	$K_{16}^{(14)} \cup \{37\}$	49	I
$K_{17}^{(15)}$	$\{98, 83, 70, 22\}$	$K_{16}^{(15)} \cup \{40\}$	56	I
$K_{17}^{(16)}$	$\{99, 80, 73, 21\}$	$K_{15}^{(16)} \cup \{14\}$	58	I
$K_{17}^{(17)}$	$\{100, 77, 76, 20\}$	$K_{16}^{(17)} \cup \{24\}$	61	I
$K_{17}^{(18)}$	$\{101, 74, 79, 19\}$	$K_{16}^{(18)} \cup \{9\}$	68	I
$K_{17}^{(19)}$	$\{102, 71, 82, 18\}$	$K_{16}^{(19)} \cup \{17\}$	79	I
$K_{17}^{(20)}$	$\{103, 68, 85, 17\}$	$K_{16}^{(20)} \cup \{16\}$	82	I
$K_{17}^{(21)}$	$\{104, 65, 88, 16\}$	$K_{16}^{(19)} \cup \{141\}$	92	I
$K_{17}^{(22)}$	$\{105, 62, 91, 15\}$	$K_{16}^{(20)} \cup \{141\}$	103	I

Table 13: Distinct classes and inequivalent arcs with No. of c_0 of K_{18}.

$K_{18}^{(i)}$;$i=1,\ldots,21$	$\{l_0, l_1, l_2, l_3\}$	Inequivalent arcs	No. of c_0	S.G
$K_{18}^{(1)}$	$\{80,120,33,40\}$	$K_{17}^{(1)}\cup\{124\}$	4	I
$K_{18}^{(2)}$	$\{81,117,36,39\}$	$K_{17}^{(2)}\cup\{66\}$	4	I
$K_{18}^{(3)}$	$\{82,114,39,38\}$	$K_{17}^{(3)}\cup\{56\}$	5	I
$K_{18}^{(4)}$	$\{83,11,42,37\}$	$K_{17}^{(4)}\cup\{38\}$	9	I
$K_{18}^{(5)}$	$\{84,108,45,36\}$	$K_{17}^{(5)}\cup\{69\}$	12	I
$K_{18}^{(6)}$	$\{85,105,48,36\}$	$K_{17}^{(6)}\cup\{25\}$	12	I
$K_{18}^{(7)}$	$\{86,102,51,34\}$	$K_{17}^{(7)}\cup\{33\}$	15	I
$K_{18}^{(8)}$	$\{87,99,54,33\}$	$K_{17}^{(8)}\cup\{57\}$	17	I
$K_{18}^{(9)}$	$\{88,96,57,32\}$	$K_{17}^{(9)}\cup\{85\}$	17	I
$K_{18}^{(10)}$	$\{89,93,60,31\}$	$K_{17}^{(10)}\cup\{38\}$	26	I
$K_{18}^{(11)}$	$\{90,90,63,30\}$	$K_{17}^{(11)}\cup\{46\}$	23	I
$K_{18}^{(12)}$	$\{91,87,66,29\}$	$K_{17}^{(12)}\cup\{34\}$	28	I
$K_{18}^{(13)}$	$\{92,84,69,28\}$	$K_{17}^{(13)}\cup\{46\}$	36	I
$K_{18}^{(14)}$	$\{93,81,72,27\}$	$K_{17}^{(14)}\cup\{74\}$	35	I
$K_{18}^{(15)}$	$\{94,78,75,26\}$	$K_{17}^{(15)}\cup\{97\}$	40	I
$K_{18}^{(16)}$	$\{95,75,78,25\}$	$K_{17}^{(16)}\cup\{46\}$	42	I
$K_{18}^{(17)}$	$\{96,72,81,24\}$	$K_{17}^{(17)}\cup\{209\}$	44	I
$K_{18}^{(18)}$	$\{97,69,84,23\}$	$K_{17}^{(18)}\cup\{221\}$	43	I
$K_{18}^{(19)}$	$\{98,66,87,22\}$	$K_{17}^{(19)}\cup\{141\}$	56	I
$K_{18}^{(20)}$	$\{99,63,90,21\}$	$K_{17}^{(20)}\cup\{141\}$	54	I
$K_{18}^{(21)}$	$\{100,60,93,20\}$	$K_{17}^{(22)}\cup\{34\}$	72	I

Table 14: Distinct classes and inequivalent arcs with No. of c_0 of K_{18}.

$K_{19}^{(i)}; i = 1, \ldots, 21$	$\{l_0\ , l_1, l_2, l_3\}$	**Inequivalent arcs**	**No. of c_0**	**S.G**
$K_{19}^{(1)}$	$\{76, 116, 36, 45\}$	$K_{18}^{(1)} \cup \{210\}$	1	I
$K_{19}^{(2)}$	$\{77, 113, 39, 44\}$	$K_{18}^{(2)} \cup \{165\}$	3	I
$K_{19}^{(3)}$	$\{78, 110, 42, 43\}$	$K_{18}^{(3)} \cup \{113\}$	2	I
$K_{19}^{(4)}$	$\{79, 107, 45, 42\}$	$K_{18}^{(4)} \cup \{174\}$	6	I
$K_{19}^{(5)}$	$\{80, 104, 48, 41\}$	$K_{18}^{(5)} \cup \{81\}$	5	I
$K_{19}^{(6)}$	$\{81, 101, 51, 40\}$	$K_{18}^{(6)} \cup \{78\}$	5	I
$K_{19}^{(7)}$	$\{82, 98, 54, 39\}$	$K_{18}^{(8)} \cup \{158\}$	9	I
$K_{19}^{(8)}$	$\{83, 95, 57, 38\}$	$K_{18}^{(9)} \cup \{159\}$	9	I
$K_{19}^{(9)}$	$\{84, 92, 63, 36\}$	$\cup \{169\} K_{18}^{(7)}$	10	I
$K_{19}^{(10)}$	$\{85, 89, 63, 36\}$	$K_{18}^{(10)} \cup \{48\}$	14	I
$K_{19}^{(11)}$	$\{86, 86, 66, 35\}$	$K_{18}^{(11)} \cup \{61\}$	17	I
$K_{19}^{(12)}$	$\{87, 83, 69, 34\}$	$K_{18}^{(12)} \cup \{37\}$	19	I
$K_{19}^{(13)}$	$\{88, 80, 72, 33\}$	$K_{18}^{(13)} \cup \{97\}$	21	I
$K_{19}^{(14)}$	$\{89, 77, 75, 32\}$	$K_{18}^{(14)} \cup \{68\}$	21	I
$K_{19}^{(15)}$	$\{90, 74, 78, 31\}$	$K_{18}^{(15)} \cup \{35\}$	29	I
$K_{19}^{(16)}$	$\{91, 71, 81, 30\}$	$K_{18}^{(16)} \cup \{121\}$	26	I
$K_{19}^{(17)}$	$\{92, 68, 84, 29\}$	$K_{18}^{(17)} \cup \{101\}$	32	I
$K_{19}^{(18)}$	$\{93, 65, 87, 28\}$	$K_{18}^{(19)} \cup \{18\}$	32	I
$K_{19}^{(19)}$	$\{94, 62, 90, 27\}$	$K_{18}^{(20)} \cup \{176\}$	35	I
$K_{19}^{(20)}$	$\{95, 59, 93, 26\}$	$K_{18}^{(21)} \cup \{16\}$	45	I
$K_{19}^{(21)}$	$\{96, 56, 96, 25\}$	$K_{18}^{(21)} \cup \{,89\}$	48	I

Table 15: Distinct classes and inequivalent arcs with No. of c_0 of K_{20}.

$\boldsymbol{K_{20}^{(i)}};\boldsymbol{i=1,\ldots,20}$	$\{\boldsymbol{l_0}\ ,\boldsymbol{l_1},\boldsymbol{l_2},\boldsymbol{l_3}\}$	**Inequivalent arcs**	**No. of c_0**	**S.G**
$K_{20}^{(1)}$	$\{74,107,43,49\}$	$K_{19}^{(1)}\cup\{248\ \}$	0	I
$K_{20}^{(2)}$	$\{75,104,46,48\}$	$K_{19}^{(2)}\cup\{38\}$	2	I
$K_{20}^{(3)}$	$\{73,110,40,50\}$	$K_{19}^{(3)}\cup\{169\}$	1	I
$K_{20}^{(4)}$	$\{76,101,49,47\}$	$K_{19}^{(4)}\cup\{150\}$	2	I
$K_{20}^{(5)}$	$\{77,98,52,46\}$	$K_{19}^{(5)}\cup\{102\}$	1	I
$K_{20}^{(6)}$	$\{78,95,55,45\}$	$K_{19}^{(6)}\cup\{77\}$	4	I
$K_{20}^{(7)}$	$\{79,92,58,44\}$	$K_{19}^{(7)}\cup\{46\}$	3	I
$K_{19}^{(8)}$	$\{80,89,61,43\}$	$K_{19}^{(8)}\cup\{168]$	3	I
$K_{20}^{(9)}$	$\{81,86,64,42\}$	$K_{19}^{(9)}\cup\{160\}$	6	I
$K_{20}^{(10)}$	$\{82,83,67,41\}$	$K_{19}^{(10)}\cup\{71\}$	9	I
$K_{20}^{(11)}$	$\{83,80,70,40\}$	$K_{19}^{(12)}\cup\{121\}$	6	I
$K_{20}^{(12)}$	$\{84,77,73,39\}$	$K_{19}^{(13)}\cup\{\{105]$	10	I
$K_{20}^{(13)}$	$\{85,74,76,38\}$	$K_{19}^{(14)}\cup\{50\}$	14	I
$K_{20}^{(14)}$	$\{86,71,79,37\}$	$K_{19}^{(15)}\cup\{59\}$	16	I
$K_{20}^{(15)}$	$\{87,68,82,36\}$	$K_{19}^{(16)}\cup\ \{176\}$	14	I
$K_{20}^{(16)}$	$\{88,65,85,35\}$	$K_{19}^{(17)}\cup\{119\}$	17	I
$K_{20}^{(17)}$	$\{89,62,88,34\}$	$K_{19}^{(18)}\cup\{56]$	16	I
$K_{20}^{(18)}$	$\{90,59,91,33\}$	$K_{19}^{(19)}\cup\{25\}$	20	I
$K_{20}^{(19)}$	$\{91,56,99,32\}$	$K_{19}^{(20)}\cup\{160]$	23	I
$K_{20}^{(20)}$	$\{92,53,97,31\}$	$K_{19}^{(21)}\cup\{16\}$	27	I

Table 16: Distinct classes and inequivalent arcs with No. of c_0 of K_{21}.

$K_{21}^{(i)}; i = 1, \ldots, 19$	$\{l_0, l_1, l_2, l_3\}$	**Inequivalent arcs**	**No. of c_0.**	**S.G**
$K_{21}^{(1)}$	$\{71, 102, 45, 55\}$	$K_{20}^{(3)} \cup \{150\}$	0	I
$K_{21}^{(2)}$	$\{72, 99, 48, 54\}$	$K_{20}^{(4)} \cup \{76\}$	0	I
$K_{21}^{(3)}$	$\{73, 96, 51, 53\}$	$K_{20}^{(5)} \cup \{259\}$	0	I
$K_{21}^{(4)}$	$\{74, 03, 54, 52\}$	$K_{20}^{(7)} \cup \{141\}$	1	I
$K_{21}^{(5)}$	$\{75, 90, 57, 51\}$	$K_{20}^{(6)} \cup \{189\}$	1	I
$K_{21}^{(6)}$	$\{76, 87, 60, 50\}$	$K_{19}^{(8)} \cup \{143\}$	1	I
$K_{21}^{(7)}$	$\{77, 84, 63, 49\}$	$K_{20}^{(9)} \cup \{170\}$	4	I
$K_{21}^{(8)}$	$\{78, 81, 66, 48\}$	$K_{20}^{(10)} \cup \{109\}$	4	I
$K_{21}^{(9)}$	$\{79, 78, 69, 47\}$	$K_{20}^{(11)} \cup \{151\}$	2	I
$K_{21}^{(10)}$	$\{80, 75, 72, 47\}$	$K_{20}^{(12)} \cup \{176\}$	2	I
$K_{21}^{(11)}$	$\{81, 72, 75, 45\}$	$K_{20}^{(13)} \cup \{81\}$	8	I
$K_{21}^{(12)}$	$\{82, 69, 78, 44\}$	$K_{20}^{(14)} \cup \{38\}$	6	I
$K_{20}^{(13)}$	$\{83, 66, 81, 43\}$	$K_{20}^{(15)} \cup \{45\}$	7	I
$K_{21}^{(14)}$	$\{84, 63, 84, 42\}$	$K_{20}^{(16)} \cup \{115\}$	8	I
$K_{21}^{(15)}$	$\{85, 60, 87, 41\}$	$K_{20}^{(17)} \cup \{62\}$	9	I
$K_{21}^{(16)}$	$\{86, 57, 90, 40\}$	$K_{20}^{(19)} \cup \{74\}$	9	I
$K_{21}^{(17)}$	$\{87, 54, 93, 39\}$	$K_{20}^{(18)} \cup \{160\}$	14	I
$K_{21}^{(18)}$	$\{88, 51, 96, 38\}$	$K_{20}^{(20)} \cup \{ 149\}$	10	I
$K_{21}^{(19)}$	$\{89, 48, 99, 37\}$	$K_{20}^{(20)} \cup \{ 25\}$	16	I

Table 17: Distinct classes and inequivalent arcs with No. of c_0 of K_{22}.

$K_{22}^{(i)}; i = 1, \ldots ,16$	$\{l_0\ , l_1, l_2, l_3\}$	**Inequivalent arcs**	**No. of** $\boldsymbol{c_0}$	S.G
$K_{22}^{(1)}$	$\{70, 92, 51, 60\}$	$K_{21}^{(5)} \cup \{56\}$	4	I
$K_{22}^{(2)}$	$\{72, 86, 57, 68\}$	$K_{21}^{(5)} \cup \{191\}$	1	I
$K_{22}^{(3)}$	$\{73, 83, 60, 57\}$	$K_{21}^{(6)} \cup \{226\}$	0	I
$K_{22}^{(4)}$	$\{74, 80, 63, 56\}$	$K_{21}^{(7)} \cup \{171\}$	0	I
$K_{22}^{(5)}$	$\{76, 74, 69, 54\}$	$K_{21}^{(9)} \cup \{105\}$	0	I
$K_{22}^{(6)}$	$\{77, 71, 72, 53\}$	$K_{21}^{(8)} \cup \{262\}$	1	I
$K_{22}^{(7)}$	$\{78, 68, 75, 52\}$	$K_{21}^{(10)} \cup \{106\}$	1	I
$K_{22}^{(8)}$	$\{79, 65, 78, 51\}$	$K_{21}^{(12)} \cup \{108\}$	2	I
$K_{22}^{(9)}$	$\{80, 62, 81, 50\}$	$K_{20}^{(13)} \cup \{177\}$	3	I
$K_{22}^{(10)}$	$\{81, 59, 84, 49\}$	$K_{21}^{(15)}$A$\cup \{75\}$	4	I
$K_{22}^{(11)}$	$\{82, 56, 87, 48\}$	$K_{21}^{(16)} \cup \{55\}$	6	I
$K_{22}^{(12)}$	$\{83, 53, 90, 47\}$	$K_{21}^{(16)} \cup \{25\}$	4	I
$K_{22}^{(13)}$	$\{84, 50, 93, 46\}$	$K_{21}^{(19)} \cup \{42\}$	3	I
$K_{22}^{(14)}$	$\{85, 47, 96, 45\}$	$K_{21}^{(18)} \cup \{160\}$	2	I
$K_{22}^{(15)}$	$\{86, 4499, 44\}$	$K_{21}^{(17)} \cup \{34\}$	4	I
$K_{22}^{(16)}$	$\{87, 41, 102, 43\}$	$K_{21}^{(19)} \cup \{160\}$	9	I

Table 18: Distinct classes and inequivalent arcs with No. of c_0 of K_{23}.

$\boldsymbol{K_{23}^{(i)}; i = 1, \ldots ,12}$	$\{\boldsymbol{l_0\ , l_1, l_2, l_3}\}$	**Inequivalent arcs**	**No. of** $\boldsymbol{c_0}$	**S.G**
$K_{23}^{(1)}$	$\{68, 86, 52, 67\}$	$K_{23}^{(1)} \cup \{191\}$	0	
$K_{23}^{(2)}$	$\{72, 74, 64, 63\}$	$K_{22}^{(6)} \cup \{49\}$	0	
$K_{23}^{(3)}$	$\{75, 65, 73, 60\}$	$K_{22}^{(7)} \cup \{251\}$	0	
$K_{23}^{(4)}$	$\{76, 62, 76, 59\}$	$K_{22}^{(8)} \cup \{121\}$	1	
$K_{23}^{(5)}$	$\{77, 59, 79, 58\}$	$K_{22}^{(10)} \cup \{86\}$	1	
$K_{23}^{(6)}$	$\{78, 56, 82, 57\}$	$K_{22}^{(11)} \cup \{229\}$	2	
$K_{23}^{(7)}$	$\{79, 53, 85, 56\}$	$K_{22}^{(12)} \cup \{55\}$	2	
$K_{23}^{(8)}$	$\{80, 50, 88, 55\}$	$K_{22}^{(13)} \cup \{\ 55\}$	1	
$K_{23}^{(9)}$	$\{81, 47, 91, 54\}$	$K_{22}^{(14)} \cup \{\ 55\}$	0	
$K_{23}^{(10)}$	$\{82, 44, 94, 53\}$	$K_{22}^{(16)} \cup \{74\}$	3	
$K_{23}^{(11)}$	$\{83, 41, 97, 52\}$	$K_{22}^{(16)} \cup \{55\}$	1	
$K_{23}^{(12)}$	$\{86, 32, 106, 49\}$	$K_{22}^{(15)} \cup \{89$	1	