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Linear operator of various types and its basic properties

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Abstract

We are starting to construct a new theory of linear operators of various types defined on fuzzy normed space inspired by the theory of linear operators of various types defined on ordinary normed space. The first goal in this paper is to introduce the notion of a fuzzy bounded linear operator on a-fuzzy normed space then basic properties of this type of linear operators are proved. The second aim in this paper is to introduce the notion of a fuzzy compact linear operator on a-fuzzy normed space then we shall discuss important general properties of this type of linear operators.

Keywords: a-fuzzy normed space, Fuzzy bounded linear operator, Fuzzy compact linear operator.
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1. Introduction

We are starting to construct a new theory of fuzzy functional analysis inspired by the classical functional analysis theory. The literature review of this paper is as follows in 2011 in [3] Kider introduce his first definition of fuzzy normed space also in [4] he introduce a completion of this fuzzy normed space. Again in [5] Kider introduce a new definition of fuzzy normed space. In [12, 2] Kider and Kadhum introduce the fuzzy norm of a linear fuzzy bounded operators then they proved basic properties of this fuzzy norm of fuzzy bounded operators. Kider and Ali in [1, 8] introduce the definition of fuzzy normed algebra and they discuss important general properties of this type of algebra.

Kider and Gheeab in [9, 10] introduced the notion general fuzzy normed space after that they proved basic and important properties of this type of fuzzy normed spaces. In [13], Kider and

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Kadhum proved basic properties of fuzzy compact linear operators on fuzzy normed spaces. In [7] Kider introduce a new type of fuzzy metric he call it fuzzy soft metric and he proved some properties of fuzzy soft metric space also Kider in the same year in [6] he introduced the notion of a-fuzzy metric and he proved some properties of a-fuzzy metric space. Kider and M. Gheeab in [11] proved some properties of the adjoint operator of a general fuzzy bounded operator. Khudhair and Kider in [14, 16, 17] introduce the notion a-fuzzy norm then they discuss important general properties of a-fuzzy normed space.

The organization of this paper are as follows, the aim of section two is to recall the a-fuzzy normed space with its basic properties and the goal of section three is to investigate important properties when the linear operator is fuzzy closed. Finally the main results in this paper are in section four which is designed to investigate important properties when the linear operator is fuzzy compact.

2. The a-fuzzy normed space and its basic properties

Remark 2.1. *[15] If $\circledcirc$ is a t-conorm then*

(i)for all $0 < a < 1, 0 < b < 1$, with $a > b$, there exists c, 0<c<1, satisfies $a > b \circledcirc c$.

(ii) for all $a, 0 < a < 1$, there exists $0 < b < 1$, satisfies $b \circledcirc b \le a$.

Example 2.2. *[15] $a \circledcirc b = a + b - ab$, the algebra product is a continuous t-conorm, for all $0 \le a, b \le 1$.*

Definition 2.3. *[14] If $\circledcirc$ is a continuous t-conorm and $a_{\mathbb{R}} : \mathbb{R} \to I$ is a fuzzy set then $a_{\mathbb{R}}$ is called* ***a-fuzzy absolute value*** *on $\mathbb{R}$ if the following are satisfied, for all $r, s \in \mathbb{R}$:*

(i) $a_{\mathbb{R}}(r) \in (0, 1]$,

(ii) $a_{\mathbb{R}}(r) = 0 \iff$ if $r = 0$,

(iii) $a_{\mathbb{R}}(\sigma r) \le a_{\mathbb{R}}(\sigma) a_{\mathbb{R}}(r)$,

(iv) $a_{\mathbb{R}}(r + s) \le a_{\mathbb{R}}(r) \circledcirc a_{\mathbb{R}}(s)$.

Then $(\mathbb{R}, a_{\mathbb{R}}, \circledcirc)$ is called ***a-fuzzy absolute value space****.*

Definition 2.4. *[14] Any arbitrary fuzzy Cauchy sequence (r_k) in $\mathbb{R}$ is fuzzy, if there exists $r \in \mathbb{R}$, satisfying $r_k \to r$, then $\mathbb{R}$ is called fuzzy complete.*

Definition 2.5. *[14] Let $\circledcirc$ be a continuous t-conorm and let Z be a vector space over $\mathbb{R}$. If $(\mathbb{R}, a_{\mathbb{R}}, \circledcirc)$ is a-fuzzy absolute value space and n_Z: $Z \to [0, 1]$ be a fuzzy set then n_Z is called* ***a-fuzzy norm on Z*** *if it satisfy the following conditions for all z, w $\in$ Z and for all $0 \neq \alpha \in \mathbb{R}$:*

(i) $n_Z(z) \in (0, 1]$,

(ii) $n_Z(z) = 0 \iff z=0$,

(iii) $n_Z(\sigma z) \le a_{\mathbb{R}}(\sigma)\, n_Z(z)$,

(iv) $n_Z(z + w) \le n_Z(z) \circledcirc n_Z(w)$.

Then $(Z, n_Z, \circledcirc)$ is called ***a-fuzzy normed space****.*

Example 2.6. *[14] If $t \circledcirc s = t + s - ts$ for all t, s $\in I$, Z=C[p, b], and $(\mathbb{R}, a_{\mathbb{R}}, \circledcirc)$ is a-fuzzy absolute space define $n_Z(z) = max_{r \in [p,\ b]}\, a_{\mathbb{R}}[z(r)]$, for all $z \in Z$. Then $(Z, n_Z, \circledcirc)$ is a-fuzzy normed space.*

Lemma 2.7. *[14] $n_Z(z - w) = n_Z(w - z)$, for all z, w $\in$ Z, when (Z, n, $\circledcirc$) is a-fuzzy normed space.*

Definition 2.8. *[14] Let $(z_k) \in Z$ where $(Z, n_Z, \circledcirc)$ is a-fuzzy normed space. Then (z_k) is fuzzy converges to the limit z as $k \to \infty$ if, for all, s $\in$ (0,1), there exists $N \in \mathbb{N}$ such that $n_Z(\mathrm{z_k} - z) < s$, for all $k \ge N$. If (z_k) is fuzzy converge to the limit z or $\mathrm{lim}_{n \to \infty}\, n_Z(\mathrm{z_k} - \mathrm{z}) = 0$. For simply we write $\mathrm{lim}_{k \to \infty}\, \mathrm{z_k} = z$ or $\mathrm{z_k} \to z$ as $k \to \infty$.*

Definition 2.9. *[14] fb(w, r)= {z ∈Z: n(w−z) < t} is* ***open fuzzy ball*** *and fb[w, r]= {z ∈ Z: n(w−z) ≤ t} is the* ***open closed fuzzy ball*** *with the center u ∈Z and radius t, where (Z, n, ⊚) is a-fuzzy normed space.*

Lemma 2.10. *[14] The function $z \longmapsto n_Z(z)$ is a fuzzy continuous function from Z into ℝ when (Z, n, ⊚) and (ℝ, n, ⊚) are a-fuzzy normed spaces.*

Lemma 2.11. *[14] If ∀ s ∈ (0, 1), ∃ N ∈ ℕ such that $n_Z(\mathrm{z_k}-\mathrm{z_m}) < s$, ∀ k, m ≥N then (z_k) is a fuzzy Cauchy sequence in a-fuzzy normed space (Z, n_Z, ⊚).*

Definition 2.12. *[14] If fb(w, j) ⊆W for any arbitrary w∈W and for some j ∈(0, 1), then W⊆Z is* ***fuzzy open*** *in the a-fuzzy normed space that (Z, n_Z , ⊚). Also, D⊆Z is* ***fuzzy closed*** *if D^C is fuzzy open.*

Moreover the ***fuzzy closure*** *of D, $\overline{\mathrm{D}}$ is defined to be the smallest fuzzy closed set contains D.*

Definition 2.13. *[14] If $\overline{\mathrm{B}}$ = Z, where B⊆Z, then B is* ***fuzzy dense*** *in the a-fuzzy normed space (Z, n_Z , ⊚).*

Theorem 2.14. *[14] fb(z, j) is a fuzzy open set in a-fuzzy normed space (Z, n_Z, ⊚).*

Definition 2.15. *[14] If for any arbitrary fuzzy Cauchy sequence (z_k) in a-fuzzy normed space (Z, n_Z, ⊚) there exists z ∈ Z with $z_k \to z$, then Z is known as fuzzy complete.*

Definition 2.16. *[14] If $z_k \to z \in Z$, then (z_k) is fuzzy Cauchy in a-fuzzy normed space (Z, n_Z, ⊚).*

Theorem 2.17. *[14] d∈$\overline{\mathrm{D}}$ if and only if there exists($\mathrm{d_k}$) ∈ D with $\mathrm{d_k} \to$ d when D⊂Z and (Z, n_Z, ⊚) is a-fuzzy normed space.*

Theorem 2.18. *[14] If (Z_1, n_1, ⊚), (Z_2, n_2, ⊚), . . . , (Z_k, n_k, ⊚) are a-fuzzy normed spaces then (Z, n, ⊚) is fuzzy complete a-fuzzy normed space if and only if (Z_1, n_1, ⊚), (Z_2, n_2, ⊚), . . . , (Z_k, n_k, ⊚) are fuzzy complete, where Z= $Z_1 \times Z_2 \times \ldots \times Z_k$ and n[(z_1, z_2, . . . , z_k)]=$n_1(z_1)$ ⊚ $n_2(z_2)$ ⊚, . . . , ⊚ $n_k(z_k)$ for all (z_1, z_2, . . . ,z_k) ∈ Z.*

Definition 2.19. *[16] If (Z, $\mathrm{n_Z}$, ⊚) and (W, $\mathrm{n_W}$, ⊚) are two a-fuzzy normed spaces. Then the operator L:Z→W is called fuzzy continuous at z∈Z. If for all r∈ (0, 1), ∃ t ∈ (0, 1), with $\mathrm{n_W}$[L(z)−L(y)] < r, for any y∈Z with $\mathrm{n_Z}$(z−y)< t. Also L is said to be fuzzy continuous on Z if it is fuzzy continuous at every point of Z.*

Theorem 2.20. *[16] If D={z∈Z: 0<n(z)≤1} is a fuzzy closed fuzzy in Z and is compact then Z must be finite dimension where (Z, n, ⊚) is a-fuzzy normed space*

Definition 2.21. *[16] The operator L:D(L)→Y is said to be* ***fuzzy bounded*** *if there exists s ∈ (0, 1) with*

$$n_Y[L(z)] < s n_Z(z),\ \textit{for each}\ \ z{\in}D(L), \tag{2.1}$$

where (Z, n_Z, ⊚) and (Y, n_Y, ⊚) are two a-fuzzy normed spaces.

Example 2.22. *[16] Let Z be that vector space of all polynomials on [0,1] with $n_Z(u) = max_{t\in I} a_{\mathbb{R}}[u(t)]$. Then (Z, n_Z, ⊚) is a-fuzzy normed space. Let L:Z→Z be defined by L[u(t)]=$u(t)'$ then L is linear operator and L is not fuzzy bounded.*

Notation .[16] Suppose that (Z, n_Z, ⊚) and (W, n_W, ⊚) are two a-fuzzy normed spaces. Put afb(Z, W) = {L:Z→Z, L is a linear fuzzy bounded operator}.

Theorem 2.23. *[16] Suppose that (Z, n_Z, ⊚) and (W, n_W, ⊚) are two a-fuzzy normed spaces. Define* $n_{\mathrm{afb(Z,\ W)}}(L) = \sup_{\mathrm{z\in D(L)}} \mathrm{n_W}\,(\mathrm{Lz})$ *for all L∈ afb(Z, W). Then [afb(Z, W),* $n_{\mathrm{afb(Z,\ W)}}$*, ⊚] is a-fuzzy normed space.*

Remark 2.24. *[16] Equation* (2.1) *can be written by*

$$n_W[L(u)] < n_{afb(Z,\ W)}[L]n_Z(u). \tag{2.2}$$

Theorem 2.25. *[16] If W is fuzzy complete then afb(Z, W) is fuzzy complete where (Z, n_Z, ⊚) and (W, n_W, ⊚) are two a-fuzzy normed spaces.*

Theorem 2.26. *[16] Suppose that (Z, n_Z, ⊚) and (W, n_W, ⊚) are two a-fuzzy normed spaces such that L:D(L)→W is a linear operator where* D (T) ⊆*W. Then L is fuzzy continuous if and only if L is fuzzy bounded .*

Definition 2.27. *[16] An a-fuzzy normed space (Z, n, ⊚) is said to be **fuzzy compact** if there exist* $\{\mathrm{U_1,\ U_2,\ U_3,\ldots,\ U_k}\}\subseteq \Omega$ *such that* $Z = \bigcup_{\mathrm{j=1}}^{\mathrm{k}} \mathrm{U_j}$.

Theorem 2.28. *[16] The a-fuzzy normed space (Z, n, ⊚) is fuzzy compact if and only if for every arbitrary sequence (z_k) in Z has a subsequence (z_{k_j}) with $z_{k_j} \to z\in Z$.*

Theorem 2.29. *[16] Let (Z, n_Z, ⊚) be fuzzy complete a-fuzzy normed space and M ⊆Z. Then W is fuzzy complete if and only if W is fuzzy closed.*

Proposition 2.30. *[16] Z is fuzzy totally bounded if (Z, n, ⊚) is a fuzzy compact a-fuzzy normed space.*

Definition 2.31. *[14] A sequence (z_k) in a-fuzzy normed space (Z, n_Z, ⊚) is said to **fuzzy weakly convergent** if there exists z ∈ Z with every h ∈ afb(Z, ℝ),* $\lim_{\mathrm{k\to\infty}} \mathrm{h}(z_k)=h(z)$*. The vector z is said to be the weak limit to* ($\mathrm{z_k}$) *and* ($\mathrm{z_k}$) *is said to be fuzzy converges weakly to z. This is written* $\mathrm{z_k}\to^{\mathrm{w}} z$.

Theorem 2.32. *[14] Suppose that* ($\mathrm{z_k}$) *is weak converges to z in a-fuzzy normed space (Z, n_Z, ⊚). Then*

(1) the weak limit z is unique.
(2) every subsequence ($\mathrm{z_{k_j}}$) *of* ($\mathrm{z_k}$) *converges weakly to z.*
(3) the sequence ($n_Z(\mathrm{z_k})$) *is fuzzy bounded.*

Theorem 2.33. *[14] Suppose that* ($\mathrm{z_k}$) *is in a-fuzzy normed space (Z, n_Z, ⊚). 1. If* $\mathrm{z_k} \to z$ *then* $\mathrm{z_k}\to^{\mathrm{w}} z$ *.2.* $\mathrm{z_k}\to^{\mathrm{w}} z$ *implies* $\mathrm{z_k} \to z$ *when dim Z=m for some m∈ ℕ.*

Theorem 2.34. *[14] Suppose that (Z, n_Z, ⊚) and (W, n_W, ⊚) are a-fuzzy normed spaces with dim Z=k, then every linear operator L:Z→W is fuzzy bounded.*

3. When the linear operator is fuzzy closed

Definition 3.1. *Let (Z, n_Z, ⊚), (W, n_W, ⊚) be two a-fuzzy normed spaces and S:Z→W be a linear operator. Then S is called* ***fuzzy closed*** *if the graph of S, G(S)={(z, w): z∈Z, w=S(z)} is closed in the (Z× W, n, ⊚), n[(z, w)]= $n_Z(z)$ ⊚ $n_W(w)$ for all (z, w) ∈ Z× W.*

Theorem 3.2. *Let (Z, n_Z, ⊚), (W, n_W, ⊚) be two fuzzy complete a-fuzzy normed spaces and S:Z→W be a fuzzy closed operator. If D(S) is fuzzy closed in W then S is fuzzy bounded.*

Proof . We know that from Theorem (2.18), $(Z \times W, n, \circledcirc)$ is fuzzy complete. Now suppose that $G(S)$ is fuzzy closed in $Z \times W$ and $D(S)$ is fuzzy closed in Z. Hence $G(S)$ and $D(S)$ are fuzzy complete. Now consider the mapping $P : G(S) \to D(S)$ defined by $P[z, S(z)] = z$ so P is linear and is fuzzy bounded because $n(P[(z, S(z))]) = n_Z(z) \leq n_Z(z) \circledcirc n_W[S(z)] = n[(z, S(z))]$. But P is bijective, so $P^{-1} : D(S) \to G(S)$ defined by $P^{-1}(z) = [z, S(z)]$ we see that P^{-1} is fuzzy bounded say $n[(z, S(z))] \leq t.n_Z(z)$, for some $t \in (0, 1)$ and all $z \in D(S)$. Hence S is fuzzy bounded since n_W[S(z)]≤ n_W[S(z)] ⊚ n_Z(z) = n[(z, S(z))]≤t.$n_Z(z)$, for all $z \in D(S)$. □

Theorem 3.3. *The operator S is fuzzy closed if and only if $z_k \to z$ where $z_k \in D(S)$ and $S(z_k) \to w$ then z∈ D(S) and S(z)=w., where (Z, n_Z, ⊚), (W, n_W, ⊚) are two a-fuzzy normed spaces and S:Z→W is a linear operator.*

Proof . By definition of $G(S)$ it is fuzzy closed if and only if $u = (z, w) \in \overline{G(S)}$ if and only if there are $u_k = (z_k, S(z_k)) \in G(S)$ such that $z_k \to z, S(z_k) \to w$ and $u = (z, w) \in G(S)$ if and only if $z \in D(S)$ and $w = S(z)$. This complete the prove. □

Here we give an example of a fuzzy bounded operator but not fuzzy closed that is fuzzy closedness does not imply fuzzy boundedness for a linear operator.

Example 3.4. *Let Z=C[0, 1] and S:Z→Z be an operator defined by: S(z(t))=$z(t)'$ where the prime denotes differentiation and D(S) is a subspace of Z consist of all functions z(t) ∈ Z which have a continuous derivative. Then S is not fuzzy bounded but it is fuzzy closed.*

Proof . We know that from Example(2.22), that S is not fuzzy bounded. Now we prove that S is fuzzy closed by applying Theorem (3.3) Let $(z_k) \in D(S)$ be such that both (z_k), $S(z_k)$ fuzzy converges say, $z_k \to z$ and S(z_k)=$z_k\,' \to w$. Since fuzzy converges in the a-fuzzy norm of $C[0, 1]$ is uniform fuzzy convergence on $[0, 1]$ from $z_k\,' \to w$ we have $\int_0^t w(\tau)d(\tau) = \int_0^t \lim_{k\to\infty} z_k\,(\tau)'d(\tau) = \lim_{k\to\infty} \int_0^t z_k\,(\tau)'d(\tau)$ = z(t) −z(0), that is, z(t) = z(0) + $\int_0^t w(\tau)d(\tau)$. This shows that $z \in D(S)$ and $z' = w$. Now Theorem (2.2) implies that S is fuzzy closed. □

In the following example we see that fuzzy boundedness does not imply fuzzy closedness of linear operator.

Example 3.5. *Let S:D(S)→D(S) where D(S)⊆ Z be the identity operator on D(S) where D(S) is a proper dense subspace of the a-fuzzy normed space (Z, n_Z, ⊚). Then it is trivial that S is linear and fuzzy bounded. However S is not fuzzy closed.*

Proof . This follows immediately from Theorem (3.3) if we take an $z \in Z - D(S)$ and sequence (z_k) in $D(S)$ which fuzzy converges to z. □

Proposition 3.6. *Let S:Z→W be a fuzzy bounded linear operator where (Z, n_Z, ⊚), (W, n_W, ⊚) are two a-fuzzy normed spaces. Then*

(i) S is fuzzy closed if D(S) is fuzzy closed subset of Z.

(ii) If S is fuzzy closed and (W, n_W, ⊚) is fuzzy complete then D(S) is fuzzy closed subset of Z.

Proof . (i) If $(z_k) \in D(S)$ and it is fuzzy converges say, $z_k \to z$ also $S(z_k)$ fuzzy converges, then $z \in \overline{D(S)} = D(S)$, because $D(S)$ is fuzzy closed and $S(z_k) \to w$. Since S is fuzzy continuous, by Theorem (3.2), S is fuzzy closed.

(ii) For $z \in \overline{D(S)}$, we can find (z_k) in $D(S)$, $z_k \to z$, since S is fuzzy bounded $n_W[S(z_k) - S(z_m)] = n_W[S(z_k - z_m)] \leq n_{afb(Z,W)}(S).n_Z[z_k - z_m]$. This shows that $(S(z_k))$ is fuzzy Cauchy so $(S(z_k))$ is fuzzy converges say, $S(z_k) \to w \in W$, because W is fuzzy complete.

Now since S is fuzzy closed $z \in D(S)$, by Proposition (3.6) and $S(z) = w$. Thus $\overline{D(S)} \subseteq D(S)$ but $D(S) \subseteq \overline{D(S)}$. Hence $D(S)$ is fuzzy closed. □

4. When the linear operator is fuzzy compact

Definition 4.1. *A is called **relatively fuzzy compact** if $\overline{A}$ is fuzzy compact, where $(Z, n_Z, \circledcirc)$ is a-fuzzy normed space and $A \subseteq Z$*

Definition 4.2. *The linear operator S:$Z \to W$ is called **fuzzy compact** operator if S maps every fuzzy bounded subset A of Z to a relatively fuzzy compact $S(A)$ of W. Where $(Z, n_Z, \circledcirc)$ and $(W, n_W, \circledcirc)$ are a-fuzzy normed spaces.*

Theorem 4.3. *Every fuzzy compact linear operator S:$Z \to W$ is fuzzy bounded, hence fuzzy continuous, whenever $(Z, n_Z, \circledcirc)$ and $(W, n_W, \circledcirc)$ are a-fuzzy normed spaces.*

Proof . The fuzzy ball $B = \{z \in Z : n_Z(z) = 1\}$ is fuzzy bounded. Since S is fuzzy compact, $\overline{S(B)}$ is fuzzy compact and is fuzzy bounded by Proposition (2.29). So that $sup_{z \in B} n_W[S(z)] < \infty$. Hence, S is fuzzy bounded and by Theorem (2.26) it is fuzzy continuous. □

Lemma 4.4. *If $(Z, n_Z, \circledcirc)$ a-fuzzy normed space with dim $Z < \infty$ then the identity operator is not fuzzy compact.*

Proof . Since the fuzzy closed unit fuzzy ball $B = \{z \in Z : n_Z(z) \leq 1\}$ is fuzzy bounded. If $\dim Z = \infty$, then Theorem (2.20) implies that B cannot be fuzzy compact. Thus $I(B) = B = \overline{B}$ is not relatively fuzzy compact. □

Theorem 4.5. *The operator S is fuzzy compact if and only if $(S(z_k))$ has a fuzzy convergent subsequence for every fuzzy bounded sequence (z_k) in Z. Where $(Z, n_Z, \circledcirc)$ and $(W, n_W, \circledcirc)$ are a-fuzzy normed spaces and S:$Z \to W$ is a linear operator.*

Proof . If S is fuzzy compact and the sequence (z_k) is fuzzy bounded in Z, the closure of $(S(z_k))$ is fuzzy compact in W and Definition (4.1) shows that $(S(z_k))$ has a convergent subsequence. For the converse suppose that every fuzzy bounded sequence (z_k) in Z contains a subsequence (z_{k_j}) such that $(S(z_{k_j}))$ fuzzy converges in W. Consider any fuzzy bounded subset B of Z and let (w_k) be any sequence in $S(B)$, then $S(z_k) = w_k$, for some $z_k \in B$ and (z_k) is fuzzy bounded, because B is fuzzy bounded. By our assumption $(S(z_k))$ contains a fuzzy convergent subsequence. Hence $\overline{\text{S(B)}}$ is fuzzy compact by Definition (4.1), because (w_k) was arbitrary in $S(B)$. Hence S is fuzzy compact. □

Proposition 4.6. *Let $(Z, n_Z, \circledcirc)$ and $(W, n_W, \circledcirc)$ be a-fuzzy normed spaces and S_1:$Z \to W$, S_2:$Z \to W$ be two a linear operators. If S_1 and S_2 are fuzzy compact operators then $(S_1 + S_2)$ is a fuzzy compact operator.*

Proof . It is clear that $(S_1+ S_2)$ is a linear operator. Let (z_k) be a fuzzy bounded sequence in Z, then by Theorem (4.5), $S_1(z_k)$ has a fuzzy convergent subsequence, say $S_1(z_{k_j})$ and $S_2(z_k)$ has a fuzzy convergent subsequence, say $S_2(z_{k_j})$ but the sum of two fuzzy convergent sequence is again fuzzy convergent this implies that, $S_1(z_k) + S_2(z_k)$=[$S_1+ S_2$](z_k) contains a fuzzy convergent subsequence. Hence $(S_1+ S_2)$ is fuzzy compact operator by Theorem (4.5). □

Proposition 4.7. *If S is fuzzy compact operator then αS is a fuzzy compact operator. Where (Z, n_Z, ⊚) and (W, n_W, ⊚) are a-fuzzy normed spaces and S:Z→ W is a linear operators.*

Proof . It is clear that αS is a linear operator. Let (z_k) be a fuzzy bounded sequence in Z then by Theorem 4.5, S(z_k) has a fuzzy convergent subsequence, say, S(z_{k_j}) this implies that $\alpha S(z_{k_j})$ is fuzzy convergent subsequence of $\alpha S(z_k)$. Hence αS is a fuzzy compact operator by Theorem (4.5). □

Notation. Let $afc(Z, W) = \{SjS : Z \to W$ is a linear fuzzy compact operator$\}$. From Proposition (4.6) and Proposition (4.7), we have next result

Proposition 4.8. *If (Z, n_Z, ⊚) and (W, n_W, ⊚) are a-fuzzy normed spaces then afc(Z, W) is a subspace of afb(Z,W).*

Theorem 4.9. *If W is a fuzzy complete, then $afc(Z, W)$ is a fuzzy closed subspace of $afb(Z, W)$, where $(Z, n_Z, ⊚)$ and $(W, n_W, ⊚)$ are a-fuzzy normed spaces*

Proof . Let $S \in \overline{\text{afc(Z, W)}}$. Then there exists (S_k) in $afc(Z, W)$ such that $S_k \to S$. S is a linear operator, because

$$S(\alpha z_1 + \delta z_2) = \lim_{k\to\infty} S_k(\ \alpha z_1 + \delta z_2) = \alpha \lim_{k\to\infty} S_k(\ z_1) + \delta \lim_{k\to\infty} S_k(\ z_2) = \alpha S(z_1) + \delta S(z_2).$$

Now we will show that S is fuzzy compact operator. Let (z_j) be a fuzzy bounded sequence in Z, then by Theorem (4.6), $S_k(z_j)$ contains a fuzzy convergent subsequence. Hence

$$S(z_j) = \lim_{k\to\infty} S_k(z_j)$$

contains a convergent subsequence. Thus $S \in afc(Z, W)$. This implies that $afc(Z, W) = \overline{afc(Z,\ W)}$. Hence $afc(Z, W)$ is a fuzzy closed subspace of $afb(Z, W)$. □

The next result follows from Theorem (4.9) and Theorem (2.28).

Theorem 4.10. *If W is a fuzzy complete then afc(Z,W) is a fuzzy complete when (Z, n_Z, ⊚) and (W, n_W, ⊚) are a-fuzzy normed spaces.*

Theorem 4.11. *If S is fuzzy bounded and dim S(Z) is finite then the operator S is fuzzy compact. When (Z, n_Z, ⊚) and (W, n_W, ⊚) are a-fuzzy normed spaces and S:Z→ W, be a linear operators*

Proof . Let (z_k) be a fuzzy bounded sequence in Z then from the inequality n_W[S(z_k)] $\le n_{fb(Z,\ W)}$(S). $n_Z[z_k]$. We have $(S(z_k))$ is fuzzy bounded hence $(S(z_k))$ is relatively compact by Theorem (4.5), because $\dim S(Z)$ is finite. It follows that $(S(z_k))$ has a convergent subsequence but (z_k) was an arbitrary fuzzy bounded sequence in Z. Hence S is fuzzy compact operator by Theorem (4.5). □

Theorem 4.12. *If S is fuzzy bounded and dim Z is finite then the operator S is fuzzy compact. When (Z, n_Z, ⊚) and (W, n_W, ⊚) are a-fuzzy normed spaces and S:Z→ W, be a linear operators.*

Proof . Since $\dim Z$ is finite implies $\dim S(Z)$ is finite and $\dim Z$ is finite implies fuzzy boundedness of S by Theorem (2.33). Now the operator S is fuzzy compact follows from Theorem (4.11). □

Theorem 4.13. *Suppose that $(S_k) \in afc(Z, W)$ if $S_k \to S$, then $S \in$ afc(Z, W) when (Z, n_Z, ⊚) and (W, n_W, ⊚) are a-fuzzy normed spaces with W is fuzzy complete.*

Proof . Since S_1 is fuzzy compact, (z_k) has a subsequence ($z_k^{(1)}$) such that (S_1 ($z_k^{(1)}$)) is fuzzy Cauchy. Similarly ($z_k^{(1)}$) has a subsequence ($z_k^{(2)}$) such that (S_2 ($z_k^{(2)}$)) is fuzzy Cauchy. By continuing in this way we see that (y_m)= ($z_m^{(m)}$)is a subsequence of (z_k) such for every fixed positive integer j the sequence $(S_j(y_m))_{m=1}^{\infty}$ is a fuzzy Cauchy. Since (z_k) is fuzzy bounded say, $n_Z(z_k) \leq t$, for some $t \in (0, 1)$. Let $r \in (0, 1)$, since $S_k \to S$, there is $k = p$ such that $n_{fb(Z,\ W)}[S - S_p] < t$, but $(S_p(y_m))_{m=1}^{\infty}$ is fuzzy Cauchy so there is N such that $n_W[S_p(y_j) - S_p(y_k)] < r$, for all $j, k \geq N$. Hence, for all $j, k \geq N$,

$$\begin{aligned} n_W[S(y_j) - S(y_k)] &\leq n_W[S(y_j) - S_p(y_j)] \circledcirc n_W[S_p(y_j) - S_p(y_k)] \circledcirc n_W[S_p(y_k) - S(y_k)] \\ &\leq n_{fb(Z,\ W)}[S - S_p].\ n_Z(y_j) \circledcirc r \circledcirc n_{fb(Z,\ W)}[S_p - S].\ n_Z(y_k) \\ &\leq t.c \circledcirc r \circledcirc t.c. \end{aligned}$$

We can find $q \in (0, 1)$ such that $[t.c \circledcirc r \circledcirc t.c] < q$. Thus $n_W[S(y_j) - S(y_k)] < q$. This shows that $(S(y_m))$ is a fuzzy Cauchy sequence and fuzzy converges in W since W is fuzzy complete but (y_m) is a subsequence of the arbitrary fuzzy bounded sequence (z_k). Now Theorem (4.5) implies fuzzy compactness of the operator S. □

Another basic property of fuzzy compact linear operator is that it transforms weakly fuzzy convergent sequences into strongly fuzzy convergent sequences as follows in the next result.

Theorem 4.14. *If (z_k) in Z with $z_k \to^w z$ then ($S(z_k)$) is strongly fuzzy converge in W and $S(z_k) \to w = S(z)$, whenever (Z, n_Z, ⊚) and (W, n_W, ⊚) are a-fuzzy normed spaces and $S: Z \to W$ is a fuzzy compact linear operator.*

Proof . Let $S(z_k) = w_k$ and $w = S(z)$. First we will prove that

$$w_k \to^w w \tag{4.1}$$

then

$$w_k \to w. \tag{4.2}$$

Let g be any fuzzy bounded linear functional on W. We define functional f on Z by $f(z) = g(S(z))$, for all $z \in Z$. Then f is linear and is fuzzy bounded, because S is fuzzy compact. Now

$$\begin{aligned} a_{\mathbb{R}}[f(z)] &= a_{\mathbb{R}}[g(S(z))] \\ &\leq n_{fb(W,\mathbb{R})}[g].n_W[S(z)] \\ &\leq n_{fb(W,\mathbb{R})}[g].n_{fb(Z,\ W)}[S].n_Z[z]. \end{aligned}$$

By definition $z_k \to^w z$ implies $f(z_k) \to^w f(z)$, hence by definition $g[S(z_k) \to^w g[S(z)]$. That is $g(w_k) \to g(w)$. Since g was arbitrary, this proves $w_k \to^w w$. Now to prove (4.2), suppose that (w_k) has a subsequence w_{k_j} such that

$$n_W[w_{k_j} - w] \geq t \text{ for some } \ t \in (0, 1). \tag{4.3}$$

Since (z_k) is weakly fuzzy convergent, (z_k) is fuzzy bounded by Theorem 2.38 and so is (z_{k_j}). Now fuzzy compactness of S implies by Theorem (4.5) that (Sz_{k_j}) has a convergent subsequence say, $(\widetilde{w_j})$. Let $\widetilde{w_j} \to \tilde{w}$ so $\widetilde{w_j} \to^w \tilde{w}$. Hence $\tilde{w} = w$ by (2.2) and Theorem (2.32), we have $n_W[\widetilde{w_j} - w] \to 0$ but this contradicts with $n_W[\widetilde{w_j} - w] \geq t > 0$ by (4.2). So that (4.2) must be hold. □

5. Conclusion

In this work we present the definition of fuzzy bounded linear operator on a-fuzzy normed space then basic properties of this type of operators are proved. After that we introduce the definition of fuzzy bounded linear operator on a-fuzzy normed space then basic properties of this type of operators are proved. It is known that each study left out some things and needed further elaboration throughout this study we come up with the fact that there are many notions for a linear operator one can introduced for future work such as fuzzy open linear operator, fuzzy compact linear operator, . . . , etc.

References

[1] A. Ali, *Properties of Complete Fuzzy Normed Algebra*, M.Sc. Thesis, University of Technology, Iraq, 2018.

[2] N. Kadhum, *On Fuzzy Norm of a Fuzzy Bounded Operator on Fuzzy Normed Spaces*, M.Sc. Thesis, University of Technology, Iraq, 2017.

[3] J.R. Kider, *On fuzzy normed spaces*, Eng. Tech. J. 29(9) (2011) 1790–1795.

[4] J.R. Kider, *Completion of fuzzy normed spaces*, Eng. Tech. J. 29(10) (2011) 2004–2012.

[5] J.R. Kider, *New fuzzy normed spaces*, J. Baghdad Sci. 9 (2012) 559–564.

[6] J.R. Kider, *Some properties of algebra fuzzy metric space*, J. Al-Qadisiyah Comput. Sci. Math. 12(2) (2020) 43–56.

[7] J.R. Kider, *Some properties of fuzzy Soft metric space*, Al-Qadisiyah J. Pure Sci. 25(2) (2020) 1–13.

[8] J.R. Kider and A. Ali, *Properties of fuzzy absolute value on and properties finite dimensional fuzzy normed space*, Iraqi J. Sci. 59(2B) (2018) 909–916.

[9] J.R. Kider and M. Gheeab, *Properties of a general fuzzy normed space*, Iraqi J. Sci. 60(4)(2019) 847–855.

[10] J.R. Kider and M. Gheeab, *Properties of the space GFB(V, U)*, J. AL-Qadisiyah Comput. Sci. Math. 11(1) (2019).

[11] J.R. Kider and M. Gheeab, *Properties of the adjoint operator of a general fuzzy bounded operator*, Baghdad Sci. J. 18(1) (2021) 790–796.

[12] J.R. Kider and N. Kadhum, *Properties of fuzzy norm of fuzzy bounded operators*, Iraqi J. Sci. 58(3A) (2017) 1237–1281.

[13] J.R. Kider and N. Kadhum, *Properties of fuzzy compact linear operators on fuzzy normed spaces*, Baghdad Sci. J. 16(1) (2019) 104–110.

[14] Z.A. Khudhair, *Some Properties of Algebra Fuzzy Normed Space*, M.Sc. Thesis, University of Technology, Iraq, 2021.

[15] Z.A. Khudhair and J.R. Kider, *Some properties of algebra fuzzy absolute value space and algebra fuzzy normed space*, Al-Qadisiyah J. Pure Sci. 25(4) (2020) 46–58

[16] Z.A. Khudhair and J.R. Kider, *Some properties of fuzzy compact algebra fuzzy normed spaces and finite dimensional algebra fuzzy normed spaces*, J. Phys. Conf. Ser. 1879 (2021) 1–11.

[17] Z.A. Khudhair and J.R. Kider, *The Hausdorff a-fuzzy distance and its basic properties*, Engin. Technol. J. 39(7) (2021) 1185–1194.