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Fourth-order differential subordination and superordination results of meromorphic multivalent functions defined by multiplier transformation

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Abstract

In this paper, we obtain some applications of fourth-order differential subordination and superordination results involving multiplier transformation $H_p(\tau, \psi)$, for p-valent functions. Also, we obtain several sandwich-type results.

Keywords: Hadamard Product, The Multiplier Transformations, Differential Subordination, Differential Superordination, Fourth-Order.
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1. Introduction

Denote by C be a complex plane and $J = J(U)$ be the class of functions which are analytic in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$. For a positive integer number n and $a \in C$, we suppose that $J[a, n]$ be the subclass of J consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ..., \quad (z \in U),$$

and $J_1 = [1, 1]$. Let f and F be members of J. The function f is said to be subordinate to F, written $f \prec F$, or $f(z) \prec F(z)$, if there exists a Schwarz function $w(z)$, analytic in U with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$, $(z \in U)$.

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Furthermore, if the function F is univalent in U, then we have the following equivalence (see [16, 17]):

$$f(z) \prec F(z) \ (z \in U) \leftrightarrow f(0) = F(0) \text{ and } f(U) \subset F(U).$$

Assume that B_p denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{n=0}^{\infty} a_n z^n, \tag{1.1}$$

let $f \in B_p$ given by (1.1) and $g \in B_p$, defined by

$$g(z) = z^{-p} + \sum_{n=0}^{\infty} b_n z^n,$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is defined as follows:

$$(f * g)(z) = z^{-p} + \sum_{n=0}^{\infty} a_n b_n z^n = (g * f)(z).$$

For any $\tau \in N_0 = N \cup \{0\}$, Cho and Yoon introduced the multiplier transformations $H_p(\tau, \psi)$, of functions $f \in B_p$, by: (see [15])

$$H_p(\tau, \psi) f(z) = z^{-p} + \sum_{n=0}^{\infty} \left(\frac{n+p+\psi}{\psi} \right)^{\tau} a_n z^n, \quad (\psi > 0, \ z \in U).$$

Obviously, we have

$$H_p(v, \psi)(H_p(\tau, \psi) f(z)) = H_p(v + \tau, \psi) f(z),$$

for all nonnegative integers v and τ. The operators $H_1(\tau, \psi)$ and $H_1(\tau, 1)$ are the multiplier transformation introduced and studied by Sarangi and Uralegaddi [18], and Uralegaddi and Somanatha [19, 20], respectively. Now, we define the multiplier transformation $H_p(\tau, \psi)$ of functions $f, \ g \in B_p$, by

$$H_p(\tau, \psi)(f * g)(z) = z^{-p} + \sum_{n=2}^{\infty} \left(\frac{n+p+\psi}{\psi} \right)^{\tau} a_n b_n z^n, \quad (\psi > 0, \ z \in U). \tag{1.2}$$

It is easily verified from (1.2), that

$$z \left(H_p(\tau, \psi)(f * g)(z) \right)' = \psi (H_p(\tau + 1, \psi)(f * g)(z) - (\psi + p) H_p(\tau, \psi)(f * g)(z). \tag{1.3}$$

In recent years, there are many authors presented and dealing with the theory of second-order differential subordination and superordination for example ([1, 2, 3, 5, 7, 10, 11, 12, 13, 14, 16]). Also, many authors discussed the theory of the third-order differential subordination and superordination for example [4, 8]. In 2011, Antonino and Miller [4] presented basic concepts and extended the theory of the second-order differential subordination in the open unit disk introduced by Miller and Mocanu [16] to the third case. Atshan et al. [9, 6] extended the third-order case to fourth-order differential subordination and determined properties of functions g that satisfy the following fourth-order differential subordination:

$$\phi(g(z), zg'(z),\ z^2g''(z),\ z^3g'''(z), z^4g''''(z); z) \prec h(z),$$

where h be analytic univalent function in U, g is analytic function and $\phi : C^5 \times U \to C$. Now, we extended the third-order case to fourth-order differential superordination and determined properties of the function g that satisfy the following fourth-order differential superordination

$$h(z) \prec \phi(g(z), zg'(z),\ z^2g''(z),\ z^3g'''(z), z^4g''''(z); z),$$

where h be analytic univalent function in U, g is analytic function $\phi : C^5 \times U \to C$. To prove our main result, we need the basic concepts in the theory of fourth-order.

Definition 1.1. *[4] Let Q be the set of all analytic and univalent functions q on the set $\underline{U}\backslash E(q)$, where $E(q) = \{\zeta \in \partial U : q(z) = \infty\}$, such that, $|q'(z)| = \rho > 0$ for $\zeta \in \underline{U}\backslash E(q)$. Further, let the subclass of Q for which $q(0) = a$, be denoted by $Q(a)$ with $Q(0) = Q_0$ and $Q(1) = Q_1$, $Q_1 = \{q \in Q : q(0) = 1\}$.*

Definition 1.2. *[9, 6] Let $\phi : C^5 \times U \to C$ and suppose $h(z)$ be univalent function in U. If $g(z)$ is analytic function in U and satisfies the following fourth-order differential subordination:*

$$\phi(g(z), zg'(z),\ z^2g''(z),\ z^3g'''(z), z^4g''''(z); z) \prec h(z), \tag{1.4}$$

then $g(z)$ is called a solution of the differential subordination (1.4)*. A univalent function $q(z)$ is called a dominant of the solutions of* (1.4)*, or, more simply, a dominant $q(z)$ if $g(z) \prec q(z)$ for all $g(z)$ satisfying* (1.4)*. A dominant $\check{q}(z)$ which satisfies $\check{q} \prec q(z)$ for all dominants $q(z)$ of* (1.4) *is said to be the best dominant.*

Definition 1.3. *[6] Let $\phi : C^5 \times U \to C$ and suppose that $h(z)$ be analytic function in U. If $g(z)$ and*

$$\phi(g(z), zg'(z),\ z^2g''(z),\ z^3g'''(z), z^4g''''(z); z),$$

are univalent functions in U and satisfies the following fourth-order differential superordination

$$h(z) \prec \phi(g(z), zg'(z),\ z^2g''(z),\ z^3g'''(z), z^4g''''(z); z), \tag{1.5}$$

then $g(z)$ is called a solution of the differential superordination (1.5)*. An analytic function $q(z)$ is called a subordinant of the solution of* (1.5)*, or, more simply, a subordinant, if $q(z) \prec g(z)$ for all $g(z)$ satisfying* (1.5)*. A univalent subordinant $\check{q}(z)$ which satisfies $q(z) \prec \check{q}(z)$ for all subordinants of* (1.5) *is said to be the best subordinant. We note that the best subordinant is unique up to rotation of U.*

Definition 1.4. *[9, 6] Let Ω be a set in C, $q \in Q$ and $n \in N\backslash\{2\}$. The class $\Psi_n[\Omega, q]$ of admissible functions consists of those functions $\phi : C^5 \times U \to C$ that satisfy the following admissibility condition:*

$$\phi(u, v, x, y, g; \zeta) \notin \Omega,$$

whenever

$$u = q(\zeta), \quad v = k\zeta q'(\zeta), \quad Re\left\{\frac{x}{v} + 1\right\} \geqq kRe\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

and

$$Re\left\{\frac{y}{v}\right\} \geqq k^2 Re\left\{\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right\}, \quad Re\left\{\frac{g}{v}\right\} \geqq k^3 Re\left\{\frac{\zeta^3 q''''(\zeta)}{q'(\zeta)}\right\},$$

where $z \in U$, $\zeta \in \partial U \backslash E(q)$ *and* $k \geqq n$.

Definition 1.5. *[6] Let* Ω *be a set in* C, $q(z) \in J[a,n]$ *with* $q'(z) \neq 0$. *The class* $\Psi'_n[\Omega, q]$ *of admissible functions consists of those functions* $\phi : C^5 \times U \to C$ *that satisfy the following admissibility condition:*

$$\phi(u, v, x, y, g; \zeta) \notin \Omega,$$

whenever

$$u = q(z), \ v = \frac{zq'(z)}{m}, \ Re\left\{\frac{x}{v} + 1\right\} \geqq \frac{1}{m} Re\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

and

$$Re\left\{\frac{y}{v}\right\} \leqq \frac{1}{m^2} Re\left\{\frac{z^2 q'''(z)}{q'(z)}\right\}, \quad Re\left\{\frac{g}{v}\right\} \geqq \frac{1}{m^3} Re\left\{\frac{z^3 q''''(z)}{q'(z)}\right\},$$

where $z \in U$, $\zeta \in \partial U$ *and* $m \geqq n \geqq 3$.

The next lemma is the foundation result in theory of fourth-order differential subordination.

Lemma 1.6. *[9] Let* $g \in J[a,n]$ *with* $n \in N \backslash \{2\}$, *and* $q \in Q(a)$ *and satisfy the following conditions:*

$$Re\left\{\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right\} \geqq 0, \ \text{and} \ \left|\frac{\zeta^2 p''(z)}{q'(\zeta)}\right| \leqq k^2,$$

where $z \in U$, $\zeta \in \partial U \backslash E(q)$ *and* $k \geqq n$. *If* Ω *is a set in* C, $\phi \in \Psi_n[\Omega, q]$ *and*

$$\phi(g(z), zg'(z), \ z^2 g''(z), \ z^3 g'''(z), z^4 g''''(z); z) \in \Omega$$

then

$$g(z) \prec q(z), \quad (z \in U)$$

The next lemma is the foundation result in the theory of fourth-order differential superordination.

Lemma 1.7. *[6] Let* $g(z) \in J[a,n]$ *with* $\phi \in \Psi'_n[\Omega, q]$. *if*

$$\phi(g(z), zg'(z), \ z^2 g''(z), \ z^3 g'''(z), z^4 g''''(z); z),$$

is univalent in U *and* $g(z) \in Q(a)$ *satisfy the following admissibility conditions:*

$$Re\left\{\frac{z^2 q'''(z)}{q'(\zeta)}\right\} \geqq 0, \ \text{and} \ \left|\frac{z^2 p''(z)}{q'(\zeta)}\right| \leqq \frac{1}{m^2},$$

where $z \in U$, $\zeta \in \partial U$ *and* $m \geqq n \geqq 3$, *then*

$$\Omega \subset \{\phi(g(z), zg'(z), \ z^2 g''(z), \ z^3 g'''(z), z^4 g''''(z); z) : z \in U\},$$

implies that

$$q(z) \prec g(z), \quad (z \in U)$$

In this paper, we present some results for differential subordination and superordination for analytic functions defined by a multiplier transformation $H_p(\tau, \psi)(f * g)(z)$.

2. Fourth-Order Differential Subordination Results Using $H_p(\tau, \psi)(f * g)(z)$

First, we define the following class of admissible functions, which are needed in proving the differential theorems associated with $H_p(\tau, \psi)(f * g)(z)$ defined by (1.2).

Definition 2.1. *Let Ω be a set in C, and $q \in Q_0 \cap J_0$. The class $A_j[\Omega, q]$ of admissible functions consists of those functions $\phi : C^5 \times U \to C$, that satisfy the following admissibility condition:*

$$\phi(u, v, x, y, g; z) \notin \Omega,$$

whenever

$$u = q(\zeta),\ v = \frac{k\zeta q'(\zeta) + (\psi + p)q(\zeta)}{\psi},$$

$$Re\left(\frac{\psi(\psi+1)(x+v) - (2(\psi+p)+1)(\psi+p-1)(u-v) + (\psi+p)u}{\psi v + (\psi+p)u}\right) \geqq kRe\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

$$Re(\frac{\psi^3 y - (3(\psi+p)+1)\psi^2 x - (\psi+p)^2(\psi(\psi+p)+1)u}{\psi v + (\psi+p)u} - [(\psi+p)((3(\psi+p)+5)+2)]) \geqq k^2 Re\left\{\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right\}$$

and

$$Re(\frac{\psi^4 g - 4\psi^3(\psi+p+1)y + \psi^2[6(\psi+p)^2 - 3(\psi+p) - 2]x + [(\psi+p)(8(\psi+p)+9)+2](u-v) + (\psi+p)^2[(\psi+p)^2 + (\psi(\psi+p)+1)^2]u}{\psi v + (\psi+p)u} + \psi(\psi+p)[(\psi+p+1)((2(\psi+p)+5)+1)]) \geqq k^3 Re\left\{\frac{\zeta^3 q''''(\zeta)}{q'(\zeta)}\right\}$$

where $z \in U,\ \zeta \in \partial U \backslash E(q)$ and $k \geqq n$.

Theorem 2.2. *Let $\phi \in A_j[\Omega, q]$. If $f,\ g \in B_p$ and $q \in Q_0$ and satisfy the following conditions*

$$Re\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right) \geqq 0, \quad \left|\frac{H_p(\tau+2, \psi)(f * g)(z)}{q'(\zeta)}\right| \leqq k^2, \tag{2.1}$$

and

$$\{\phi(H_p(\tau,\psi)(f*g)(z), H_p(\tau+1,\psi)(f*g)(z), H_p(\tau+2,\psi)(f*g)(z), H_p(\tau+3,\psi)(f*g)(z), H_p(\tau+4,\psi)(f*g)(z))\} \subset \Omega, \tag{2.2}$$

then

$$H_p(\tau, \psi)(f * g)(z) \prec q(z) \quad (z \in U).$$

Proof . Define the analytic function $p(z)$ in U by

$$p(z) = H_p(\tau, \psi)(f * g)(z) \quad (z \in U). \tag{2.3}$$

Then, differentiating (2.3) with respect to z and using (1.3), we have

$$H_p(\tau + 1, \psi)(f * g)(z) = \frac{zp'(z) + (\psi + p)p(z)}{\psi}. \tag{2.4}$$

Further computations show that

$$H_p(\tau + 2, \psi)(f * g)(z) = \frac{z^2 p''(z) + (2(\psi + p) + 1)zp'(z) + (\psi + p)^2 p(z)}{\psi^2} \tag{2.5}$$

$$H_p(\tau+3, \psi)(f*g)(z) = \frac{z^3 p^3(z) + (3(\psi + p) + 1)z^2 p''(z) + [3(\psi + p)^2 + 3(\psi + p) + 1]zp'(z) + (\psi + p)^3 p(z)}{\psi^3} \tag{2.6}$$

and

$$H_p(\tau+4,\psi)(f*g)(z)=\frac{z^4p^4(z)+4(\psi+p+1)z^3p^3(z)+[6(\psi+p)^2+10(\psi+p)+3]z^2p''(z)+[4((\psi+p)^3+(\psi+p)^2+(\psi+p))+1]zp'(z)+(\psi+p)^4p(z)}{\psi^4} \tag{2.7}$$

We now define the transformation C^5 to C,

$$\begin{aligned}
u(r, s, t, w, b) &= r,\\
v(r, s, t, w, b) &= \frac{s + (\psi + p)r}{\psi},\\
x(r, s, t, w, b) &= \frac{t + (2(\psi + p) + 1)s + (\psi + p)^2 r}{\psi^2},\\
y(r, s, t, w, b) &= \frac{w + (3(\psi + p) + 1)t + [3(\psi + p)^2 + 3(\psi + p) + 1]s + (\psi + p)^3 r}{\psi^3}
\end{aligned} \tag{2.8}$$

and

$$g(r, s, t, w, b) = \frac{b + 4(\psi + p + 1)w + [6(\psi + p)^2 + 10(\psi + p) + 3]t + [4((\psi + p)^3 + (\psi + p)^2 + (\psi + p)) + 1]s + (\psi + p)^4 r}{\psi^4}, \tag{2.9}$$

let

$$\chi(r, s, t, w, b; z) = \phi(u, v, x, y, g; z) = \phi\left(r, \frac{s + (\psi + p)r}{\psi}, \frac{t + (2(\psi + p) + 1)s + (\psi + p)^2 r}{\psi^2}, \frac{w + (3(\psi + p) + 1)t + [3(\psi + p)^2 + 3(\psi + p) + 1]s + (\psi + p)^3 r}{\psi^3}, \frac{b + 4(\psi + p + 1)w + [6(\psi + p)^2 + 10(\psi + p) + 3]t + [4((\psi + p)^3 + (\psi + p)^2 + (\psi + p)) + 1]s + (\psi + p)^4 r}{\psi^4}\right), \tag{2.10}$$

The proof will make use Lemma 1.6. Using the equations (2.4) to (2.7), we have from (2.8) that

$$\begin{aligned}&\chi(p(z), zp'(z), z^2p''(z), z^3p^3(z), z^4p^4(z); z) = \phi\left(H_p(\tau,\psi)(f*g)(z), H_p(\tau+1,\psi)(f*g)(z),\right.\\&\left.H_p(\tau+2,\psi)(f*g)(z), H_p(\tau+3,\psi)(f*g)(z), H_p(\tau+4,\psi)(f*g)(z)\right)\end{aligned} \tag{2.11}$$

Hence, clearly (2.2) becomes

$$\chi(p(z), zp'(z), z^2p''(z), z^3p^3(z), z^4p^4(z); z) \in \Omega$$

We note that

$$\frac{t}{s}+1 = \frac{\psi(\psi+1)(x+v) - (2(\psi+p)+1)(\psi+p-1)(u-v) + (\psi+p)u}{\psi v + (\psi+p)u},$$

$$\frac{w}{s} = \frac{\psi^3 y - (3(\psi+p)+1)\psi^2 x - (\psi+p)^2(\psi(\psi+p)+1)u}{\psi v + (\psi+p)u} - [(\psi+p)((3(\psi+p)+5)+2)],$$

and

$$\frac{b}{s} = \frac{\psi^4 g - 4\psi^3(\psi+p+1)y + \psi^2[6(\psi+p)^2 - 3(\psi+p) - 2]x + [(\psi+p)(8(\psi+p)+9)+2](u-v) + (\psi+p)^2 + (\psi(\psi+p)+1)^2]u}{\psi v + (\psi+p)u} + \psi(\psi+p)[(\psi+p+1)((2(\psi+p)+5)+1)].$$

Therefore, the admissibility condition for $\phi \in A_j[\Omega, q]$, in Definition 2.1 is equivalent to admissibility condition for $\chi \in \Psi_3[\Omega, q]$ as given in Definition 1.4 with $n = 3$. Therefore, by using 2.1 and Lemma 1.6, we obtain

$$p(z) = H_p(\tau,\psi)(f*g)(z) \prec q(z).$$

This completes the proof of Theorem 2.2. □

Our next corollary, is an extension of Theorem 2.2 to the case when the behavior of $q(z) \in \partial U$, is not known.

Corollary 2.3. *Let $\Omega \subset C$, and let the function $q(z)$ be univalent in U with $q(0) = 1$. Let $\phi \in A_j[\Omega, q]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If $f,\ g \in B_p$ and q_ρ satisfies the following conditions:*

$$Re\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right) \geqq 0, \quad \left|\frac{H_p(\tau+2,\psi)(f*g)(z).}{q'(\zeta)}\right| \leqq k^2, \ (z \in U,\ \zeta \in \partial U \backslash E(q_\rho) \text{ and } k \geqq n) \tag{2.12}$$

and

$$\begin{aligned}&\phi\left(H_p(\tau,\psi)(f*g)(z), H_p(\tau+1,\psi)(f*g)(z), H_p(\tau+2,\psi)(f*g)(z),\right.\\&\left.H_p(\tau+3,\psi)(f*g)(z), H_p(\tau+4,\psi)(f*g)(z)\right) \prec h(z),\end{aligned} \tag{2.13}$$

then

$$H_p(\tau,\psi)(f*g)(z) \prec q(z) \quad (z \in U).$$

and $q(z) = \left(\frac{1+z}{1-z}\right)$ is the best dominant.

Proof . By using Theorem 2.2, yield

$$H_p(\tau,\psi)(f*g)(z) \prec q_\rho(z) \quad (z \in U).$$

Then we obtain the result from

$$q_\rho(z) \prec q(z) \quad (z \in U).$$

This completes the proof of Corollary 2.3. □

If $\Omega \neq C$ is simply-connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U on to Ω. In this case, the class $A_j[h(U),q]$, is written an$A_j[h,q]$. The following two results are immediate consequence of Theorem 2.2 and Corollary 2.3.

Theorem 2.4. *Let $\phi \in A_j[h,q]$. If $f,\ g \in B_p$ and $q \in Q_0$ satisfy the following conditions* (2.1)*, and*

$$\begin{aligned}&\phi\left(H_p(\tau,\psi)(f*g)(z), H_p(\tau+1,\psi)(f*g)(z), H_p(\tau+2,\psi)(f*g)(z),\right.\\ &\left.H_p(\tau+3,\psi)(f*g)(z), H_p(\tau+4,\psi)(f*g)(z)\right) \prec h(z),\end{aligned} \tag{2.14}$$

then

$$H_p(\tau,\psi)(f*g)(z) \prec q(z) \quad (z \in U).$$

Corollary 2.5. *Let $\Omega \subset C$, and let the function q be univalent in U with $q(0)=1$. Let $\phi \in A_j[\Omega,q]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If $f,\ g \in B_p$ and q_ρ satisfies the conditions* (2.12) *and*

$$\begin{aligned}&\phi\left(H_p(\tau,\psi)(f*g)(z), H_p(\tau+1,\psi)(f*g)(z), H_p(\tau+2,\psi)(f*g)(z),\right.\\ &\left.H_p(\tau+3,\psi)(f*g)(z), H_p(\tau+4,\psi)(f*g)(z)\right) \prec h(z),\end{aligned} \tag{2.15}$$

then

$$H_p(\tau,\psi)(f*g)(z) \prec q(z) \quad (z \in U).$$

The following result yield the best dominant of differential subordination (2.12).

Theorem 2.6. *Let the function h be univalent in U. Also let*

$$\begin{aligned}&\phi\Bigg(q(z), \frac{zq'(z)+(\psi+p)q(z)}{\psi}, \frac{z^2q''(z)+(2(\psi+p)+1)zq'(z)+(\psi+p)^2q(z)}{\psi^2}, \frac{z^3q^3(z)+(3(\psi+p)+1)z^2q''(z)+[3(\psi+p)^2+3(\psi+p)+1]zq'(z)+(\psi+p)^3q(z)}{\psi^3},\\ &\frac{z^4q^4(z)+4(\psi+p+1)z^3q^3(z)+[6(\psi+p)^2+10(\psi+p)+3]z^2q''(z)+[4((\psi+p)^3+(\psi+p)^2+(\psi+p))+1]zq'(z)+(\psi+p)^4q(z)}{\psi^4};z\Bigg)=h(z),\end{aligned} \tag{2.16}$$

has a solution $q(z)$ with $q(0)=1$, which satisfies the condition (2.1)*. If $f,\ g \in B_p$, satisfies the condition* (2.12) *and if*

$$\begin{aligned}&\phi\left(H_p(\tau,\psi)(f*g)(z), H_p(\tau+1,\psi)(f*g)(z), H_p(\tau+2,\psi)(f*g)(z),\right.\\ &\left.H_p(\tau+3,\psi)(f*g)(z), H_p(\tau+4,\psi)(f*g)(z)\right) \prec h(z), \text{ is analytic in } U, \text{ then}\end{aligned}$$

$$H_p(\tau,\psi)(f*g)(z) \prec q(z) \quad (z \in U).$$

and $q(z)$ is the best dominant.

Proof . Using Theorem 2.2, that $q(z)$ is a dominant of (2.12). Since $q(z)$ satisfies (2.15), it is also a solution of (2.12). Therefore, $q(z)$ will be dominated by all dominant. Hence $q(z)$ is the best dominant. $\square$

Definition 2.7. *Let Ω a set in C and $q \in Q_1 \cap J_1$. The class $A_{j,1}[\Omega, q]$ of admissible functions consist of those functions $\phi : C^5 \times U \to C$, that satisfy the following admissibility conditions:*

$$\phi(u, v, x, y, g; z) \notin \Omega,$$

whenever

$$\begin{aligned} &u = q(\zeta),\\ &v = \frac{k\zeta q'(\zeta) + \psi q(\zeta)}{\psi},\\ &Re\left(\frac{\psi x + (\psi + 2p)u}{(v-u)} - \psi\right) \geqq kRe\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},\\ &Re\left(\frac{\psi^2 y - \psi(2\psi+3)(\psi+2p+1)(x+u) + \psi(\psi+2p)u}{(v-u)} + (2\psi p + 3p)\right) \geqq k^2 Re\left\{\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right\} \end{aligned}$$

and

$$Re\left(\frac{\psi^4 g - 2\psi^2(3\psi+7)(\psi-1)(y+x) + (\psi+2p)(3(3\psi+7)(\psi+1)+\psi^2)u(\psi^2+2)(4\psi^2+2\psi p+9\psi+7)(x-u)}{(v-u)} + [\psi^5+\psi^4-7\psi^3+8\psi^2 p+2\psi^2+14\psi p-3\psi-8]\right)$$
$$\geqq k^3 Re\left(\frac{\zeta^3 q''''(\zeta)}{q'(\zeta)}\right),$$

where $z \in U$, $\zeta \in \partial U \backslash E(q)$ and $k \geqq n$.

Theorem 2.8. *Define $\phi \in A_{j,1}[\Omega, q]$. If $f,\ g \in B_p$ and $q \in Q_1$, and satisfy the following conditions*

$$Re\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right) \geqq 0, \quad \left|\frac{z^p H_p(\tau+2, \psi)(f * g)(z).}{q'(\zeta)}\right| \leqq k^2, \tag{2.17}$$

and

$$\begin{aligned} &\{\phi\left(z^p H_p(\tau, \psi)(f * g)(z), z^p H_p(\tau+1, \psi)(f * g)(z), z^p H_p(\tau+2, \psi)(f * g)(z),\right.\\ &\left. z^p H_p(\tau+3, \psi)(f * g)(z), z^p H_p(\tau+4, \psi)(f * g)(z)\right)\} \subset \Omega, \end{aligned} \tag{2.18}$$

then

$$z^p H_p(\tau, \psi)(f * g)(z) \prec q(z) \quad (z \in U).$$

Proof . Let $p(z)$ be analytic function in U, defined by

$$p(z) = z^p H_p(\tau, \psi)(f * g)(z). \tag{2.19}$$

From equations (1.3) and (2.19), we have

$$z^p H_p(\tau+1,\psi)(f*g)(z) = \frac{zp'(z)+\psi p(z)}{\psi}. \tag{2.20}$$

By a similar argument, we get

$$z^p H_p(\tau+2,\psi)(f*g)(z) = \frac{z^2p''(z)+(\psi+1)zp'(z)-(\psi(\psi+2p))p(z)}{\psi^2}, \tag{2.21}$$

$$z^p H_p(\tau+3,\psi)(f*g)(z) = \frac{z^3p^3(z)+(2\psi+3)z^2p''(z)+(2\psi^2+2\psi p+2\psi+1)p'(z)+\psi^2(\psi+2p))p(z)}{\psi^3}, \tag{2.22}$$

and

$$z^p H_p(\tau+4,\psi)(f*g)(z)=\frac{z^4p^4(z)+(3\psi+7)z^3p^3(z)+(4\psi^4+2\psi p+9\psi+7)z^2p''(z)+(\psi^3+4\psi^2+2\psi p+3\psi+1)zp'(z)-\psi^3(\psi+2p)p(z)}{\psi^4}. \tag{2.23}$$

Define the transformation from C^5 to C by

$$\begin{aligned}
u(r,s,t,w,b) &= r,\\
v(r,s,t,w,b) &= \frac{s+\psi r}{\psi},\\
x(r,s,t,w,b) &= \frac{t+(\psi+1)s-(\psi(\psi+2p))r}{\psi^2},\\
y(r,s,t,w,b) &= \frac{w+(2\psi+3)t+(2\psi^2+2\psi p+2\psi+1)s-\psi^2(\psi+2p)r}{\psi^3},
\end{aligned} \tag{2.24}$$

and

$$\begin{aligned}
&g(r,s,t,w,b)\\
&=\frac{b+(3\psi+7)w+(4\psi^4+2\psi p+9\psi+7)t+(\psi^3+4\psi^2+2\psi p+3\psi+1)s-\psi^3(\psi+2p)r}{\psi^4}
\end{aligned} \tag{2.25}$$

Let

$$\phi\left(r,\frac{s+\psi r}{\psi},\frac{t+(\psi+1)s-(\psi(\psi+2p))r}{\psi^2},\frac{w+(2\psi+3)t+(2\psi^2+2\psi p+2\psi+1)s-\psi^2(\psi+2p)r}{\psi^3}\right.$$
$$\left.\frac{b+(3\psi+7)w+(4\psi^4+2\psi p+9\psi+7)t+(\psi^3+4\psi^2+2\psi p+3\psi+1)s-\psi^3(\psi+2p)r}{\psi^4};z\right). \tag{2.26}$$

The proof will make use of Lemma 1.6. Using the equations (2.19) to (2.23) we have from (2.26), that

$$\begin{aligned}
&\chi(p(z)+zp'(z),z^2p''(z),z^3p^3(z),z^4p^4(z);z)=\phi\left(z^pH_p(\tau,\psi)(f*g)(z),z^pH_p(\tau+1,\psi)(f*g)(z),\right.\\
&\left.z^pH_p(\tau+2,\psi)(f*g)(z),z^pH_p(\tau+3,\psi)(f*g)(z),z^pH_p(\tau+4,\psi)(f*g)(z)\right)
\end{aligned} \tag{2.27}$$

Hence (2.18) becomes

$$\chi(p(z) + zp'(z), z^2p''(z), z^3p^3(z), z^4p^4(z); z) \in \Omega$$

We note that

$$\frac{t}{s} + 1 = \frac{\psi x + (\psi + 2p)u}{(v - u)} - \psi,$$

$$\frac{w}{s} = \frac{\psi^2 y - \psi(2\psi + 3)(\psi + 2p + 1)(x + u) + \psi(\psi + 2p)u}{(v - u)} + (2\psi p + 3p),$$

and

$$\frac{b}{s} = \frac{\psi^4 g - 2\psi^2(3\psi+7)(\psi-1)(y+x) + (\psi+2p)(3(3\psi+7)(\psi+1)+\psi^2)u(\psi^2+2)(4\psi^2+2\psi p+9\psi+7)(x-u)}{(v-u)} + [\psi^5+\psi^4-7\psi^3+8\psi^2p+2\psi^2+14\psi p-3\psi-8].$$

Thus clearly, the admissibility condition for $\phi \in A_{j,1}[\Omega, q]$, in Definition 2.7 is equivalent to admissibility condition for $\chi \in \Psi_3[\Omega, q]$ as given in Definition 1.4 with $n = 3$. Therefore, by using (2.17) and Lemma 1.6, we obtain

$$p(z) = z^p H_p(\tau, \psi)(f * g)(z) \prec q(z).$$

This completes the proof of Theorem 2.8. □

In the our corollary we obtain an extension of Theorem 2.8, to the case when the behavior of $q(z)$ on ∂U is not known.

Corollary 2.9. *Let $\Omega \subset C$ and let the function $q(z)$ be univalent in U with $q(0) = 1$. Let $\phi \in A_{j,1}[\Omega, q]$ for some $\rho(0, 1)$, where $q_\rho(z) = q(\rho z)$. If $f,\ g \in B_p$ and q_ρ, satisfies the following conditions:*

$$Re\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right) \geqq 0,\ \left|\frac{H_p(\tau + 2, \psi)(f * g)(z).}{q'(\zeta)}\right| \leqq k^2,\ (z \in U, \zeta \in \partial U \backslash E(q_\rho)\ \textit{and}\ k \geqq n) \tag{2.28}$$

and

$$\begin{aligned}&\phi\left(H_p(\tau, \psi)(f * g)(z), H_p(\tau + 1, \psi)(f * g)(z), H_p(\tau + 2, \psi)(f * g)(z), H_p(\tau + 3, \psi)(f * g)(z),\right.\\ &\left.H_p(\tau + 4, \psi)(f * g)(z)\right) \prec h(z),\end{aligned} \tag{2.29}$$

then

$$z^p H_p(\tau, \psi)(f * g)(z) \prec q(z) \quad (z \in U).$$

Proof . By using Theorem 2.8, yield

$$z^p H_p(\tau, \psi)(f * g)(z) \prec q_\rho(z) \quad (z \in U).$$

Then we obtain the result from

$$q_\rho(z) \prec q(z) \quad (z \in U).$$

This completes the proof of Corollary 2.9. □

If $\Omega \neq C$ is simply-connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U on to Ω. In this case, the class $A_{j,1}[h(U), q]$, is written an $A_{j,1}[h, q]$. The following two results are immediate consequence of Theorem 2.8 and Corollary 2.9.

Theorem 2.10. *Define $\phi \in A_{j,1}[\Omega, q]$. If $f,\ g \in B_p$ and $q \in Q_1$, satisfy the conditions (2.17) and*

$$\begin{aligned}&\phi\left(H_p(\tau,\psi)(f*g)(z), H_p(\tau+1,\psi)(f*g)(z), H_p(\tau+2,\psi)(f*g)(z), H_p(\tau+3,\psi)(f*g)(z),\right.\\&\left.H_p(\tau+4,\psi)(f*g)(z)\right) \prec h(z),\end{aligned} \tag{2.30}$$

then

$$z^p H_p(\tau,\psi)(f*g)(z) \prec q(z) \quad (z \in U).$$

Corollary 2.11. *Let $\Omega \subset C$ and let the function $q(z)$ be univalent in U with $q(0) = 1$. Let $\phi \in A_{j,1}[\Omega, q]$ for some $\rho(0,1)$, where $q_\rho(z) = q(\rho z)$. If $f,\ g \in B_p$ and q_ρ, satisfies the conditions (2.17)*

$$Re\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right) \geqq 0, \ \left|\frac{H_p(\tau+2,\psi)(f*g)(z).}{q'(\zeta)}\right| \leqq k^2, \ (z \in U, \zeta \in \partial U \backslash E(q_\rho) \text{ and } k \geqq n) \tag{2.31}$$

then

$$z^p H_p(\tau,\psi)(f*g)(z) \prec q(z) \quad (z \in U).$$

In the following result yield the best dominant of differential subordination (2.29).

Theorem 2.12. *Let the function h be univalent in U. Also let $\phi : C^5 \times U \to C$, and suppose that the following differential equation*

$$\begin{aligned}&\phi\Bigg(q(z), \frac{zq'(z)+\psi q(z)}{\psi}, \frac{z^2q''+(\psi+1)zq'(z)-(\psi(\psi+2p))q(z)}{\psi^2}, \frac{z^3q^3(z)+(2\psi+3)z^2q''(z)+(2\psi^2+2\psi p+2\psi+1)q'(z)-\psi^2(\psi+2p)q(z)}{\psi^3},\\&\frac{z^4q^4(z)+(3\psi+7)z^3q^3(z)+(4\psi^4+2\psi p+9\psi+7)z^2q''+(\psi^3+4\psi^2+2\psi p+3\psi+1)zq'(z)-\psi^3(\psi+2p)q(z)}{\psi^4};z\Bigg)=h(z)\end{aligned}$$

has a solution $q(z)$ with $q(0) = 1$, which satisfies the condition (2.28). If $f,\ g \in B_p$, then

$$z^p H_p(\tau,\psi)(f*g)(z) \prec q(z) \quad (z \in U).$$

Proof . From Theorem 2.8, then $z^p H_p(\tau,\psi)(f*g)(z) \prec q_\rho(z) \quad (z \in U)$. The result asserted by Corollary 2.3, is now deduced from the following subordination property $q_\rho(z) \prec q(z) \quad (z \in U)$. □

Also, here the fourth-order differential superordination thermos for the multiplier transformation $H_p(\tau,\psi)(f*g)(z)$ defined in 1.2 is investigated. For the purpose, we considered the following admissible functions.

3. Fourth-Order Differential Superordination Results Using $H_p(\tau,\psi)(f*g)(z)$

Definition 3.1. *Let Ω be a set in C, and $q \in Q_0$ with $q'(z) \neq 0$. The class $A_j[\Omega, q]$ of admissible functions consists of those functions $\phi : C^5 \times \underline{U} \to C$, which satisfy the following admissibility condition*

$$\phi(u, v, x, y, g; \zeta) \in \Omega,$$

whenever

$$\begin{aligned}
&u = q(z),\\
&v = \frac{k\zeta q'(z) + (\psi + p)q(z)}{\psi},\\
&Re\left(\frac{\psi(\psi+1)(x+v) - (2(\psi+p)+1)(\psi+p-1)(u-v) + (\psi+p)u}{\psi v + (\psi+p)u}\right) \leqq \frac{1}{m} Re\left\{\frac{zq''(z)}{q'(z)} + 1\right\},\\
&Re\left(\frac{\psi^3 - (3(\psi+p)+1)\psi^2 x - (\psi+p)^2(\psi(\psi+p)+1)u}{\psi v + (\psi+p)u} - [(\psi+p)((3(\psi+p)+5)+2)]\right)\\
&\leqq \frac{1}{m^2} Re\left\{\frac{\zeta^2 q'''(z)}{q'(z)}\right\},
\end{aligned}$$

and

$$Re\left(\frac{\psi^4 g - 4\psi^3(\psi+p+1)y + \psi^2[6(\psi+p)^2 - 3(\psi+p) - 2]x + [(\psi+p)(8(\psi+p)+9)+2](u-v) + (\psi+p)^2[(\psi+p)^2 + (\psi(\psi+p)+1)^2]u}{\psi v + (\psi+p)u} + \psi(\psi+p)[(\psi+p+1)((2(\psi+p)+5)+1)]\right)$$
$$\leqq \frac{1}{m^3} Re\left(\frac{\zeta^3 q''''(\zeta)}{q'(\zeta)}\right),$$

where $z \in U,\ \zeta \in \partial U \backslash E(q)$ *and* $k \geqq n$.

Theorem 3.2. *Let* $\phi \in A_j[\Omega, q]$. *If* $f,\ g \in B_p$ *and* $H_p(\tau,\psi)(f*g)(z) \in Q_0$ *with* $q'(z) \neq 0$, *satisfying the following conditions:*

$$Re\left(\frac{z^2 q'''(z)}{q'(z)}\right) \geqq 0,\ \left|\frac{H_p(\tau+2,\psi)(f*g)(z).}{q'(z)}\right| \leqq \frac{1}{m^2}, \tag{3.1}$$

and the function

$$\phi\left(H_p(\tau,\psi)(f*g)(z), H_p(\tau+1,\psi)(f*g)(z), H_p(\tau+2,\psi)(f*g)(z), H_p(\tau+3,\psi)(f*g)(z),\right.$$
$$\left. H_p(\tau+4,\psi)(f*g)(z); z\right),\ \text{ is univalent in } U$$

and

$$\Omega \subset \{\phi\left(H_p(\tau,\psi)(f*g)(z), H_p(\tau+1,\psi)(f*g)(z), H_p(\tau+2,\psi)(f*g)(z), H_p(\tau+3,\psi)(f*g)(z),\right.$$
$$\left. H_p(\tau+4,\psi)(f*g)(z); z\right) : z \in U\} \tag{3.2}$$

implies that

$$q(z) \prec H_p(\tau,\psi)(f*g)(z) \quad (z \in U).$$

Proof . Let the function $p(z)$ be defined by (2.3) and χ by (2.10), since $\phi \in A_j[\Omega, q]$. Thus from (2.11) and (3.2) yield

$$\Omega \subset \left\{\chi(p(z) + zp'(z), z^2p''(z), z^3p^3(z), z^4p^4(z); z) : z \in U\right\}.$$

From (2.8) and (2.9), we see that the admissibility for $\phi \in A_j[\Omega, q]$, in Definition 2.1 is equivalent to the admissibility condition for χ as given in Definition 1.5, with $n = 3$. Hence $\chi \in \Psi_3[\Omega, q]$, and by using Lemma 1.7 and (3.2), we have

$$q(z) \prec H_p(\tau, \psi)(f * g)(z).$$

If $\Omega \neq C$ is simply-connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U on to Ω. In this case, the class $A_j[h(U), q]$, is written an $A_j[h, q]$. □

The next theorem is directly consequence of Theorem 3.2.

Theorem 3.3. *Let $\phi \in A_j[\Omega, q]$ and h be analytic function in U. If $f,\ g \in B_p$, $H_p(\tau, \psi)(f * g)(z) \in Q_0$ and $q \in J_0$ with $q'(z) \neq 0$, satisfying the following conditions* (3.1) *and satisfying the following conditions:*

$$\phi\left(H_p(\tau, \psi)(f * g)(z), H_p(\tau+1, \psi)(f * g)(z), H_p(\tau+2, \psi)(f * g)(z), H_p(\tau+3, \psi)(f * g)(z),\right.$$
$$\left.H_p(\tau+4, \psi)(f * g)(z); z\right), \text{ is univalent in } U, \text{ then}$$

$$h(z) \prec \phi\left(H_p(\tau, \psi)(f * g)(z), H_p(\tau+1, \psi)(f * g)(z), H_p(\tau+2, \psi)(f * g)(z), H_p(\tau+3, \psi)(f * g)(z),\right.$$
$$\left.H_p(\tau+4, \psi)(f * g)(z); z\right), \tag{3.3}$$

implies that

$$q(z) \prec H_p(\tau, \psi)(f * g)(z) \quad (z \in U).$$

Theorem 3.4. *Let h be analytic function in U, and let $\phi : C^5 \times \underline{U} \to C$ and χ be given by* (2.10). *Suppose that the following differential equation:*

$$\chi(p(z) + zp'(z), z^2p''(z), z^3p^3(z), z^4p^4(z); z) = h(z), \tag{3.4}$$

has a solution $q(z) \in Q_0$. If $f,\ g \in B_p$, $H_p(\tau, \psi)(f * g)(z) \in Q_0$ and $q \in J_0$ with $q'(z) \neq 0$, satisfying the following conditions (3.1) *and*

$$\phi\left(H_p(\tau, \psi)(f * g)(z), H_p(\tau+1, \psi)(f * g)(z), H_p(\tau+2, \psi)(f * g)(z), H_p(\tau+3, \psi)(f * g)(z),\right.$$
$$\left.H_p(\tau+4, \psi)(f * g)(z); z\right), \text{ is univalent in } U, \text{ then}$$

$$h(z) \prec \phi\left(H_p(\tau, \psi)(f * g)(z), H_p(\tau+1, \psi)(f * g)(z), H_p(\tau+2, \psi)(f * g)(z), H_p(\tau+3, \psi)(f * g)(z),\right.$$
$$\left.H_p(\tau+4, \psi)(f * g)(z); z\right),$$

implies that

$$q(z) \prec H_p(\tau, \psi)(f * g)(z) \quad (z \in U).$$

and $q(z)$ is the best subordinant of (3.3).

Proof . In view of Theorems 3.2 and 3.3, we note that q is a subordination of (3.3). Since q satisfies (3.4), that also a solution of (3.3) and therefore q will be subordinated by all subordinants. Hence q is the best subordinant. □

Definition 3.5. *Let Ω be a set in C, and $q \in Q_1$ with $q'(z) \neq 0$. The class $A_{j,1}[\Omega, q]$ of admissible functions consists of those functions $\phi : C^5 \times \underline{U} \to C$, which satisfy the following admissibility condition*

$$\phi(u, v, x, y, g; \zeta) \in \Omega,$$

whenever

$$u = q(z),$$
$$v = \frac{k\zeta q'(z) + \psi q(z)}{\psi},$$
$$Re\left(\frac{\psi x + (\psi + 2p)u}{(u - v)} - \psi\right) \geqq kRe\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$
$$Re\left(\frac{\psi^2 y - \psi(2\psi + 3)(\psi + 2p + 1)(x + u) + \psi(\psi + 2p)u}{(v - u)} + (2\psi p + 3p)\right) \geqq k^2 Re\left\{\frac{z^2 q'''(z)}{q'(z)}\right\}$$

and

$$Re\left(\frac{\psi^4 g - 2\psi^2(3\psi+7)(\psi-1)(y+x) + (\psi+2p)(3(3\psi+7)(\psi+1)+\psi^2)u + (\psi^2+2)(4\psi^2+2\psi p+9\psi+7)(x-u)}{(v-u)} + [\psi^5+\psi^4-7\psi^3+8\psi^2 p+2\psi^2+14\psi p-3\psi-8]\right)$$
$$\geqq k^3 Re\left(\frac{z^3 q''''(z)}{q'(z)}\right),$$

where $z \in U$, $\zeta \in \partial U \backslash E(q)$ and $k \geqq n$.

Theorem 3.6. *Let $\phi \in A_{j,1}[\Omega, q]$. If f, $g \in B_p$ and $z^p H_p(\tau, \psi)(f * g)(z) \in Q_1$ with $q'(z) \neq 0$, satisfying the following conditions:*

$$Re\left(\frac{z^2 q'''(z)}{q'(z)}\right) \geqq 0, \quad \left|\frac{z^p H_p(\tau + 2, \psi)(f * g)(z).}{q'(z)}\right| \leqq \frac{1}{m^2}, \tag{3.5}$$

and the function

$$\phi\left(z^p H_p(\tau, \psi)(f * g)(z), z^p H_p(\tau + 1, \psi)(f * g)(z), z^p H_p(\tau + 2, \psi)(f * g)(z), z^p H_p(\tau + 3, \psi)(f * g)(z),\right.$$
$$\left. z^p H_p(\tau + 4, \psi)(f * g)(z); z\right), \text{ is univalent in } U$$

and

$$\Omega \subset \left\{\phi\left(z^p H_p(\tau, \psi)(f * g)(z), z^p H_p(\tau + 1, \psi)(f * g)(z), z^p H_p(\tau + 2, \psi)(f * g)(z), z^p H_p(\tau + 3, \psi)(f * g)(z),\right.\right.$$
$$\left.\left. z^p H_p(\tau + 4, \psi)(f * g)(z); z\right) : z \in U\right\}, \tag{3.6}$$

implies that

$$q(z) \prec z^p H_p(\tau, \psi)(f * g)(z) \quad (z \in U).$$

Proof . Let the function $p(z)$ be defined by (2.19) and χ by (2.26), since $\phi \in A_j[\Omega, q]$. Thus from (2.27) and (3.6) yield

$$\Omega \subset \{\chi(p(z), zp'(z), z^2p''(z), z^3p^3(z), z^4p^4(z); z) : z \in U\}.$$

From (2.24) and (2.25), we see that the admissibility for $\phi \in A_j[\Omega, q]$, in Definition 2.1 is equivalent to the admissibility condition for χ as given in Definition 1.5, with $n = 3$. Hence $\chi \in \Psi_3[\Omega, q]$, and by using Lemma 1.7 and (3.6), we have

$$q(z) \prec z^p H_p(\tau, \psi)(f * g)(z).$$

□

If $\Omega \neq C$ is simply-connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U on to Ω. In this case, the class $A_j[h(U), q]$, is written an $A_j[h, q]$. The next theorem is directly consequence of Theorem 3.6.

Theorem 3.7. *Let $\phi \in A_{j,1}[\Omega, q]$ and h be analytic function in U. If f, $g \in B_p$ and $q \in Q_1$ with $q'(z) \neq 0$, satisfying the following conditions* (3.5) *and*

$$\phi\left(z^p H_p(\tau, \psi)(f * g)(z), z^p H_p(\tau+1, \psi)(f * g)(z), z^p H_p(\tau+2, \psi)(f * g)(z), z^p H_p(\tau+3, \psi)(f * g)(z),\right.$$
$$\left. z^p H_p(\tau+4, \psi)(f * g)(z); z\right),$$

is univalent in U, then

$$h(z) \prec \phi\left(z^p H_p(\tau, \psi)(f * g)(z), z^p H_p(\tau+1, \psi)(f * g)(z), z^p H_p(\tau+2, \psi)(f * g)(z), z^p H_p(\tau+3, \psi)(f * g)(z),\right.$$
$$\left. z^p H_p(\tau+4, \psi)(f * g)(z); z\right), \tag{3.7}$$

implies that

$$q(z) \prec z^p H_p(\tau, \psi)(f * g)(z) \quad (z \in U).$$

Combining Theorems 2.4 and 3.3, we obtain the following sandwich-type theorem.

Theorem 3.8. *Let h_1 and q_1, be univalent in U, h_2 be univalent function in U, $q_2 \in Q_0$, with $q_1(0) = q_2(0) = 1$ and $\phi \in A_j[h_1, q_1] \cap A_j[h_2, q_2]$. If $f, g \in B_p$, $H_p(\tau, \psi)(f * g)(z) \in Q_0 \cap J_0$, and*

$$\phi\left(H_p(\tau, \psi)(f * g)(z), H_p(\tau+1, \psi)(f * g)(z), H_p(\tau+2, \psi)(f * g)(z), H_p(\tau+3, \psi)(f * g)(z),\right.$$
$$\left. H_p(\tau+4, \psi)(f * g)(z); z\right),$$

is univalent in U, and the conditions (2.1) *and* (3.1) *are satisfied, then*

$$q_1(z) \prec H_p(\tau, \psi)(f * g)(z) \prec q_2(z).$$

4. Sandwich Results

Combining Theorems (2.1) and (2.6), we obtain the following sandwich-type theorem.

Theorem 4.1. *Let* h_1 *and* q_1*, be univalent in* U*,* h_2 *be univalent function in* U*,* $q_2 \in Q_1$*, with* $q_1(0) = q_2(0) = 1$ *and* $\phi \in A_{j,1}[h_1, q_1] \cap A_{j,1}[h_2, q_2]$*. If* $f, g \in B_p$*,* $z^p H_p(\tau, \psi)(f * g)(z) \in Q_1 \cap J_1$*, and*

$$\phi\left(z^p H_p(\tau,\psi)(f*g)(z), z^p H_p(\tau+1,\psi)(f*g)(z), z^p H_p(\tau+2,\psi)(f*g)(z), z^p H_p(\tau+3,\psi)(f*g)(z),\right.$$
$$\left. z^p H_p(\tau+4,\psi)(f*g)(z); z\right),$$

is univalent in U*, and the conditions* (2.17) *and* (3.5) *are satisfied, then*

$$q_1(z) \prec z^p H_p(\tau,\psi)(f*g)(z) \prec q_2(z).$$

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