Some types of Smarandache filters of a Smarandache BH-algebra

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(Communicated by Ali Jabbari)

Abstract

In this paper, the notions of a Smarandache p-filter, a Smarandache \( n \)-fold \( p \)-filter, Smarandache \( q \)-filter, a Smarandache-\( n \)-fold \( q \)-filter of a Smarandache BH-Algebra are introduced. Some properties of them with some theorems, proportions and examples are given.

\textit{Keywords:} BCK-algebra, BH-algebra, Smarandache filter.

\textit{2020 MSC:} 13L99

1. Introduction

The idea of BCK-algebras was formulated first in \cite{4, 5}. In the same year another algebraic structure called BCI-algebra which was a popularization of a BCK-algebra was given by K. Iséki \cite{6}. In 1983, Hu and Li introduced the notion of a BCH-algebra which was a popularization of BCK/BCI-algebras \cite{8, 11}. Hoo show that the notions of an ideal and a filter in a BCI-algebra \cite{7}. A BH-algebra is an algebraic structure introduced by Jun et al in \cite{10} which was a popularization of BCH/BCI/BCK-algebras. The notions of a Smarandache BCI-algebra, Smarandache ideal of a Smarandache BCI-algebra are given by Jun in \cite{9}. Abbass and Dahham introduced the concept of completely closed filter of a BH-algebra in \cite{11}. Abbass and Luhaib introduced the idea of Smarandache filter of a Smarandache BH-Algebra in \cite{3}. In this paper, the notions of a Smarandache-\( p \)-filter, a Smarandache-\( n \)-fold \( p \)-filter, Smarandache \( q \)-filter, a Smarandache-\( n \)-fold \( q \)-filter and of a Smarandache BH-Algebra are given.
2. Preliminaries

In this section, several basic connotations about a BCI-algebra, a BCK-algebra, a Smarandache BH-algebra, and a Smarandache filter of a Smarandache are reviewed.

Definition 2.1. A BCI-algebra is an algebra $\langle X, \Box, 0 \rangle$, where $X$ is a nonempty set, $\Box$ is a binary operation and $0$ is a constant, for all $x, y, z \in X$, satisfying the following axioms:

- i. $(x \Box y) \Box (z \Box y) = 0$,
- ii. $(x \Box (y \Box z)) \Box y = 0$,
- iii. $x \Box x = 0$,
- iv. $x \Box y = 0$ and $y \Box x = 0$ imply $x = y$.

Definition 2.2. A BCK-algebra is a BCI-algebra satisfying the axiom: $0 \Box x = 0$, for all $x \in X$.

Definition 2.3. A BH-algebra is a nonempty set $X$ with a constant $0$ and a binary operation $\Box$ satisfying the following conditions:

- i. $x \Box x = 0$, for all $x \in X$.
- ii. $x \Box y = 0$ and $y \Box x = 0$ imply $x = y$, for all $x, y \in X$.
- iii. $x \Box 0 = x$, for all $x \in X$.

Definition 2.4. A nonempty subset $S$ of a BH-algebra $X$ is called a subalgebra of $X$ if $x \Box y \in S$, for all $x, y \in S$.

Definition 2.5. A filter of a BH-algebra $X$ is a non-empty subset $F$ of $X$ such that:

- (F1) if $x \in F$ and $y \in F$, then $y \Box (x \Box y) \in F$ and $x \Box (x \Box y) \in F$.
- (F2) If $x \in F$ and $x \Box y = 0$ then $y \in F$ for all $y \in X$.

Further $F$ is a closed filter if $0 \Box x \in F$, for all $x \in F$.

Definition 2.6. Let $X$ be a BH-algebra and $F$ be a filter of $X$. Then $F$ is called a $p$-filter denoted by $p-f$ if it satisfies:

$$\text{if } x, y \in F \text{ imply } (x \Box z) \Box (y \Box z) \in F \text{ for all } y, z \in X.$$  

Definition 2.7. Let $F$ be a filter of a BH-algebra $X$. If $x, y \in F$ and there exists a fixed $n \in N$ such that $z^n \in X$ imply $(x \Box z^n) \Box (y \Box z^n) \in F$, for all $z \in X$. Then $F$ is said to be a $n$-fold $p$-filter of $X$.

Definition 2.8. Let $X$ be a BH-algebra and $F$ be a filter of $X$. Then $F$ is called a $q$-filter denoted by $q-f$ if it satisfies:

$$\text{If } x \Box z \in F, y \in F \text{ imply } x \Box (y \Box z) \in F, \text{ for all } x, z \in X.$$  

Definition 2.9. Let $X$ be a BH-algebra, $F$ be a filter of $X$, and there exists a fixed $n \in N$ such that $x \Box z^n \in F, y \in F$, for all $x, z \in X$ imply $x \Box (y \Box z^n) \in F$. Then $F$ is called a $n$-fold $q$-filter of $X$.

Definition 2.10. A Smarandache BH-algebra is defined to be a BH-algebra $X$ in which there exists a proper subset $Q$ of $X$ denoted by $S$, BH-algebra such that

- i. $0 \in Q$ and $|Q| \geq 2$.
- ii. $Q$ is a BCK-algebra under the operation of $X$. 

Definition 2.11. A non-empty subset $F$ of a $S$. BH-algebra $X$ is called a **Smarandache filter** of $X$ denoted by $S.f$, if it satisfies $(F_1)$ and

$$(F_3) \text{ If } x \in F \text{ and } x \triangleleft y = 0 \text{ then } y \in F, \forall y \in Q.$$ 

Proposition 2.12. Let $X$ be a $S$. BH-algebra and let $\{F_\beta, \beta \in \Omega\}$ be a family of $S.f$ of $X$. Then $\bigcap_{\beta \in \Omega} F_\beta$ is an $S.f$ of $X$.

Proposition 2.13. Let $X$ be a $S.f$ and let $\{F_i, i \in \lambda\}$ be a chain of $S.f$ of $X$. Then $\bigcup_{\beta \in \Omega} F_\beta$ is a $S.f$ of $X$.

Theorem 2.14. Let $X$ be a $S$. BH-algebra, and $F$ be a $S.f$ of $X$ such that $x \triangleleft y \neq 0$, for all $y \notin F$ and $x \in F$. Then $F$ is a filter of $X$.

3. Main Results

In this section, the notions of a Smarandache-$p$-filter, a Smarandache $n$-fold $p$-filter, Smarandache $q$-filter, a Smarandache-$n$-fold $q$-filter and of a Smarandache BH-Algebra of a Smarandache BH-Algebra are introduced. Also, some properties of these notions are studied.

Definition 3.1. Let $X$ be a $S$. BH-algebra and $F$ be a Smarandache filter of $X$. Then $F$ is called a **Smarandache $p$-filter** of $X$ and denoted by $S.p.f$ of $X$ if it satisfies:

$$\text{If } x, y \in F \text{ imply } (x \triangleleft z) \circ (y \triangleleft z) \in F \text{ for all } z \in Q.$$ 

Further $F$ is a Smarandache closed $p$-filter if $0 \triangleleft x \in F$, for all $x \in F$.

Example 3.2. Let $X = \{0, 1, 2, 3\}$. Define $\triangleleft$ as follows:

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0 & 0 & 0 & 2 & 3 \\
1 & 1 & 0 & 1 & 2 \\
2 & 2 & 2 & 0 & 1 \\
3 & 3 & 3 & 2 & 0 \\
\end{array}
\]

where $Q = \{0, 1\}$, the subset $F = \{0, 1, 2\}$ is a $S. p.f$ of $X$. But is not $p.f$ of $X$, since $z = 3, x = 3, y = 0, (3 \triangleleft 3) \circ (0 \triangleleft 3) = 3 \notin F$.

Proposition 3.3. Let $X$ be a $S$. BH-algebra and $F$ be a $p$-f of $X$. Then $F$ is a $S.p.f$ of $X$.

Proof. Directly since $Q \subseteq X$. □

Theorem 3.4. Let $X$ be a $S$. BH-algebra, and $F$ be a $S.p.f$ of $X$ such that $x \triangleleft y \neq 0$, $y \notin F$ if $(x \triangleleft z) \circ (y \triangleleft z) \notin F$ and $x \in F, z \in X$. Then $F$ is a $p.f$ of $X$. 
Proof. Let $F$ be a S.p-f of $X$ it follows that By Definition 3.1 is a S.f of $X$. Since $x \boxdot y \neq 0, y \notin F, x \in F$, by Theorem 2.14, $F$ is a filter of $X$.

Now, let $x, y \in F, z \in X$, then we have two cases:

Case (I): If $z \in Q$, imply $(x \boxdot z) \cap (y \boxdot z) \in F$ because by definition 3.1 $F$ is S.p-f of $X$.

Cases (II): If $z \notin Q$, then either $(x \boxdot z) \cap (y \boxdot z) \notin F$ or $(x \boxdot z) \cap (y \boxdot z) \in F$.

Suppose $(x \boxdot z) \cap (y \boxdot z) \notin F$, then $y \notin F$, this is a contradiction. Thus $(x \boxdot z) \cap (y \boxdot z) \in F$.

Therefore, is a p.f of $X$. □

Proposition 3.5. Let $X$ be a Smarandache BH-algebra, and let $\{F_\beta, \beta \in \Omega\}$ be a family of S.p-fs of $X$. Then $\bigcap_{\beta \in \Omega} F_\beta$ is a S.p-f of $X$.

Proof. Let $\{F_\beta, \beta \in \Omega\}$ be a family of S.p-fs of $X$, imply $\{F_\beta, \beta \in \Omega\}$ be a family of Smarandache filters of $X$. Hence, By Proposition 2.12 $\bigcap_{\beta \in \Omega} F_\beta$ is a S.f of $X$. Now, let $x, y \in \bigcap_{\beta \in \Omega} F_\beta$ and $z \in Q$. Then $x, y \in F_\beta$ and $z \in Q, \forall \beta \in \Omega$ implies that $(x \boxdot z) \cap (y \boxdot z) \in F_\beta, \forall \beta \in \Omega$, because $F_\beta$ is a S.p-f of $X$, for all $\beta \in \Omega$, this mean that $(x \boxdot z) \cap (y \boxdot z) \in \bigcap_{\beta \in \Omega} F_\beta$. Therefore $\bigcap_{\beta \in \Omega} F_\beta$ is a S.p-f of $X$. □

Example 3.6. Let $X = \{0, 1, 2, 3, 4, 5\}$. Define $\boxdot$ as follows:-

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where $Q = \{0, 2\}$. The subset $F_1 = \{0, 2, 3\}$ and $F_2 = \{0, 2, 5\}$ are two S.p-f of $X$, but $F_1 \cup F_2 = \{0, 2, 3, 5\}$ is not a S.p-f of $X$, since $x = 3, y = 5, z = 0 \notin Q$ but $(3 \boxdot 0) \cap (5 \boxdot 0) = 1 \notin F_1 \cup F_2$,

Proposition 3.7. Let $X$ be a S. BH-algebra, and let $\{F_\beta, \beta \in \Omega\}$ be a chain of S.P.f of $X$. Then $\bigcup_{\beta \in \Omega} F_\beta$ is a S.P.f of $X$.

Proof. Let $\{F_\beta, \beta \in \Omega\}$ be a chain of S.P.f of $X$. it follows that $\{F_\beta, \beta \in \Omega\}$ be a chain of Smarandache filters of $X$ [By definition 3.1]. This together with Proposition 2.13 implies that $\bigcup_{\beta \in \Omega} F_\beta$ is a Smarandache filter of $X$.

Now, let $x, y \in \bigcup_{\beta \in \Omega} F_\beta, z \in Q$, then there exists $F_n, F_m \in \{F_\beta, \beta \in \Omega\}, \text{such that } x \in F_j$ and $y \in F_k$. Then either $F_n \subseteq F_m$ or $F_m \subseteq F_n$. If $F_n \subseteq F_m$, it follows that $x, y \in F_m$ and $z \in Q$. So, there exists $m \in \Omega$ such that $(x \boxdot z) \cap (y \boxdot z) \in F_m$, because $F_i$ is a S.P.f of $X, (\forall \beta \in \Omega)$. Then $(x \boxdot z) \cap (y \boxdot z) \in \bigcup_{\beta \in \Omega} F_\beta$. Similarly, $F_m \subseteq F_n$ implies that $\bigcup_{\beta \in \Omega} F_\beta$ is a S.P.f of $X$. □
**Definition 3.8.** Let \( F \) be a Smarandache filter of a S. BH-algebra \( X \). If \( x, y \in F \) and there exists a fixed \( n \in N \) such that \( z^n \in Q \) imply \( x \triangle z^n \triangle (y \triangle z^n) \in F \), for all \( z \in Q \). Then \( F \) is said to be a Smarandache \( n \)-fold p-filter of \( X \), denoted by a **S. \( n \)-fold. p-f** of \( X \).

**Example 3.9.** Let \( X = \{0, 1, 2, 3, 4\} \) be as in example 3.6. The filter \( F = \{0, 2, 3\} \) is a S. 2-fold. p-f of \( X \).

**Theorem 3.10.** Let \( X \) be a S. BH-algebra, and \( F \) be a S. \( n \)-fold. p-f of \( X \) such that \( x \triangle y \neq 0, y \notin F \) if \( x \triangle z^n \triangle (y \triangle z^n) \notin F \) and \( x \in F, z^n \in X \), for a fixed \( n \in N \). Then \( F \) is a \( n \)-fold p-filter of \( X \).

**Proof.** Let \( F \) be a S. \( n \)-fold. P.f of \( X \), then By Definition 3.8, \( F \) is a S.f of \( X \). Since \( x \triangle y \neq 0, y \notin F, x \in F \), By Theorem 2.14, \( F \) is a filter of \( X \). Now, let \( x, y \in F, z^n \in X \), then we have the following two cases:

- **Case (I):** If \( z^n \in Q \), then \( (x \triangle z^n) \triangle (y \triangle z^n) \in F \), because by Definition 3.8, \( F \) is S. \( n \)-fold. P.f of \( X \).
- **Cases (II):** If \( z^n \notin Q \), then either \( (x \triangle z^n) \triangle (y \triangle z^n) \notin F \) or \( (x \triangle z^n) \triangle (y \triangle z^n) \in F \).

Suppose that \( (x \triangle z^n) \triangle (y \triangle z^n) \notin F \), then \( y \notin F \), this a contradiction. Thus \( (x \triangle z^n) \triangle (y \triangle z^n) \in F \), consequently \( F \) is a \( n \)-fold p-filter of \( X \). \( \square \)

**Proposition 3.11.** Let \( X \) be a S. BH-algebra, and let \( \{F_\beta, \beta \in \Omega\} \) be a family of S. \( n \)-fold. p-f of \( X \). Then \( \bigcap_{\beta \in \Omega} F_\beta \) is a S. \( n \)-fold. p-f of \( X \).

**Proof.** Straightforward. \( \square \)

**Proposition 3.12.** Let \( X \) be a Smarandache BH-algebra, and let \( \{F_\beta, \beta \in \Omega\} \) be a chain of S. \( n \)-fold. p-f of \( X \). Then \( \bigcup_{\beta \in \Omega} F_\beta \) is a S. \( n \)-fold. p-f of \( X \).

**Proof.** Straightforward. \( \square \)

**Definition 3.13.** Let \( X \) be a S. BH-algebra and \( F \) be a Smarandache filter of \( X \). Then \( F \) is called a **Smarandache q-filter** and denoted by a **S.q-f** of \( X \) if it satisfies: If \( x \triangle z \in F, y \in F \) imply \( x \triangle (y \triangle z) \in F \) for all \( x, z \in Q \).

**Example 3.14.** Let \( X = \{0, 1, 2, 3, 4\} \). Define \( \triangle \) as follows:

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Where \( Q = \{0, 2\} \). The subset \( F = \{0, 1, 2\} \) is a S.q-f of \( X \) but it is not a q-filter of \( X \). Since \( x = 3, y = 0, z = 3 \) and \( 3 \triangle (0 \triangle 3) = 3 \notin F \).
Proposition 3.15. Let $X$ be a $S$. BH-algebra and $F$ is a q-filter of $X$. Then $F$ is a S.q.f of $X$.

Proof. Since $Q \subseteq X$, the proof is clear. \qed

Remark 3.16. Consider the $Q_1 - S$. BH-algebra and $Q_2$-Smarandache BH-algebra $X$ such that $Q_1 \subseteq Q_2$. The $Q_1$-Smarandache q-filter of $X$ may be not a $Q_2$-Smarandache q-filter of $X$ as in the following example. Consider $X = \{0, 1, 2, 3\}$ in example 3.14, where $Q_1 = \{0, 1\}$, $Q_2 = \{0, 2, 3\}$ are BCK-algebras and $Q_1 \subseteq Q_2 : F = \{0, 1, 2\}$ is a $Q_1$-Smarandache q-filter of $X$, but it is not $Q_2$-Smarandache q-filter of $X$. Since $x = 3, y = 2, z = 3$ implies that $3 \cdot (2 \cdot 3) = 3, 3 \notin F$.

Proposition 3.17. Let $X$ be a $S$. BH-algebra and $F$ be a S.q.f of $X$, such that $F \subseteq Q$. Then $F$ is a subalgebra of $X$.

Proof. Let $x, y \in F$. Since $z \in Q$, choose $z = 0$, we have $x = x \cdot 0 \in F, y \in F, x, 0 \in Q$, because $F \subseteq Q$. This implies that $x \cdot (y \cdot 0) \in F$, because by Definition 3.13 $F$ is a S.q.f of $X$. Then $x \cdot y \in F$. Hence, $F$ is a subalgebra. \qed

Theorem 3.18. Let $X$ be a $S$. BH-algebra, and be a S.q.f of $X$ such that $x \cdot y \neq 0, x \cdot z \notin F$, and $y \notin F \cap x \cdot (y \cdot z) \notin F$ and $x \in F, z \in X$. Then $F$ is a q-filter of $X$.

Proof. Let $F$ be a S.q.f of $X$, then by Definition 3.13 it is a S.f of $X$. Since $x \cdot y \neq 0, y \notin F, x \in F$, By Theorem 2.14 $F$ is a filter of $X$.

Now, let $x \cdot z \in F, y \in F, x, z \in X$, then we have the following two cases:

Case (I): If $x, z \in Q$, then by Definition 3.13 $x \cdot (y \cdot z) \in F$.

Cases (II): If $x, z \notin Q$, then either $x \cdot (y \cdot z) \notin F$ or $x \cdot (y \cdot z) \in F$.

If $x \cdot (y \cdot z) \notin F$, then $y \in F$, or $x \cdot z \notin F$, contradiction. Since $x \cdot z \in F, y \in F$, we have $x \cdot z \notin F$. Hence, it is a q-filter of $X$. \qed

Proposition 3.19. Let $X$ be a $S$. BH-algebra, and let $\{F_\beta, \beta \in \Omega\}$ be a family of S.q.f of $X$. Then $\bigcap_{\beta \in \Omega} F_\beta$ is a S.q.f of $X$.

Proof. Let $\{F_\beta, \beta \in \Omega\}$ be a family of S.q.fs of $X$, then by Definition 3.13 $\{F_\beta, \beta \in \Omega\}$ be a family of S.f of $X$. Thus, By Proposition 2.12 $\bigcap_{\beta \in \Omega} F_\beta$ is a S.f of $X$.

Now, let $x \cdot z \in \bigcap_{\beta \in \Omega} F_\beta$, $y \in \bigcap_{\beta \in \Omega} F_\beta$ such that $x, z \in Q$, it follows that $x \cdot z \in F_\beta, y \in F_\beta$, such that $x, z \in Q$, imply $x \cdot (y \cdot z) \in F_\beta, (\forall \beta \in \Omega)$, because $F_\beta$ is a S.q.f of $X$. Hence, $x \cdot (y \cdot z) \in \bigcap_{\beta \in \Omega} F_\beta$.

Therefore, $\bigcap_{\beta \in \Omega} F_\beta$ is a S.q.f of $X$. \qed

Remark 3.20. Let $X$ be a $S$. BH-algebra and let $F_1, F_2$ be a S.q.f of $X$. Then $F_1 \cup F_2$ is not necessary a S.q.f of $X$.

Example 3.21. Consider $X = \{0, 1, 2, 3, 4, 5\}$ be as in example 3.6, where $Q = \{0, 1\}$. The subset $F_1 = \{0, 1, 3\}$ and $F_2 = \{0, 1, 4\}$ are two S.q.s of $X$, but $F_1 \cup F_2 = \{0, 1, 3, 4\}$ is not a S.q.f of $X$, because $3, 4 \in F_1 \cup F_2$, but $3 \cdot (3 \cdot 4) = 2 \notin F_1 \cup F_2$. Then $F_1 \cup F_2$ it is not a S.q.f.
Proposition 3.22. Let $X$ be a S. BH-algebra and let $\{F_\beta, \beta \in \Omega\}$ be a chain of S.q.f of $X$. Then $\bigcup_{\beta \in \Omega} F_\beta$ is a S.q.f of $X$.

**Proof.** Let $\{F_\beta, \beta \in \Omega\}$ be a chain of S.q.f of $X$. Then by Definition 3.13 $\{F_\beta, \beta \in \Omega\}$ is a chain of S.f of $X$. Thus, by Proposition 3.13 $\bigcup_{\beta \in \Omega} F_\beta$ is a S.f of $X$.

Now, let $x \sqcap z \in \bigcup_{\beta \in \Omega} F_\beta$, $y \in \bigcup_{\beta \in \Omega} F_\beta$, such that $x, z \in Q$, then there exist $F_n, F_m \in \{F_\beta : \beta \in \Omega\}$, such that $x \sqcap z \in F_n$ and $y \in F_m$. Thus either $F_n \subseteq F_m$ or $F_m \subseteq F_n$.

If $F_n \subseteq F_m$, then $x \sqcap z \in F_m$, $y \in F_m$, such that $x, z \in Q$, thus there exists $m \in \Omega$ such that $x \sqcap (y \sqcap z) \in F_m$, because $F_\beta$ is a S.q.f of $X$, for all $\beta \in \Omega$. Consequently, $x \sqcap (y \sqcap z) \in \bigcup_{\beta \in \Omega} F_\beta$.

Similarly, $F_m \subseteq F_n$. Hence, $\bigcup_{\beta \in \Omega} F_\beta$ is a S.q.f of $X$. \qed

References


