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Sandwich theorems for analytic univalent functions defined by Hadamard product operator

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Abstract

In the present paper, we obtain some subordination and superordination results involving the Hadamard product operator $D_{\alpha,c}^{\mu,b}$ for certain normalized analytic univalent functions in the open unit disk. These results are applied to obtain sandwich results.

Keywords: Analytic function, Integral Operator, Differential Subordination, Superordination, Sandwich results
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1. Introduction

Let $H = H(U)$ be the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For n a positive integer and $a \in \mathbb{C}$ Let $H[a \cdot n]$ be the subclass of $f \in H$ of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots \quad (a \in \mathbb{C},\ N = \{1, 2, 3, \dots\}) \tag{1.1}$$

Let T denote the subclass of H of functions f of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ (z \in U) \tag{1.2}$$

If $f \in T$ is given by (1.2) and $g \in T$ given by

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$$g(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ (z \in U)$$

The Hadamard product (or the convolution) of f and g is defined by

$$(f * g)(z) = z \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z)$$

If f and g are analytic functions in H. We say that f is subordinate to g in U and write $f \prec g$, if there exists a Shwarz function w, which is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f(z) = g(w(z)), \ (z \in U)$.

Furthermore, if the function g is univalent in U, we have the following equivalence relationship (cf. ,e.g. [10, 13, 14])

$$f(z) \prec g(z) \leftrightarrow f(0) = g(0) \ and \ f(U) \subset g(U), \ z \in U.$$

Definition 1.1. *[13] Let $\varphi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$ and let $h(z)$ be analytic in U. If l and $\varphi(l(z), zl'(z), z^2 l''(z); z)$ are univalent in U and if l satisfies the second-order differential superordination,*

$$h(z) \prec \varphi(l(z), zl'(z), z^2 l''(z); z), \quad (z \in U) \tag{1.3}$$

then l is called a solution of the differential superordination (1.3). *An analytic function $q(z)$ which is called a subordinate of the solutions of the differential superordination* (1.3) *or more simply a subordinate, if $l \prec q$ for all l satisfying* (1.3). *A univalent subordinate $\tilde{q}(z)$ that satisfies $q \prec \tilde{q}$ for all subordinants q of* (1.3) *is said to be the best subordinate.*

Definition 1.2. *[13] Let $\varphi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$ and let $h(z)$ be univalent in U. If l is analytic in U and satisfies the second-order differential subordination,*

$$\varphi(l(z), zl'(z), z^2 l''(z); z) \prec h(z), \quad (z \in U) \tag{1.4}$$

then l is called a solution of the differential subordination (1.4). *The univalent function q is called a dominant of the solution of the differential subordination* (1.4) *or more simply a dominant, if $l \prec q$ for all l satisfying* (1.4). *A dominant $\tilde{q}(z)$ that satisfies $q \prec \tilde{q}$ for all dominant q of* (1.4) *is said to be the best dominant.*

Recently, several authors, like, [1, 2, 7, 13, 15] obtained sufficient conditions on the functions h, l and φ for which the following implication holds

$$h(z) \prec \varphi(l(z), zl'(z), z^2 l''(z); z) \to q(z) \prec l(z), \quad (z \in U) \tag{1.5}$$

By using results (see [3, 4, 9, 14]) to obtain sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. Also, several authors (see [1, 3, 4, 5, 6, 15]) derived some differential subordination and superordination results with some sandwich theorems.

Choi and Srivastava [12] found several interesting properties of Hurwitz-Lerch zeta function $\varphi(z,s,a)$ defined by

$$\varphi(z,s,a)=\sum_{n=0}^{\infty}(\frac{z}{(n+a)^s}) \tag{1.6}$$

$$a\in\mathbb{C}\backslash\{0,-1,-2,...\},\ s\in\mathbb{C},\ Re(s)>1 \text{ and } |z|=1$$

In [16] Srivastava-Attiya introduced the following operator $F_{\mu,b}:T\to T$

$$F_{\mu,b}(z)=(1+b)^{\mu}[\varphi(z,\mu,b)-b^{-\mu}]$$

which has the following form:

$$F_{\mu,b}f(z)=z+\sum_{n=2}^{\infty}\left(\frac{1+b}{n+b}\right)^{\mu}a_nz^n \tag{1.7}$$

$$b\in\mathbb{C}\backslash\{0,-1,-2,...\},\ \mu\in\mathbb{C},\ z\in U,\ f\in T$$

For $f\in T$. Carlson and Shaffer [11] defined the following integral operator $T_\alpha f(z)$ by

$$T_\alpha f(z)=z+\sum_{n=2}^{\infty}\frac{(\alpha)_{n-1}}{(c)_{n-1}}a_nz^n \tag{1.8}$$

Atshan et.al Defined the operator $D^{\mu,b}_{\alpha,c}f(z)$ [8],

$$D^{\mu,b}_{\alpha,c}f(z)=F_{\mu,b}(z)*T_\alpha f(z)=z+\sum_{n=2}^{\infty}\left(\frac{1+b}{n+b}\right)^{\mu}\frac{(\alpha)_{n-1}}{(c)_{n-1}}a_nz^n \tag{1.9}$$

Moreover, from (1.9), it follows that

$$z\left(D^{\mu+1,b}_{\alpha,c}f(z)\right)'=(1+b)D^{\mu,b}_{\alpha,c}f(z)-bD^{\mu+1,b}_{\alpha,c}f(z) \tag{1.10}$$

The main object here to find sufficient conditions for certain normalized analytic function f to satisfy:

$$q_1(z)\prec\left[\frac{D^{\mu+1,b}_{\alpha,c}f(z)}{z}\right]^{\gamma}\prec q_2(z)$$

and

$$q_1(z)\prec\left[\frac{tD^{\mu+1,b}_{\alpha,c}f(z)+(1-t)D^{\mu,b}_{\alpha,c}f(z)}{z}\right]^{\gamma}\prec q_2(z)$$

where q_1 and q_2 are given univalent functions in U with $q_1(0)=q_2(0)=1$.

In this paper, we derive some differential subordination, superordination and sandwich results involving the operator $D^{\mu,b}_{\alpha,c}f(z)$.

2. Preliminaries

We need the following definitions and lemmas to prove our results.

Definition 2.1. *[10] Let Q the set of all functions $f(z)$ that are analytic and injective on $\bar{U}|E(q)$, where $\bar{U} = U \cup \{z \in \partial U\}$, and*

$$E(f) = \{\varepsilon \in \partial U : \lim_{z \to \varepsilon} f(z) = \infty\}$$

and are such that $f'(\varepsilon) \neq 0$ for $\varepsilon \in \partial U|E(f)$. Further, let the subclass of Q for which $f(0) = a$ be denoted by $Q(a)$, and $Q(0) = Q_0, q(1) = Q_1 = \{f \in Q : f(0) = 1\}$.

Lemma 2.2. *[13] Let q be a convex univalent function in U and let $\alpha \in \mathbb{C},\ \beta \in \mathbb{C}|\{0\}$ with*

$$Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -Re(\frac{\alpha}{\beta})\right\}.$$

If l is analytic in U and

$$\alpha l(z) + \beta z l'(z) \prec \alpha q(z) + \beta z q'(z), \tag{2.1}$$

then $l \prec q$ and q is the best dominant.

Lemma 2.3. *[14] Let q be univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$, when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that*

- *$Q(z)$ is starlike univalent in U,*
- *$Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ for $z \in U$.*

If l is analytic in U, with $p(0) = q(0),\ p(U) \subseteq D$ and

$$\theta(l(z)) + zl'(z)\phi(l(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{2.2}$$

then $l \prec q$ and q is the best dominant.

Lemma 2.4. *[14] Let q be univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$. Suppose that*

- *$Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} > 0$ for $z \in U$,*
- *$Q(z) = zq'(z)\phi(q(z))$ is starlike univalent in U.*

If $l \in H[q(0), 1] \cap Q$, with $l(U) \subset D$, $\theta(l(z)) + zl'(z)\phi(l(z))$ is univalent in U and

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(l(z)) + zl'(z)\phi(l(z)), \tag{2.3}$$

then $q \prec l$ and q is the best subordinant.

Lemma 2.5. *[10] Let q be a convex univalent in U and $q(0) = 1$ and let $\beta \in \mathbb{C}$, that $Re(\beta) > 0$. If $l \in H[q(0), 1] \cap Q$ and $l(z) + \beta z l'(z)$ is univalent in U, then*

$$q(z) + \beta z q'(z) \prec l(z) + \beta z l'(z), \tag{2.4}$$

which implies that $q \prec l$ and q is the best subordinant.

3. Subordination Results

Now, we discuss some differential subordination results by using the Hadamard product operator $D^{\mu,b}_{\alpha,c}f(z)$.

Theorem 3.1. *Let q be convex univalent function in U with $q(0) = 1, 0 \neq \varepsilon \in \mathbb{C},\ \gamma > 0$ and suppose that q satisfies:*

$$Re\left\{1 - \frac{zq''(z)}{q'(z)}\right\} > \max\{0, -Re\left(\frac{\gamma}{\varepsilon}\right)\}. \tag{3.1}$$

If $f \in T$ satisfies the subordination

$$\left[\frac{D^{\mu+1,b}_{\alpha,c}f(z)}{z}\right]^{\gamma} + \varepsilon(b+1)\left[\frac{D^{\mu+1,b}_{\alpha,c}f(z)}{z}\right](\frac{D^{\mu,b}_{\alpha,c}f(z)}{D^{\mu+1,b}_{\alpha,c}f(z)} - 1) \prec q(z) + \frac{\varepsilon}{\gamma}zq'(z), \tag{3.2}$$

then

$$\left[\frac{D^{\mu+1,b}_{\alpha,c}f(z)}{z}\right]^{\gamma} + \varepsilon(b+1)\left[\frac{D^{\mu+1,b}_{\alpha,c}f(z)}{z}\right] \prec q(z), \tag{3.3}$$

and q is the best dominant.

Proof . Define the function l by

$$l(z) = \left[\frac{D^{\mu+1,b}_{\alpha,c}f(z)}{z}\right]^{\gamma}, \tag{3.4}$$

then the function $l(z)$ is analytic in U and $l(0) = 1$, therefore, differentiating (3.4) with respect to z and using the identity (1.10) in the resulting equation, we obtain

$$\frac{zl'(z)}{l(z)} = \gamma\left[\left(\frac{z(D^{\mu+1,b}_{\alpha,c}f(z))'}{D^{\mu+1,b}_{\alpha,c}f(z)} - 1\right)\right]. \tag{3.5}$$

Now, in view of (3.5), we obtain

$$\frac{zl'(z)}{\gamma} = \left[\frac{D^{\mu+1,b}_{\alpha,c}f(z)}{z}\right]^{\gamma}\left(b\left(\frac{D^{\mu,b}_{\alpha,c}f(z)}{D^{\mu+1,b}_{\alpha,c}f(z)} - 1\right) + \left(\frac{D^{\mu,b}_{\alpha,c}f(z)}{D^{\mu+1,b}_{\alpha,c}f(z)} - 1\right)\right). \tag{3.6}$$

The subordination (3.2) from the hypothesis becomes

$$l(z) + \frac{\varepsilon}{\gamma}zl'(z) \prec q(z) + \frac{\varepsilon}{\gamma}zq'(z).$$

An application of Lemma 2.2 with $\beta = \frac{\varepsilon}{\gamma}$ and $\alpha = 1$, we obtain (3.3). □

Putting $q(z) = \left(\frac{1+z}{1-z}\right)$ in Theorem 3.1,we obtain the following corollary.

Corollary 3.2. *Let $0 \neq \varepsilon \in \mathbb{C},\ \gamma > 0$ and*

$$Re\left\{1 + \frac{2z}{1-z}\right\} > \max\{0, -Re\left(\frac{\gamma}{\varepsilon}\right)\}.$$

If $f \in T$ satisfies the subordination

$$\left[\frac{D_{\alpha,c}^{\mu+1,b}f(z)}{z}\right]^{\gamma}+\varepsilon(b+1)\left[\frac{D_{\alpha,c}^{\mu+1,b}f(z)}{z}\right]^{\gamma}(\frac{D_{\alpha,c}^{\mu,b}f(z)}{D_{\alpha,c}^{\mu+1,b}f(z)}-1)\prec\left(\frac{1-z^2+2\frac{\varepsilon}{\gamma}z}{(1-z)^2}\right).$$

then

$$\left[\frac{D_{\alpha,c}^{\mu+1,b}f(z)}{z}\right]^{\gamma}\prec\left(\frac{1+z}{1-z}\right)$$

and $q(z)=\left(\frac{1+z}{1-z}\right)$ is the best dominant.

Theorem 3.3. *Let q be convex univalent function in U with $q(0)=1,\ q'(z)\neq 0\ (z\in U)$ and assume that q satisfies*

$$Re\{q(z)+z\frac{q''(z)}{q'(z)}-z\frac{q'(z)}{q(z)}\}>0. \tag{3.7}$$

Suppose that $z\frac{q'(z)}{q(z)}$ is starlike univalent in U. If $f\in A$ satisfies

$$p(z)\prec t+q(z)+z\frac{q'(z)}{q(z)}, \tag{3.8}$$

where,

$$p(z)=t+\left[\frac{tD_{\alpha,c}^{\mu+1,b}f(z)+(1-t)D_{\alpha,c}^{\mu,b}f(z)}{z}\right]^{\gamma}+\gamma\left[\frac{tz(D_{\alpha,c}^{\mu+1,b}f(z))'+(1-t)z(D_{\alpha,c}^{\mu,b}f(z))'}{tD_{\alpha,c}^{\mu+1,b}f(z)+(1-t)D_{\alpha,c}^{\mu,b}f(z)}-1\right] \tag{3.9}$$

then

$$\left[\frac{tD_{\alpha,c}^{\mu+1,b}f(z)+(1-t)D_{\alpha,c}^{\mu,b}f(z)}{z}\right]^{\gamma}\prec q(z) \tag{3.10}$$

and q is the best dominant.

Proof . Define analytic function $l(z)$ by

$$l(z)=\left[\frac{tD_{\alpha,c}^{\mu+1,b}f(z)+(1-t)D_{\alpha,c}^{\mu,b}f(z)}{z}\right]^{\gamma}. \tag{3.11}$$

Then the function $l(z)$ is analytic in U and $l(0)=1$ differenitating (3.10) with respect to z, and using the identity (1.10) we get,

$$\frac{zl'(z)}{l(z)}=\gamma\left[\frac{tz(D_{\alpha,c}^{\mu+1,b}f(z))'+(1-t)z(D_{\alpha,c}^{\mu,b}f(z))'}{tD_{\alpha,c}^{\mu+1,b}f(z)+(1-t)D_{\alpha,c}^{\mu,b}f(z)}+1\right]. \tag{3.12}$$

By setting
$\theta(w)=1+w$ and $\phi(w)=\frac{1}{w},\ w\neq 0$
we see that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w)\neq 0$, $w\in\mathbb{C}\backslash\{0\}$. Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = z\frac{q'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = t + q(z) + z\frac{q'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent in U,

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\} = Re\left\{q(z) + z\frac{q''(z)}{q'(z)} - z\frac{q'(z)}{q(z)}\right\} > 0.$$

By a straightforward computation, we obtain

$$l(z) = t + l(z) + z\frac{l'(z)}{l(z)}. \tag{3.13}$$

By making use of (3.9), we obtain

$$t + l(z) + z\frac{l'(z)}{l(z)} \prec t + q(z) + z\frac{q'(z)}{q(z)}. \tag{3.14}$$

Therefore, by Lemma 2.3, we get $l(z) \prec q(z)$. By using (3.9), we obtain the result. □
Putting $q(z) = \left(\frac{1+Az}{1+Bz}\right)$, $(-1 \leq B < A \leq 1)$ in Theorem 3.3, we obtain the following corollary:

Corollary 3.4. *Let* $-1 \leq B < A \leq 1$ *and*

$$Re\left\{\frac{1+Az}{1+Bz} + \frac{2Bz}{1+Bz} + \frac{(A-B)z}{(1+Bz)(1+Az)}\right\} > 0$$

where $t \in \mathbb{C}$ *and* $z \in U$*, if* $f \in T$ *satisfies*

$$l(z) \prec t + \frac{1+Az}{1+Bz} + \frac{(A-B)z}{(1+Bz)(1+Az)},$$

where is given $l(z)$ *by* (3.10)*, then*

$$\left[\frac{tD^{\mu+1,b}_{\alpha,c}f(z) + (1-t)D^{\mu,b}_{\alpha,c}f(z)}{z}\right]^{\gamma} \prec \left(\frac{1+Az}{1+Bz}\right),$$

and $q(z) = \left(\frac{1+Az}{1+Bz}\right)$ *is the best dominant.*

Taking the function $q(z) = \left(\frac{1+z}{1-z}\right)^{\rho}$ $(-1 \leq \rho \leq 1)$ in Theorem 3.3, we obtain the following corollary:

Corollary 3.5. *Let* $-1 \leq \rho \leq 1$ *and*

$$Re\left\{(\frac{1+z}{1-z})^{\rho} + \frac{2\rho z}{1+z^2} + \frac{2z^2}{1+z^2}\right\} > 0$$

where $t \in \mathbb{C}$ *and* $z \in U$*, if* $f \in T$ *satisfies*

$$l(z) \prec (\frac{1+z}{1-z})^{\rho} + \frac{2\rho z}{1+z^2} + \frac{2z^2}{1+z^2},$$

where $l(z)$ defined in (3.10),*then*

$$\left[\frac{tD_{\alpha,c}^{\mu+1,b}f(z) + (1-t)D_{\alpha,c}^{\mu,b}f(z)}{z}\right]^{\gamma} \prec \left(\frac{1+z}{1-z}\right)^{\rho},$$

and $q(z) = \left(\frac{1+z}{1-z}\right)^{\rho}$ is the best dominant.

4. Superordination Results

Theorem 4.1. *Let q be convex univalent function in U with $q(0) = 1$, $\gamma > 0$ and $Re\{\varepsilon\} > 0$. Let $f \in T$ satisfies*

$$\left[\frac{D_{\alpha,c}^{\mu+1,b}f(z)}{z}\right]^{\gamma} \in H[q(0), 1] \cap Q$$

and

$$\left[\frac{D_{\alpha,c}^{\mu+1,b}f(z)}{z}\right]^{\gamma} + \varepsilon(b+1)\left[\frac{D_{\alpha,c}^{\mu+1,b}f(z)}{z}\right]^{\gamma}(\frac{D_{\alpha,c}^{\mu,b}f(z)}{D_{\alpha,c}^{\mu+1,b}f(z)} - 1)$$

be univalent in U. If

$$q(z) + \frac{\varepsilon}{\gamma}zq'(z) \prec \left[\frac{D_{\alpha,c}^{\mu+1,b}f(z)}{z}\right]^{\gamma} + \varepsilon(b+1)\left[\frac{D_{\alpha,c}^{\mu+1,b}f(z)}{z}\right]^{\gamma}(\frac{D_{\alpha,c}^{\mu,b}f(z)}{D_{\alpha,c}^{\mu+1,b}f(z)} - 1), \tag{4.1}$$

then

$$q(z) \prec \left[\frac{D_{\alpha,c}^{\mu+1,b}f(z)}{z}\right]^{\gamma} \tag{4.2}$$

and q is the best subordinant of (4.1).

Proof . Define the function l by

$$l(z) = \left[\frac{D_{\alpha,c}^{\mu+1,b}f(z)}{z}\right]^{\gamma}. \tag{4.3}$$

Differentiating (4.3) with respect to z, we get

$$\frac{zl'(z)}{l(z)} = \gamma\left[\frac{z(D_{\alpha,c}^{\mu+1,b}f(z))'}{D_{\alpha,c}^{\mu+1,b}f(z)} - 1\right]. \tag{4.4}$$

After some computations and using (1.10), from (4.4), we obtain

$$\left[\frac{D_{\alpha,c}^{\mu+1,b}f(z)}{z}\right]^{\gamma} + \varepsilon(b+1)\left[\frac{D_{\alpha,c}^{\mu+1,b}f(z)}{z}\right]^{\gamma}(\frac{D_{\alpha,c}^{\mu,b}f(z)}{D_{\alpha,c}^{\mu+1,b}f(z)} - 1) = l(z) + \frac{\varepsilon}{\gamma}zl'(z),$$

and now, by using Lemma 2.5, we get the desired result. □

Putting $q(z) = \left(\frac{1+z}{1-z}\right)$ in Theorem 4.1, we obtain the following corollary:

Corollary 4.2. *Let $\gamma > 0$ and $Re\{\varepsilon\} > 0$. If $f \in T$ satisfies*

$$\left[\frac{D^{\mu+1,b}_{\alpha,c} f(z)}{z}\right]^{\gamma} \in H[q(0),1] \cap Q$$

and

$$\left[\frac{D^{\mu+1,b}_{\alpha,c} f(z)}{z}\right]^{\gamma} + \varepsilon(b+1)\left[\frac{D^{\mu+1,b}_{\alpha,c} f(z)}{z}\right]^{\gamma} \left(\frac{D^{\mu,b}_{\alpha,c} f(z)}{D^{\mu+1,b}_{\alpha,c} f(z)} - 1\right)$$

be univalent in U. If

$$\left(\frac{1 - z^2 + 2\frac{\varepsilon}{\gamma} z}{(1-z)^2}\right) \prec \left[\frac{D^{\mu+1,b}_{\alpha,c} f(z)}{z}\right]^{\gamma} + \varepsilon(b+1)\left[\frac{D^{\mu+1,b}_{\alpha,c} f(z)}{z}\right]^{\gamma} \left(\frac{D^{\mu,b}_{\alpha,c} f(z)}{D^{\mu+1,b}_{\alpha,c} f(z)} - 1\right)$$

then

$$\left(\frac{1+z}{1-z}\right) \prec \left[\frac{D^{\mu+1,b}_{\alpha,c} f(z)}{z}\right]^{\gamma}$$

and $q(z) = \left(\frac{1+z}{1-z}\right)$ is the best subordinant.

Theorem 4.3. *Let q be convex univalent function in U, Let $t \in \mathbb{C}$, $\gamma > 0$, $q'(z) \neq 0$ and $f \in T$, suppose that*

$$Re\{zq'(z)q(z)\} > 0, \tag{4.5}$$

$$\left[\frac{tD^{\mu+1,b}_{\alpha,c} f(z) + (1-t)D^{\mu,b}_{\alpha,c} f(z)}{z}\right]^{\gamma} \in H[q(0),1] \cap Q$$

And

$$\left[\frac{tD^{\mu+1,b}_{\alpha,c} f(z) + (1-t)D^{\mu,b}_{\alpha,c} f(z)}{z}\right]^{\gamma} \neq 0.$$

If the function $l(z)$ (3.10) is univalent in U and

$$t + q(z) + z\frac{q'(z)}{q(z)} \prec l(z), \tag{4.6}$$

then

$$q(z) \prec \left[\frac{tD^{\mu+1,b}_{\alpha,c} f(z) + (1-t)D^{\mu,b}_{\alpha,c} f(z)}{z}\right]^{\gamma} \tag{4.7}$$

and q is the best subordinant.

Proof . Define the function l by

$$l(z)=\left[\frac{tD_{\alpha,c}^{\mu+1,b}f(z)+(1-t)D_{\alpha,c}^{\mu,b}f(z)}{z}\right]^{\gamma}. \tag{4.8}$$

Differentiating (4.8) with respect to z, we get

$$\frac{zl'(z)}{l(z)}=\gamma\left[\frac{tz(D_{\alpha,c}^{\mu+1,b}f(z))'+(1-t)z(D_{\alpha,c}^{\mu,b}f(z))'}{tD_{\alpha,c}^{\mu+1,b}f(z)+(1-t)D_{\alpha,c}^{\mu,b}f(z)}+1\right]. \tag{4.9}$$

By setting

$$\theta(w)=1+w \text{ and } \phi(w)=\frac{1}{w},\ w\neq 0,$$

we see that $theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w)\neq 0$, $w\in\mathbb{C}\backslash\{0\}$. Also, we get

$$Q(z)=zq'(z)\phi(q(z))=z\frac{q'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent in U,

$$Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\}=Re\{zq'(z)q(z)\}>0.$$

By making use of (4.9) the hypothesis (4.7) can equivalently written as

$$\theta(q(z))+aq'(z)\phi(q(z))\prec\theta(l(z))+al'(z)\phi(l(z)).$$

Thus, by applying Lemma 2.4, the proof is complete. □

5. Sandwich Results

Theorem 5.1. *Let q_1 be convex univalent function in U with $q_1(0)=1$, $\gamma>0$ and $Re\{\varepsilon\}>0$ and q_2 be univalent U, $q_2(0)=1$ and satisfies (3.2). Let $f\in T$ satisfies*

$$\left[\frac{D_{\alpha,c}^{\mu+1,b}f(z)}{z}\right]^{\gamma}\in H[1,1]\cap Q$$

And

$$\left[\frac{D_{\alpha,c}^{\mu+1,b}f(z)}{z}\right]^{\gamma}+\varepsilon(b+1)\left[\frac{D_{\alpha,c}^{\mu+1,b}f(z)}{z}\right]^{\gamma}(\frac{D_{\alpha,c}^{\mu,b}f(z)}{D_{\alpha,c}^{\mu+1,b}f(z)}-1)$$

be univalent in U. If

$$q_1(z)+\frac{\varepsilon}{\gamma}zq_1'(z)\prec\left[\frac{D_{\alpha,c}^{\mu+1,b}f(z)}{z}\right]^{\gamma}+\varepsilon(b+1)\left[\frac{D_{\alpha,c}^{\mu+1,b}f(z)}{z}\right]^{\gamma}(\frac{D_{\alpha,c}^{\mu,b}f(z)}{D_{\alpha,c}^{\mu+1,b}f(z)}-1)\prec q_2(z)+\frac{\varepsilon}{\gamma}zq_2'(z),$$

then

$$q_1(z) \prec \left[\frac{D_{\alpha,c}^{\mu+1,b}f(z)}{z}\right]^{\gamma} \prec q_2(z)$$

and q_1 *and* q_2 *are respectively the best subordinant and the best dominant.*

Theorem 5.2. *Let* q_1 *be convex univalent function in* U *with* $q_1(0) = q_2(0) = 1$. *Suppose* q_1 *satisfies* (4.6) *and* q_2 *satisfies* (3.9). *Let* $f \in A$ *satisfies*

$$\left[\frac{tD_{\alpha,c}^{\mu+1,b}f(z) + (1-t)D_{\alpha,c}^{\mu,b}f(z)}{z}\right]^{\gamma} \in H[1,1] \cap Q$$

$$\left[\frac{tD_{\alpha,c}^{\mu+1,b}f(z) + (1-t)D_{\alpha,c}^{\mu,b}f(z)}{z}\right]^{\gamma} \neq 0$$

$l(z)$ *is univalent in* U, *then*

$$t + q(z) + z\frac{q_1'(z)}{q_1(z)} \prec l(z) \prec t + q(z) + z\frac{q_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left[\frac{tD_{\alpha,c}^{\mu+1,b}f(z) + (1-t)D_{\alpha,c}^{\mu,b}f(z)}{z}\right]^{\gamma} \prec q_2(z)$$

and q_1 *and* q_2 *are respectively the best subordinant and the best dominant.*

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