Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 21–30
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.6226

# An application of the Elzaki homotopy perturbation method for solving fractional Burger's equations

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*(Communicated by Javad Vahidi)*

**Abstract**

In this paper, the solution of time-fractional Burgers and linked Burger's equations is obtained by using an effective analytical methodology termed the Elzaki homotopy perturbation method. Caputo sense is used to characterize the fractional derivatives. The recommended technique's answer is represented as a series that converges to the precise solution of the supplied issues. Furthermore, the outcomes of this strategy have revealed tight ties to the methods to the problems under investigation. The validity of the current strategy is demonstrated by illustrative instances.

Keywords: Elzaki homotopy perturbation method, time-fractional Burger's equations, Caputo fractional derivative.
2020 MSC: 26A33, 34A08

## 1 Introduction

Mathematical modeling is an area of practical mathematics that deals with non-integral powers in differential and integral operators. The vast range of applications of fractional calculus in rheology, viscoelasticity, electrochemistry, fluid mechanics, and other fields has made it popular. For further information, read the monographs of Kilbas et al. [27], as well as some important works on fractional calculus and the solution technique of differential equations of arbitrary real order, as well as applications of the presented methods in other domains [27, 32].

Burger's equation is a fundamental partial differential equation from fluid mechanics. It occurs in various areas of applied mathematics, such as modeling of dynamics, heat conduction, and acoustic waves. It is named for Johannes Martinis Burgers (1895–1981). It is very rare that a real life applications can be modeled by a single partial differential equation, usually it takes a system of coupled partial differential equations to yield a complete model [1]. Recently, Burger's equations were studied by several authors by using ADM [4], HPM [41], q-HATM [34], SVIM [23] SHPM [24], and SADM [25]. Many analytical and approximation approaches for solving fractional differential equations have been developed in recent years [12, 42]. As the main aim of this work the EHPM is implemented to solve fractional PDEs and nonlinear system of fractional PDEs.

The following is the outline for this article: Sect. 2 introduces some fundamental elements of fractional calculus that are relevant to the listed problems. Sects. 3 and 4 contain the Elzaki transform and the EHPM elaborated version, respectively. Two examples issues are presented in Sect. 5 to demonstrate the usefulness and precision of the suggested strategy. Finally, in Sect. 6, there is a conclusion.

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*Received:* December 2021 *Accepted:* February 2022

## 2 Preliminaries

**Definition 2.1.** The fractional integral operator of order $v \geq 0$ Riemann Liouville, of a function $\varphi(\mu) \in C_\vartheta, \vartheta \geq -1$ is defined as [27, 28, 29]

$$I^\alpha u(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(t) d\tau, & \alpha > 0, \quad t > 0. \\ u(t), & \alpha = 0 \end{cases} \tag{2.1}$$

Properties of operator $I^\alpha$:

1. $I^\alpha I^\sigma u(t) = I^{\alpha+\sigma} u(t)$.
2. $I^\alpha I^\sigma u(t) = I^\sigma I^\alpha u(t)$.
3. $I^\alpha t^m = \frac{\Gamma(m+1)}{\Gamma(\alpha+m+1)} t^{\alpha+m}$.

**Definition 2.2.** The Caputo fractional derivative (CFD) of order $\alpha$ of a function $u(t)$ is defined as [27, 28, 29]

$$D^\alpha u(t) = I^{m-\alpha} D^m u(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} u^{(m)}(\tau) d\tau \tag{2.2}$$

For $m-1 < \alpha < m$, $m \in N$, $t > 0$ and $u \in C_{-1}^m$.

The operator $D^\alpha$ fundamental attributes are as follows:

1. $D^\alpha k = 0$, where k is a constant.
2. $D^\alpha t^\sigma = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma-\alpha+1)} t^{\sigma-\alpha}$,
3. $D^\alpha D^\sigma u(t) = D^{\alpha+\sigma} u(t)$
4. $I^\alpha D^\alpha u(t) = u(t) - \sum_{k=0}^{m-1} u^{(k)}(0) \frac{t^k}{k!}$.

**Definition 2.3.** The Mittag-Leffler function $E_\alpha(z)$ with $\alpha > 0$ is defined as [27]

$$E\alpha(z) = \sum_{m=0}^{\infty} \frac{z^\alpha}{\Gamma(m\alpha+1)} \tag{2.3}$$

## 3 Elzaki transform

**Definition 3.1.** The Elzaki transform (ET) is defined over the set of functions

$$A = \left\{ u(t) : \exists \mu \ , \ k_1, k_2 > 0 \ , \ |u(t)| < \mu \ e^{\frac{|t|}{k_j}} \ , \ \tau \in (-1)^j \times [0, \infty) \right\}.$$

by the following formula:

$$E[u(t)] = T(w) = s \int_0^\infty e^{\frac{-t}{w}} u(t) dt, \ w \in [k_1, k_2] \tag{3.1}$$

Some ET Properties:-

1. E $[1] = w^2$
2. $E[t^\alpha] = \Gamma(\alpha+1) \ w^{\alpha+2}$

**Definition 3.2.** The ET of the CFD is given by:

$$E[D_t^\alpha \ \mathrm{u}(x,t)] = \frac{E[u(x,t)]}{w^\alpha} - \sum_{k=0}^{n-1} w^{2-\alpha+k} u^{(k)}(x,0), \quad n-1 < \alpha \leq n \tag{3.2}$$

## 4 Elzaki Homotopy perturbation method (EHPM)

Consider the following fractional nonlinear PDEs:

$$ {}^{c}D_t^{\alpha} u(x,t) + R\left[u(x,t)\right] + N\left[u(x,t)\right] = g(x,t), \quad t>0,\ n-1<\alpha\leqslant n \tag{4.1}$$

where ${}^{c}D_t^{\alpha} u(x,t)$ is the derivative of $u(x,t)$ in Caputo sense, $R$, $N$ differential operators, including linear and nonlinear and $g(x,t)$ is the source term.

Now by taking ET on both sides of Eq. (4.1), we obtain

$$E\left\{ {}^{c}D_t^{\alpha} u(x,t) + R[u(x,t)] + N[u(x,t)]\right\} = E\left\{g(x,t)\right\}. \tag{4.2}$$

We achieve using ET's distinction feature

$$\frac{E\left\{u(x,t)\right\}}{w^{\alpha}} - \sum_{k=0}^{n-1} w^{2-\alpha+k} u^{(k)}(x,0) = E\left\{g(x,t)\right\} - E\left\{R\left[u(x,t)\right] + N\left[u(x,t)\right]\right\}, \tag{4.3}$$

or

$$E\left\{u(x,t)\right\} = \sum_{k=0}^{n-1} w^{2+k} u^{(k)}(x,0) + w^{\alpha} E\left\{g(x,t)\right\} - w^{\alpha} E\left\{R\left[u(x,t)\right] + N\left[u(x,t)\right]\right\}. \tag{4.4}$$

Applying inverse Elzaki transform on both sides of Eq. (4.4), we find

$$u(x,t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(x,0) + E^{-1}\left(w^{\alpha} E\left\{g(x,t)\right\}\right) - E^{-1}\left(w^{\alpha} E\left\{R\left[u(x,t)\right] + N\left[u(x,t)\right]\right\}\right). \tag{4.5}$$

Now, by applying HPM to the Eq. (4.5), we have

$$u(x,t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(x,0) + E^{-1}\left(w^{\alpha} E\left\{g(x,t)\right\}\right) - p\left[E^{-1}\left(w^{\alpha} E\left\{R\left[u(x,t)\right] + N\left[u(x,t)\right]\right\}\right)\right]. \tag{4.6}$$

To broaden the solution, the homotopy parameter p is employed

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n, \tag{4.7}$$

and the nonlinear term is decomposed as

$$N\left(u(x,t)\right) = \sum_{n=0}^{\infty} p^n H_n, \tag{4.8}$$

where

$$H_n = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left( \sum_{i=0}^{n} p^i u_i \right)_{p=0}.$$

Substituting (4.7) and (4.8) in (4.6), we get

$$\begin{aligned} \sum_{n=0}^{\infty} p^n u_n = & \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(x,0) + E^{-1}\left(w^{\alpha} E\left\{g(x,t)\right\}\right) \\ & - p\left[E^{-1}\left(w^{\alpha} E\left\{R\left[\sum_{n=0}^{\infty} p^n u_n\right] + \sum_{n=0}^{\infty} p^n H_n\right\}\right)\right]. \end{aligned} \tag{4.9}$$

The following equations are generated by comparing the coefficients of equal powers of p from both sides of the equation

$$\begin{aligned} & p^0 : u_0(x,t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(x,0) + E^{-1}\left(w^{\alpha} E\left\{g(x,t)\right\}\right), \\ & p^{n+1} : u_{n+1}(x,t) = -E^{-1}\left(w^{\alpha} E\left\{R\left[u_n\right] + H_n\right\}\right), \quad n\geq 0. \end{aligned} \tag{4.10}$$

The solution is written as

$$u(x,t) = u_0 + u_1 + u_2 + \cdots.$$

## 5 Application of EHPM

**Example 5.1.** Let us consider the fractional Burger's equation

$$^{c}D_t^{\alpha} u + uu_x = 0, \qquad 0 < \alpha \leqslant 1 \tag{5.1}$$

with initial condition

$$u\left(x,0\right) = x. \tag{5.2}$$

Taking ET on both sides of Eq. (5.1) with IC (5.2), we obtain

$$\frac{E\left\{u\left(x,t\right)\right\}}{w^{\alpha}} - w^{2-\alpha}\ \mathrm{u}\left(x,0\right) = -E\left\{uu_x\right\},$$

or

$$E\left\{u\left(x,t\right)\right\} = w^2 x - w^{\alpha} E\left\{uu_x\right\}. \tag{5.3}$$

The inverse ET of Eq. (5.3) implies that

$$u\left(x,t\right) = x - E^{-1}\left(w^{\alpha} E\left\{uu_x\right\}\right). \tag{5.4}$$

Now applying the HPM, we get

$$\sum_{n=0}^{\infty} p^n u_n = x - p\left[E^{-1}\left(w^{\alpha} E\left\{\sum_{n=0}^{\infty} p^n H_n\right\}\right)\right], \tag{5.5}$$

where $H_n$ is He's polynomials which signifies the nonlinear term $uu_x$.

The first few components of He's polynomials are given as

$$\begin{aligned} H_0 &= u_0 u_{0x} \\ H_1 &= u_0 u_{1x} + u_1 u_{0x} \\ H_2 &= u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x} \\ &\vdots \end{aligned}$$

Using the He's polynomials mentioned above and comparing the coefficients of the same power of p in Eq. (5.4), we get

$$\begin{aligned} &P^0 : u_0(x,t) = x, \\ &P^1 : u_1(x,t) = -E^{-1}\left(w^{\alpha} E\left\{H_0\right\}\right) = -E^{-1}\left(w^{\alpha} E\left\{x\right\}\right) = -E^{-1}\left(w^{\alpha+2} x\right) = -\frac{xt^{\alpha}}{\Gamma(\alpha+1)} \\ &P^2 : u_2(x,t) = -E^{-1}\left(w^{\alpha} E\left\{H_1\right\}\right) = -E^{-1}\left(w^{\alpha} E\left\{-\frac{2xt^{\alpha}}{\Gamma(\alpha+1)}\right\}\right) = \frac{2xt^{2\alpha}}{\Gamma(2\alpha+1)} \\ &\quad \vdots \end{aligned}$$

So the solution $u(x,t)$ is written as

$$u\left(x,t\right) = u_0 + u_1 + u_2 + \cdots = x - \frac{xt^{\alpha}}{\Gamma\left(\alpha+1\right)} + \frac{2xt^{2\alpha}}{\Gamma\left(2\alpha+1\right)} + \cdots \tag{5.6}$$

For $\alpha = 1$, Eq. (5.6) reduce to

$$u\left(x,t\right) = x(1 - t + t^2 - \cdots) = \frac{x}{1+t}. \tag{5.7}$$

This approach is quite like the precise solution. The outcome is identical to VIM [26].

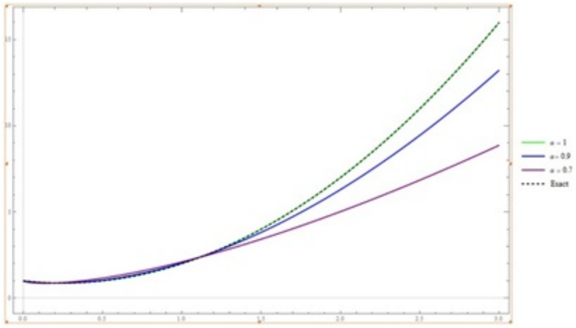

Figure 1: Plots of the exact and approximate solutions $u(x,t)$ for different values of $\alpha$ with fixed value $x = 1$.

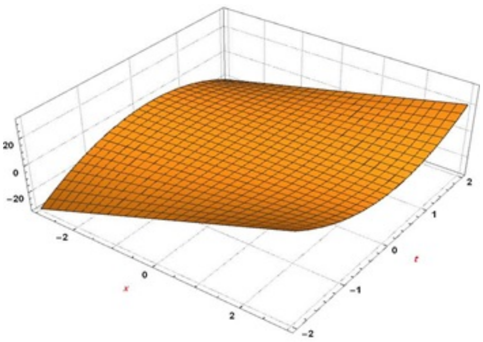

Figure 2: The surface graph of the exact solution of Eq. (5.1).

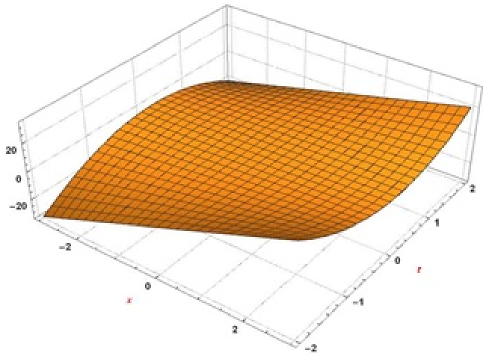

Figure 3: The surface graph of the approximate solution of Eq. (5.1) when $\alpha = 1$.

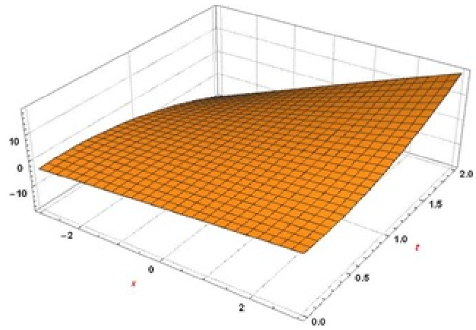

Figure 4: The surface graph of the approximate solution of Eq. (5.1) when $\alpha = 0.9$

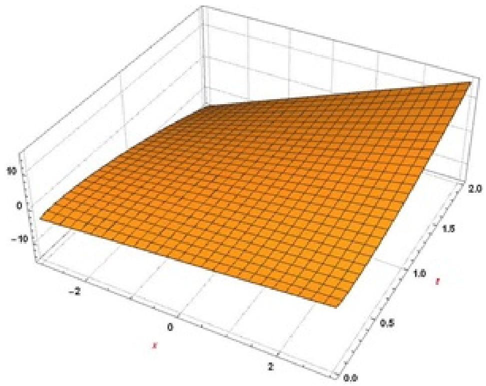

Figure 5: The surface graph of the approximate solution of Eq. (5.1) when $\alpha = 1$.

**Example 5.2.** Consider the following Burger's equations, which are time-fractional and linked

$$\begin{aligned} {}^{C}{D_t}^{\alpha} u(x,t)) - u_{xx} - 2uu_x + (uv)_x = 0 \\ {}^{C}{D_t}^{\beta} v(x,t)) - v_{xx} - 2vv_x + (uv)_x = 0, \end{aligned} \tag{5.8}$$

with initial conditions

$$u(x,0) = \sin x, \qquad v(x,0) = \sin x. \tag{5.9}$$

Taking ET on both sides of Eq. (5.8) with IC (5.9), we obtain

$$\begin{aligned} E(u(x,t)) = w^2 \ \sin x + w^{\alpha} E\{u_{xx} + 2uu_x - (uv)_x\}, \\ E(v(x,t)) = w^2 \sin x \ + w^{\beta} E\{v_{xx} + 2vv_x - (uv)_x\}. \end{aligned} \tag{5.10}$$

The inverse ET of Eq. (5.10) implies that

$$\begin{aligned} u(x,t) &= \sin x + E^{-1}\left(w^{\alpha} E\left\{u_{xx} + 2uu_x - (uv)_x\right\}\right) \\ v(x,t) &= \sin x + E^{-1}\left(w^{\beta} E\left\{v_{xx} + 2vv_x - (uv)_x\right\}\right). \end{aligned} \tag{5.11}$$

Now applying the HPM, we get

$$\begin{aligned} \sum_{n=0}^{\infty} p^n u_n &= \sin x + p\left[E^{-1}\left(w^{\alpha} E\left\{\sum_{n=0}^{\infty} p^n u_{\text{nxx}} + 2\sum_{n=0}^{\infty} p^n H_n - \sum_{n=0}^{\infty} p^n G_n\right\}\right)\right], \\ \sum_{n=0}^{\infty} p^n v_n &= \sin x + p\left[E^{-1}\left(w^{\beta} E\left\{\sum_{n=0}^{\infty} p^n v_{\text{nxx}} + 2\sum_{n=0}^{\infty} p^n E_n - \sum_{n=0}^{\infty} p^n G_n\right\}\right)\right], \end{aligned} \tag{5.12}$$

where $H_n,\ G_n$ and $E_n$ are He's polynomials which signifies the nonlinear terms $uu_x$ , $(uv)_x$ and $vv_x$ respectively. The first few components of He's polynomials are given as

$$\begin{aligned} H_0 &= u_0 u_{0x} \\ H_1 &= u_0 u_{1x} + u_1 u_{0x} \\ H_2 &= u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x} \\ &\vdots \\ G_0 &= (u_0 v_0)_x \\ G_1 &= (u_0 v_1)_x + (u_1 v_0)_x \\ G_2 &= (u_0 v_2)_x + (u_1 v_1)_x + (u_2 v_0)_x \\ &\vdots \\ E_0 &= v_0 v_{0x} \\ E_1 &= v_0 v_{1x} + v_1 v_{0x} \\ E_2 &= v_0 v_{2x} + v_1 v_{1x} + v_2 v_{0x} \\ &\vdots \end{aligned}$$

Comparing the coefficients of same power of $p$ in Eq. (5.12) we get

$$\begin{aligned} p^0 &: \begin{array}{l} u_0(x,t) = \sin x, \\ v_0(x,t) = \sin x, \end{array} \\ P^1 &: \begin{array}{l} u_1(x,t) = E^{-1}\left(w^{\alpha} E\left\{u_{0xx} + 2H_0 - G_0\right\}\right) \\ v_1(x,t) = E^{-1}\left(w^{\beta} E\left\{v_{0xx} + 2E_0 - G_0\right\}\right) \end{array} \\ P^2 &: \begin{array}{l} u_2(x,t) = E^{-1}\left(w^{\alpha} E\left\{u_{1xx} + 2H_1 - G_1\right\}\right) \\ v_2(x,t) = E^{-1}\left(w^{\beta} E\left\{v_{1xx} + 2E_1 - G_1\right\}\right) \end{array} \\ &\vdots \end{aligned}$$

The initial terms of fractional EHPM have the following shape according to the preceding formulas:

$$p^0: \begin{array}{l} u_0(x,t) = \sin x, \\ v_0(x,t) = \sin x, \end{array}$$

$$P^1: \begin{array}{l} u_1(x,t) = E^{-1}\left(w^{\alpha} E\left\{-\sin x + 2\sin x \cos x - 2\sin x\cos x\ \right\}\right) \\ v_1(x,t) = E^{-1}\left(w^{\beta} E\left\{-\sin x + 2\sin x\cos x - 2\sin x\cos x\right\}\right) \end{array}$$

$$= -\sin x\ E^{-1}\left(w^{\alpha+2}\right)$$
$$= -\sin x\ E^{-1}\left(w^{\beta+2}\right)$$
$$= -\frac{t^{\alpha}}{\Gamma(\alpha+1)}\sin x$$
$$= -\ \frac{t^{\beta}}{\Gamma(\beta+1)}\sin x$$

$$P^2: \begin{array}{l} u_2(x,t) = E^{-1}\left(w^{\alpha} E\left\{\frac{t^{\alpha}}{\Gamma(\alpha+1)}\ \sin x - 2\frac{t^{\alpha}}{\Gamma(\alpha+1)}\sin x\cos x + 2\frac{t^{\beta}}{\Gamma(\beta+1)}\sin x\cos x\right\}\right) \\ v_2(x,t) = E^{-1}\left(w^{\beta} E\left\{\frac{\tau^{\beta}}{\Gamma(\beta+1)}\ \sin x - 2\frac{t^{\beta}}{\Gamma(\beta+1)}\sin x\cos x + 2\frac{t^{\alpha}}{\Gamma(\alpha+1)}\sin x\cos x\right\}\right) \end{array}$$

$$= \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\sin x - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\sin x\cos x + \frac{2t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\sin x\cos x$$
$$= \frac{t^{2\beta}}{\Gamma(2\beta+1)}\sin x - 2\frac{t^{2\beta}}{\Gamma(2\beta+1)}\sin x\cos x + \frac{2t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\sin x\cos x$$
$$= \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\sin x + 2\sin x\cos x\left(\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right)$$
$$= \frac{t^{2\beta}}{\Gamma(2\beta+1)}\sin x + 2\sin x\cos x\left(\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - \frac{t^{2\beta}}{\Gamma(2\beta+1)}\right)$$

$$\vdots$$

So the series solution is written as

$$\begin{aligned} &= \sin x\left(1 - \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\cdots\right) + 2\sin x\cos x\left(\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\cdots\right) \\ &= \sin x\left(1 - \frac{t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)}\cdots\right) + 2\sin x\cos x\left(\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - \frac{t^{2\beta}}{\Gamma(2\beta+1)}\cdots\right) \end{aligned} \tag{5.13}$$

Setting $\alpha = \beta$ in (5.13), we obtain

$$\begin{aligned} u(x,t) &= \sin x\left[1 - \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \ldots\right] = \sin x\ E_{\alpha}(-t^{\alpha}), \\ v(x,t) &= \sin x\left[1 - \frac{t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \ldots\right] = \sin x\ E_{\beta}\left(-t^{\beta}\right). \end{aligned} \tag{5.14}$$

The Eq. (5.14) is approximate to the form $u(x,t) = v(x,t) = e^{-t}\sin x$ for $\alpha = \beta = 1$, which is the exact solution of Eq. (5.8) for $\alpha = \beta = 1$. The result is same as HPM [4].

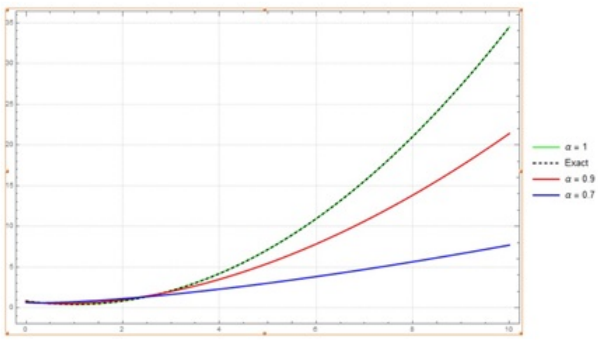


Figure 6: Plots of the exact and approximate solutions $u(x,t) = v(x,t)$ for different values of $\alpha$ with fixed value $x = 1$..

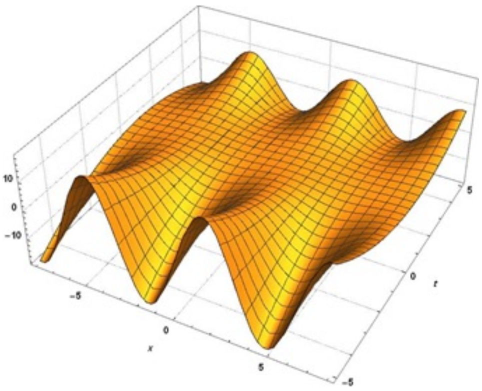


Figure 7: The surface graph of the exact solution of Eq. (5.8)

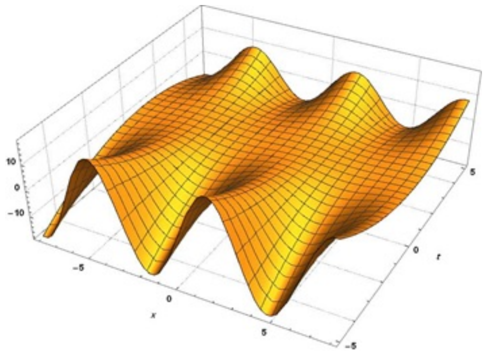


Figure 8: The surface graph of the approximate solution of Eq. (5.8) when $\alpha = 1$.

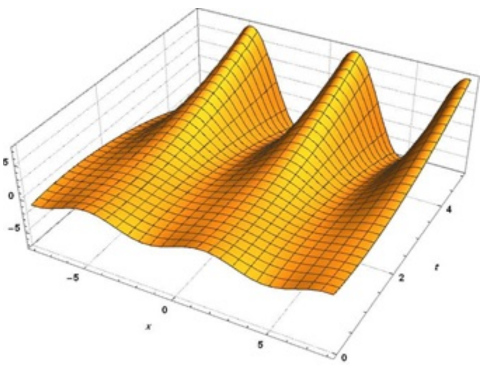


Figure 9: The surface graph of the approximate solution of Eq. (5.8) when $\alpha = 0.9$

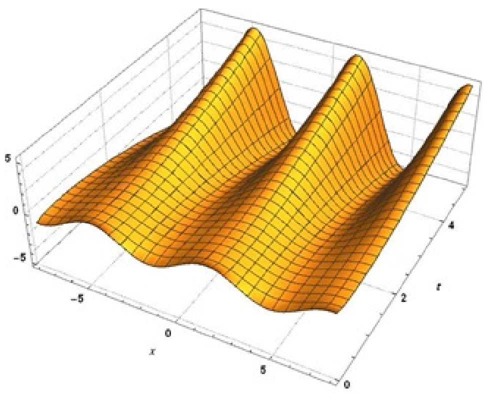


Figure 10: The surface graph of the approximate solution of Eq. (5.8) when $\alpha = 0.7$

## 6 Conclusion

The EHPM has been successfully applied to obtain the analytical solutions of time-fractional Burgers and coupled Burger's equations. The process is simple to follow because it consists of applying the Elzaki transform directly to the provided issue and then using the homotopy perturbation method. The analytical solution for the given issue is then obtained using the inverse Elzaki transform. The findings achieved by the proposed technique are in good agreement with the precise solution of Example 5.1 and 5.2 in the article, as shown in the figures. As a result, the suggested method is an appropriate analytical tool for solving fractional partial differential equations and systems of fractional PDEs.

## References

[1] M. Al-Mazmumy, *The Modified Adomian Decomposition Method for Solving Nonlinear Coupled Burger's Equations*, Nonlinear Anal. Differ. Equ. **3** (2015), 111–122.

[2] D. Baleanu and H.K. Jassim, *Approximate analytical solutions of Goursat problem within local fractional operators*, J. Nonlinear Sci. Appl. **9** (2016), 4829–4837.

[3] D. Baleanu and H.K. Jassim, *Exact solution of two-dimensional fractional partial differential equations*, Fractal Fract. **4** (2020), no. 21, 1–9.

[4] D. Baleanu and H.K. Jassim, *A Modification Fractional Homotopy Perturbation Method for Solving Helmholtz and Coupled Helmholtz Equations on Cantor Sets*, Fractal Fract. **3** (2019), no. 30, 1–8.

[5] D. Baleanu, H.K. Jassim and M. Al Qurashi, *Solving Helmholtz equation with local fractional derivative operators*, Fractal Fract. **3** (2019), no. 43, 1–13.

[6] D. Baleanu, H.K. Jassim and H. Khan, *A modification fractional variational iteration method for solving nonlinear gas dynamic and coupled KdV equations involving local fractional operators*, Thermal Sci. **22** (2018), 165–175.

[7] D. Baleanu and H.K. Jassim, *Approximate solutions of the damped wave equation and dissipative wave equation in fractal strings*, Fractal Fract. **3** (2016), no. 26, 1–12.

[8] Y. Chen, and A. Hong-Li, *Numerical solutions of coupled Burgers equations with time and space fractional derivatives*, Appl. Math. Comput. **200** (2008), 87–95.

[9] H. A. Eaued, H.K. Jassim and M.G. Mohammed, *A Novel Method for the Analytical Solution of Partial Differential Equations Arising in Mathematical Physics*, IOP Conf. Series: Mater. Sci. Engin. **928** (2020), 1–16.

[10] Z.P. Fan, H.K. Jassim, R.K. Rainna and X. J. Yang, *Adomian decomposition method for three-dimensional diffusion model in fractal heat transfer involving local fractional derivatives*, Thermal Sci. **19** (2015), 137–141.

[11] M.S. Hu, R.P. Agarwal and X.J. Yang, *Local fractional Fourier series with application to wave equation in fractal vibrating*, Abstr. Appl. Anal. **2012** (2012), 1-7.

[12] H. Jafari and H.K. Jassem, *Local fractional variational iteration method for nonlinear partial differential equations within local fractional operators*, Appl. Appl. Math. **10** (2015), 1055–1065.

[13] H. Jafari, H.K. Jassim, F. Tchier and D. Baleanu, *On the Approximate Solutions of Local Fractional Differential Equations with Local Fractional Operator*, Entropy **18** (2016), 1–12.

[14] H. Jafari, H.K. Jassim and S.P. Moshokoa, *Reduced differential transform method for partial differential equations within local fractional derivative operators*, Adv. Mech. Engin. 8 (2016), no. 4, 1–6.

[15] H. Jafari, H.K. Jassim and J. Vahidi, *Reduced differential transform and variational iteration methods for 3D diffusion model in fractal heat transfer within local fractional operators*, Thermal Sci. **22** (2018), 301–307.

[16] H.K. Jassim and D. Baleanu, *A novel approach for Korteweg-de Vries equation of fractional order*, J. Appl. Comput. Mech. 5 (2019), no. 2, 192–198.

[17] H. K. Jassim and S.A. Khafif, *SVIM for solving Burger's and coupled Burger's equations of fractional order*, Prog. Fract. Different. Appl. **7** (2021), no. 1, 1–6.

[18] H.K. Jassim, W. A. Shahab, *Fractional variational iteration method to solve one dimensional second order hyperbolic telegraph equations*, J. Phys.: Conf. Ser. **1032** (2018), no. 1, 1–9.

[19] H.K. Jassim and M.A. Shareef, *On approximate solutions for fractional system of differential equations with Caputo-Fabrizio fractional operator*, J. Math. Comput. Sci., **23** (2021), 58–66.

[20] H.K. Jassim, C. Ünlü, S.P. Moshokoa and C.M. Khalique, *Local fractional Laplace variational iteration method for solving diffusion and wave equations on cantor sets within local fractional operators*, Math. Prob. Engin. **2015** (2015), 1-7.

[21] H.K. Jassim and H.A. Kadhim, *Fractional Sumudu decomposition method for solving PDEs of fractional order*, J. Appl. Comput. Mech. **7** (2021), no. 1, 302–311.

[22] H. K. Jassim, J. Vahidi and V.M. Ariyan, *Solving Laplace equation within local fractional operators by using local fractional differential transform and Laplace variational iteration methods*, Nonlinear Dyn. Syst. Theory **20** (2020), no. 4, 388–396.

[23] H.K. Jassim and S.A. Khafif, *SVIM for solving Burger's and coupled Burger's equations of fractional order*, Prog. Fractional Different. Appl. **7** (2021), no. 1, 1–6.

[24] H.K. Jassim, *A new approach to find approximate solutions of Burger's and coupled Burger's equations of fractional order*, TWMS J. Appl. Engin. Math. **11** (2021), no. 2, 415–423.

[25] H.K. Jassim, M.G. Mohammed and S.A. Khafif, *The Approximate solutions of time-fractional Burger's and coupled time-fractional Burger's equations*, Int. J. Adv. Appl. Math. Mech. **6** (2019), no. 4, 64–70.

[26] H.K. Jassim, *Analytical Approximate Solutions for Local Fractional Wave Equations*, Math. Meth. Appl. Sci. **43** (2020), no. 2, 939–947.

[27] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and applications of fractional differential equations*, Else-

vier, Amsterdam, 2006.

[28] V. Kiryakova, *Generalized fractional calculus and applications. Pitman research notes in mathematics series*, Longman Scientific & Technical, Harlow, 1994.

[29] V. Lakshmikantham and A. S. Vatsala, *Basic theory of fractional differential equations*, Nonlinear Anal. Theory, Meth. Appl. **69** (2008), no. 8, 2677–2682.

[30] Y. Li, L.F. Wang and S.J. Yuan, *Reconstructive schemes for variational iteration method within Yang-Laplace transform with application to fractal heat conduction problem*, Thermal Science **17** (2013), 715–721.

[31] K.S. Miller and B. Ross, *An introduction to the fractional calculus and differential equations*, Wiley, New York, 1993.

[32] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, CA., 1999.

[33] J. Singh, H.K.Jassim and D. Kumar, *An efficient computational technique for local fractional Fokker-Planck equation*, Phys. A: Statist. Mech. Appl. 555(2020) 1-8.

[34] J. Singh, D. Kumar and R. Swroop, *Numerical solution of time- and space-fractional coupled Burger's equations via homotopy algorithm*, Alexandria Engin. J. **55** (2016), 1753–1763.

[35] A.M. Wazwaz, *Partial Differential Equations and Solitary Waves Theory*, Beijing and Springer-Verlag Berlin Heidelberg, 2009.

[36] S. Xu, X. Ling, Y. Zhao and H.K. Jaseem, *A novel schedule for solving the two-dimensional diffusion in fractal heat transfer*, Thermal Sci. **19** (2015), 99–103.

[37] X.J. Yang, *Local fractional functional analysis and its applications*, Asian Academic, Hong Kong, China, 2011.

[38] X.J. Yang, J.A. Machad and H.M. Srivastava, *A new numerical technique for solving the local fractional diffusion equation: Two-dimensional extended differential transform approach*, Appl. Math. Comput. **274** (2016), 143–151.

[39] A.M. Yang, X. Yang and Z. Li, *Local fractional series expansion method for solving wave and diffusion equations Cantor sets*, Abstr. Appl. Anal. **2013** (2013), 1–5.

[40] S.P. Yan, H. Jafari and H.K. Jassim, *Local fractional Adomian decomposition and function decomposition methods for solving Laplace equation within local fractional operators*, Adv. Math. Phys. **2014** (2014), 1-7.

[41] A. Yildirim and A. Kelleci, *Homotopy perturbation method for numerical solutions of coupled Burgers equations with time-space fractional derivatives*, Int. J. Numerical Meth. Heat Fluid Flow **20** (2010, 897-909.

[42] A. Yildirim and A. Kelleci, *Homotopy perturbation method for numerical solutions of coupled Burgers equations with time-space fractional derivatives*, Int. J. Numerical Meth. Heat Fluid Flow **20** (2010), 897–909.

[43] Y. Zhang, X.J. Yang and C. Cattani, *Local fractional homotopy perturbation method for solving nonhomogeneous heat conduction equations in fractal domain*, Entropy **17** (2015), 6753–6764.

[44] C.G. Zhao, A.M. Yang, H. Jafari and A. Haghbin, *The Yang-Laplace transform for solving the IVPs with local fractional derivative*, Abstr. Appl. Anal. **2014** (2014), 1–5.