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Characterization and stability of multi-cubic mappings

Ali Reza Neisi[a], Mohammad Sadegh Asgari[a,]

[a]*Department of Mathematics, Faculty of Science, Central Tehran Branch, Islamic Azad University, Tehran, Iran*

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Abstract

In this article, we introduce a new class of multi-cubic mappings and then unify a system of cubic functional equations defining a multi-cubic mapping to an equation, as the multi-cubic functional equation. Moreover, we show that the mentioned equation describes the multi-cubic mappings. Furthermore, we prove the Hyers-Ulam stability of multi-cubic mappings in non-Archimedean normed spaces by applying a known fixed point theorem.

Keywords: Fixed point, Hyers-Ulam stability, multi-cubic mapping, non-Archimedean normed space.
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1 Introduction

Let V and W be linear spaces, and $n \geq 2$ be a natural number. A mapping $f : V^n \longrightarrow W$ is called *multi-additive* if it is additive (satisfies Cauchy's functional equation $A(x + y) = A(x) + A(y)$) in each variable; see [13] and [15]. Some facts on such mappings can be found in [24] and many other sources. Moreover, f is said to be *multi-quadratic* if it is quadratic in each variable [26], namely, it satisfies quadratic functional equation

$$Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y) \tag{1.1}$$

in each variable; refer to [3], [8], [10], [28], [29] and [33] more details about some forms of multi-quadratic mappings.

J. M. Rassias was the first author who defined cubic functional equations in [27] as follows:

$$C(x + 2y) = 3C(x + y) + C(x - y) - 3C(x) + 6C(y). \tag{1.2}$$

After that, Jun and Kim introduced the cubic equation

$$C(2x + y) + C(2x - y) = 2C(x + y) + 2C(x - y) + 12C(x) \tag{1.3}$$

in [21] and then different form of cubic functional equation defined by them in [22] is

$$C(x + 2y) + C(x - 2y) = 4C(x + y) + 4C(x - y) - 6C(x). \tag{1.4}$$

Equations (1.3) and (1.4) were generalized by Bodaghi in [4] as follows:

$$f(rx + sy) + f(rx - sy) = rs^2[f(x + y) + f(x - y)] + 2r(r^2 - s^2)f(x) \tag{1.5}$$

Email addresses: alireza_neisi@yahoo.com (Ali Reza Neisi), m.s.asgari19@gmail.com (Mohammad Sadegh Asgari)

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where r, s are integer numbers with $r \pm s \neq 0$; see also [7].

Ghaemi et al., in [19] introduced the multi-cubic mappings for the first time. In fact, they considered a mapping $f : V^n \longrightarrow W$ which satisfies (1.5) in each variable. Next, a special case of such mappings is studied in [11]. In fact, a mapping $f : V^n \longrightarrow W$ is called *multi-cubic* if it is cubic in each variable, i.e., satisfies (1.3) in each variable. In [11], the authors unified the system of functional equations defining a multi-cubic mapping to a single equation, namely, multi-cubic functional equation. Furthermore, the general system of cubic functional equations which was defined in [19], characterized as a single equation in [18].

In two last decades, the stability problem for functional equations which was initiated by Ulam [30] for group homomorphisms, answered and studied for several variables mappings. Indeed, a functional equation Γ is said to be *stable* if any function f satisfying the equation Γ approximately must be near to an exact solution.

The Hyers-Ulam stability of multi-quadratic mappings in various Banach spaces have been studied in [8], [9], [14], [16], [17] and [33]. In [11], it is shown that every multi-cubic functional equation is stable; for the miscellaneous versions of multi-cubic mappings and their stabilities in modular spaces, we refer to [25]), respectively. For the structure and stability of multi-additive-quadratic and multi-quadratic-cubic, we refer to [1] and [6], respectively.

Using equation (1.2), in this paper, we define a new form of multi-cubic mappings which are different from [11] and [18] and then present a characterization of such mappings. In other words, we reduce the system of n equations defining the multi-cubic mappings to obtain a single functional equation. We also prove the Hyers-Ulam stability and hyperstabilty for multi-cubic functional equations in non-Archimedean normed spaces by applying a known fixed point theorem which has been introduced and studied in [12]; for more applications of this technique see [2], [31] and [32].

2 Characterization of multi-cubic mappings

Throughout this paper, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are the set of all positive integers, integers and rational numbers, respectively, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}, \mathbb{R}_+ := [0, \infty)$. For any $l \in \mathbb{N}_0, n \in \mathbb{N}$, $t = (t_1, \ldots, t_n) \in \{-1, 1\}^n$ and $x = (x_1, \ldots, x_n) \in V^n$ we write $lx := (lx_1, \ldots, lx_n)$ and $tx := (t_1x_1, \ldots, t_nx_n)$, where lx stands, as usual, for the scaler product of l on x in the linear space V.

Definition 2.1. Let V and W be vector spaces over $\mathbb{Q}$, $n \in \mathbb{N}$. A several variables mapping $f : V^n \longrightarrow W$ is called *n-cubic* or *multi-cubic* if f satisfies *(1.2)* in each variable.

Let $n \in \mathbb{N}$ with $n \geq 2$ and $x_i^n = (x_{i1}, x_{i2}, \ldots, x_{in}) \in V^n$, where $i \in \{1, 2\}$. We shall denote x_i^n by x_i if there is no risk of mistake. For $x_1, x_2 \in V^n$ and $p_l \in \mathbb{N}_0$ with $0 \leq p_l \leq n$, where $l \in \{1, 2, 3\}$. Set

$$\mathbb{A}^n = \{\mathfrak{A}_n = (A_1, \ldots, A_n)|\ A_j \in \{x_{1j} \pm x_{2j}, x_{1j}, x_{2j}\}\},$$

for all $j \in \{1, \ldots, n\}$. The subset $\mathbb{A}^n_{(p_1,p_2,p_3)}$ of $\mathbb{A}^n$ is considered as follows:

$$\begin{aligned}\mathbb{A}^n_{(p_1,p_2,p_3)} : &= \{\mathfrak{A}_n \in \mathbb{A}^n|\ \mathrm{Card}\{A_j :\ A_j = x_{1j}\} = p_1,\\ &\mathrm{Card}\{A_j :\ A_j = x_{2j}\} = p_2, \mathrm{Card}\{A_j :\ A_j = x_{1j} + x_{2j}\} = p_3, \}.\end{aligned}$$

The following notations can be used for the multi-cubic mappings.

$$f\left(\mathbb{A}^n_{(p_1,p_2,p_3)}\right) := \sum_{\mathfrak{A}_n \in \mathbb{A}^n_{(p_1,p_2,p_3)}} f\left(\mathfrak{A}_n\right), \tag{2.1}$$

and

$$f\left(\mathbb{A}^n_{(p_1,p_2,p_3)}, z\right) := \sum_{\mathfrak{A}_n \in \mathbb{A}^n_{(p_1,p_2,p_3)}} f\left(\mathfrak{A}_n, z\right) \qquad (z \in V).$$

The notation $\left(\begin{array}{c} n \\ k \end{array}\right)$ is the binomial coefficient which is defined for all $n, k \in \mathbb{N}_0$ with $n \geq k$ by $\frac{n!}{k!(n-k)!}$.

Definition 2.2. Consider a mapping $f : V^n \longrightarrow W$. Then, it

(i) satisfies *zero condition* if $f(x)=0$ for any $x\in V^n$ with at least one component which is equal to zero.
(ii) is odd in the jth variable if

$$f(z_1,\dots,z_{j-1},-z_j,z_{j+1},\dots,z_n)=-f(z_1,\dots,z_{j-1},z_j,z_{j+1},\dots,z_n).$$

Here, we bring the following lemma, presented in [5].

Lemma 2.3. Let $n,k,p_l\in\mathbb{N}_0$, such that $k+\sum_{l=1}^m p_l\leq n$, where $l\in\{1,\dots,m\}$. Then

$$\begin{aligned}&\binom{n-k}{n-k-\sum_{l=1}^m p_l}\binom{\sum_{l=1}^m p_l}{\sum_{l=1}^{m-1} p_l}\cdots\binom{p_1+p_2}{p_1}\\&=\binom{n-k}{p_1}\binom{n-k-p_1}{p_2}\cdots\binom{n-k-\sum_{l=1}^{m-1}p_l}{p_m}.\end{aligned}$$

Let $0\leq k\leq n-1$. Put $\mathbf{n}:=\{1,\dots,n\}$, $n\in\mathbb{N}$. For a subset $T=\{j_1,\dots,j_i\}$ of $\mathbf{n}$ with $1\leq j_1<\dots<j_i\leq n$ and $x=(x_1,\dots,x_n)\in V^n$,

$$_Tx:=(0,\dots,0,x_{j_1},0,\dots,0,x_{j_i},0,\dots,0)\in V^n$$

denotes the vector which coincides with x in exactly those components, which are indexed by the elements of T and whose other components are set equal zero. Note that $_0x=0$, $_{\mathbf{n}}x=x$. We use these notations in the proof of upcoming lemma.

We need the next lemma in reaching our goal in this section. The idea of proof is taken form [6, Lemma 2.3].

Lemma 2.4. Suppose that a mapping $f:V^n\longrightarrow W$ satisfies the equation

$$f(x_1+2x_2)=\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}\sum_{p_3=0}^{n-p_1-p_2}(-3)^{p_1}6^{p_2}3^{p_3}f\left(\mathbb{A}^n_{(p_1,p_2,p_3)}\right),\tag{2.2}$$

for all $x_1,x_2\in V^n$, where $f\left(\mathbb{A}^n_{(p_1,p_2,p_3)}\right)$ is defined in *(2.1)*. Then, it has zero condition.

Proof . Given an arbitrary and fixed $x\in V^n$. We argue by induction on k that for each $_kx$, $f(_kx)=0$ when $0\leq k\leq n-1$. Putting $x_1=x_2={_0x}$ in (2.2), we have

$$\begin{aligned}&f(_0x)\\&=\left[\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}\sum_{p_3=0}^{n-p_1-p_2}\binom{n}{n-p_1-p_2-p_3}\binom{p_1+p_2+p_3}{p_1+p_2}\binom{p_1+p_2}{p_1}(-3)^{p_1}6^{p_2}3^{p_3}\right]f(_0x).\end{aligned}\tag{2.3}$$

Using Lemma 2.3 for $k=0$, we have

$$\begin{aligned}&\sum_{p_1=0}^{n}\binom{n}{n-p_1-p_2-p_3}\binom{p_1+p_2+p_3}{p_1+p_2}\binom{p_1+p_2}{p_1}(-3)^{p_1}6^{p_2}3^{p_3}\\&=\sum_{p_1=0}^{n}\binom{n}{p_1}(-3)^{p_1}\sum_{p_2=0}^{n-p_1}\binom{n-p_1}{p_2}6^{p_2}\sum_{p_3=0}^{n-p_1-p_2}\binom{n-p_1-p_2}{p_3}1^{n-p_1-p_2-p_3}\times 3^{p_3}\\&=\sum_{p_1=0}^{n}\binom{n}{p_1}(-3)^{p_1}\sum_{p_2=0}^{n-p_1}\binom{n-p_1}{p_2}6^{p_2}\times 4^{n-p_1-p_2}\\&=\sum_{p_1=0}^{n}\binom{n}{p_1}(-3)^{p_1}10^{n-p_1}=7^n.\end{aligned}\tag{2.4}$$

It follows from relations (2.3) and (2.4) that $f(_0x)=7^nf(_0x)$, and so $f(_0x)=0$. Assume that for each $_{k-1}x$, $f(_{k-1}x)=0$. We show that $f(_kx)=0$. Without loss of generality, we assume that the first k variables are non-zero.

By our assumption, replacing (x_1, x_2) by $({}_kx_1, 0)$ in equation (2.2), we have

$$f({}_kx) = \left[\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-k-p_1}\sum_{p_3=0}^{n-k-p_1-p_2}\binom{n-k}{n-k-p_1-p_2-p_3}\binom{p_1+p_2+p_3}{p_1+p_2}\binom{p_1+p_2}{p_1}(-3)^{p_1}6^{p_2}3^{p_3}\right]f({}_kx).$$

Similar the above and by using Lemma 2.3, we can obtain $f({}_kx) = 7^{n-k}f({}_kx)$ and this implies that $f({}_kx) = 0$ and so the proof is now finished. □

In the following result which is our aim in this section, we unify the general system of cubic functional equations defining a multi-cubic mapping to a equation and indeed this functional equation describes a multi-cubic mapping.

Proposition 2.5. A mapping $f : V^n \longrightarrow W$ is multi-cubic if and only if it satisfies equation *(2.2)*.

Proof . First, suppose that f is a multi-cubic mapping. We obtain this implication by induction on n. For $n = 1$, it is trivial that f satisfies equation (1.2). If (2.2) is true for some positive integer $n > 1$, then,

$$\begin{aligned}
&f\left(x_1^{n+1} + 2x_2^{n+1}\right) = f\left(x_1^n + 2x_2^n, x_{1,n+1} + 2x_{2,n+1}\right)\\
&= 3f\left(x_1^n + 2x_2^n, x_{1,n+1} + x_{2,n+1}\right) + f\left(x_1^n + 2x_2^n, x_{1,n+1} - x_{2,n+1}\right)\\
&- 3f\left(x_1^n + 2x_2^n, x_{1,n+1}\right) + 6f\left(x_1^n + 2x_2^n, x_{2,n+1}\right)\\
&= 3\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}\sum_{p_3=0}^{n-p_1-p_2}(-3)^{p_1}6^{p_2}3^{p_3}f\left(\mathbb{A}^n_{(p_1,p_2,p_3)}, x_{1,n+1} + x_{2,n+1}\right)\\
&+ \sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}\sum_{p_3=0}^{n-p_1-p_2}(-3)^{p_1}6^{p_2}3^{p_3}f\left(\mathbb{A}^n_{(p_1,p_2,p_3)}, x_{1,n+1} - x_{2,n+1}\right)\\
&- 3\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}\sum_{p_3=0}^{n-p_1-p_2}(-3)^{p_1}6^{p_2}3^{p_3}f\left(\mathbb{A}^n_{(p_1,p_2,p_3)}, x_{1,n+1}\right)\\
&+ 6\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}\sum_{p_3=0}^{n-p_1-p_2}(-3)^{p_1}6^{p_2}3^{p_3}f\left(\mathbb{A}^n_{(p_1,p_2,p_3)}, x_{2,n+1}\right)\\
&= \sum_{p_1=0}^{n+1}\sum_{p_2=0}^{n+1-p_1}\sum_{p_3=0}^{n+1-p_1-p_2}(-3)^{p_1}6^{p_2}3^{p_3}f\left(\mathbb{A}^{n+1}_{(p_1,p_2,p_3)}\right).
\end{aligned}$$

This means that (2.2) holds for $n + 1$.

Conversely, let $j \in \{1, \ldots, n\}$ be arbitrary and fixed. Set

$$f_j^*(z) := f\left(z_1, \ldots, z_{j-1}, z, z_{j+1}, \ldots, z_n\right).$$

Putting $x_{2k} = 0$ for all $k \in \{1, \ldots, n\}\backslash\{j\}$, $x_{2j} = w$ and $x_1 = (z_1, \ldots, z_{j-1}, z, z_{j+1}, \ldots, z_n)$ in (2.2), using Lemma 2.4, we get

$$\begin{aligned}
f_j^*(z+2w) &= \left[\sum_{p_1=0}^{n-1}\sum_{p_3=1}^{n-p_1}\binom{n-1}{p_1}\binom{n-1-p_1}{p_3-1}(-3)^{p_1}3^{p_3}\right]f_j^*(z+w)\\
&+ \left[\sum_{p_1=0}^{n-1}\sum_{p_3=0}^{n-1-p_1}\binom{n-1}{p_1}\binom{n-1-p_1}{p_3}(-3)^{p_1}3^{p_3}\right]f_j^*(z-w)\\
&+ \left[\sum_{p_1=1}^{n}\sum_{p_3=0}^{n-p_1}\binom{n-1}{p_1-1}\binom{n-p_1}{p_3}(-3)^{p_1}3^{p_3}\right]f_j^*(z)\\
&+ 6\left[\sum_{p_1=0}^{n-1}\sum_{p_3=0}^{n-1-p_1}\binom{n-1}{p_1}\binom{n-1-p_1}{p_3}(-3)^{p_1}3^{p_3}\right]f_j^*(w)
\end{aligned}$$

$$
\begin{aligned}
&= \left[\sum_{p_1=0}^{n-1} \binom{n-1}{p_1} (-3)^{p_1} \sum_{p_3=0}^{n-1-p_1} \binom{n-1-p_1}{p_3} 3^{p_3+1} \times 1^{n-1-p_3}\right] f_j^*(z+w)\\
&+ \left[\sum_{p_1=0}^{n-1} \binom{n-1}{p_1} (-3)^{p_1} \sum_{p_3=0}^{n-1-p_1} \binom{n-1-p_1}{p_3} 3^{p_3} \times 1^{n-1-p_3}\right] f_j^*(z-w)\\
&+ \left[\sum_{p_1=0}^{n-1} \binom{n-1}{p_1} (-3)^{p_1+1} \sum_{p_3=0}^{n-1-p_1} \binom{n-1-p_1}{p_3} 3^{p_3} \times 1^{n-1-p_3}\right] f_j^*(z)\\
&+ 6\left[\sum_{p_1=0}^{n-1} \binom{n-1}{p_1} (-3)^{p_1} \sum_{p_3=0}^{n-1-p_1} \binom{n-1-p_1}{p_3} 3^{p_3} \times 1^{n-1-p_3}\right] f_j^*(w)\\
&= 3\left[\sum_{p_1=0}^{n-1} \binom{n-1}{p_1} (-3)^{p_1} 4^{n-1-p_1}\right] f_j^*(z+w)\\
&+ \left[\sum_{p_1=0}^{n-1} \binom{n-1}{p_1} (-3)^{p_1} 3^{p_3} 4^{n-1-p_1}\right] f_j^*(z-w)\\
&- 3\left[\sum_{p_1=0}^{n-1} \binom{n-1}{p_1} (-3)^{p_1+1} 4^{n-1-p_1}\right] f_j^*(z)\\
&+ 6\left[\sum_{p_1=0}^{n-1} \binom{n-1}{p_1} (-3)^{p_1} 3^{p_3} 4^{n-1-p_1}\right] f_j^*(w)\\
&= 3(4-3)^{n-1} f_j^*(z+w) + (4-3)^{n-1} f_j^*(z-w) - 3(4-3)^{n-1} f_j^*(z) + 6(4-3)^{n-1} f_j^*(w)\\
&= 3f_j^*(z+w) + f_j^*(z-w) - 3f_j^*(z) + 6f_j^*(w).
\end{aligned}
$$

This finishes the proof. □

3 Stability of multi-cubic mappings

In this section, we prove the Hyers-Ulam stability of equation (2.2) in non-Archimedean normed spaces. The proof is based on a fixed point result that can be derived from [12, Theorem 1]. Before that, we bring some basic facts concerning non-Archimedean spaces and some preliminary results. Recall that a metric d on a nonempty set X is said to be *non-Archimedean* (or an *ultrametric*) provided $d(x,z) \leq \max\{d(x,y), d(y,z)\}$ for $x,y,z \in X$. By a non-Archimedean field we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ into $[0,\infty)$ such that

(i) $|a| = 0$ if and only if $a = 0$;
(ii) $|ab| = |a||b|$ for all $a, b \in \mathbb{K}$;
(iii) $|a+b| \leq \max\{|a|, |b|\}$ for all $a, b \in \mathbb{K}$.

It is clear that $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{Z}$. A trivial valuation on any field $\mathbb{K}$ is defined by the following for $a \in \mathbb{K}$

$$|a| := \begin{cases} 0 & a = 0,\\ 1 & a \neq 0. \end{cases}$$

For a nontrivial non-Archimedean valuation on $\mathbb{Q}$, assume that p is prime number. It is known that any non-zero rational number r can be uniquely written as $r = \frac{m}{n}p^s$, where $m, n, s \in \mathbb{Z}$ in which m and n are integers not divisible by p. It easily verified that the function $|\cdot|_p : \mathbb{Q} \longrightarrow [0,\infty)$ given through

$$|r|_p := \begin{cases} 0 & a = 0,\\ p^{-s} & a \neq 0, \end{cases}$$

is a nontrivial non-Archimedean valuation on $\mathbb{Q}$.

Let V be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : V \longrightarrow \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|ax\| = |a|\|x\|, \quad (x \in V, a \in \mathbb{K})$;
(iii) the strong triangle inequality (ultrametric); namely,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\} \qquad (x, y \in V).$$

Then, $(V, \|\cdot\|)$ is said to be a *non-Archimedean normed space*.

A sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space $\mathcal{X}$. Indeed, the above definition is taken from the fact that

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\|; m \leq j \leq n-1\} \quad (n \geq m).$$

A non-Archimedean normed space is complete if every Cauchy sequence is convergent. If $(V, \|\cdot\|)$ is a non-Archimedean normed space, then it is easy to check that the function $d_V : V \times V \longrightarrow \mathbb{R}_+$, defined via $d_V(x, y) := \|x - y\|$, is a non-Archimedean metric on V that is invariant (i.e., $d_V(x+z, y+z) = d_V(x, y)$ for $x, y, z \in X$). In other words, every non-Archimedean normed space is a special case of a metric space with invariant metrics.

The most interesting example of non-Archimedean normed spaces is p-adic numbers which have gained the interest of physicists because of their connections with some problems coming from quantum physics, p-adic strings and superstrings [23]. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: for all $x, y > 0$, there exists an integer n such that $x < ny$; for more details we refer to [20].

We recall that for a field $\mathbb{K}$ with multiplicative identity 1, the characteristic of $\mathbb{K}$ is the smallest positive number n such that $\overbrace{1 + \cdots + 1}^{n-times} = 0$.

Throughout, for two sets A and B, the set of all mappings from A to B is denoted by B^A. Here, we indicate the next theorem which is a fundamental result in fixed point theory [12, Theorem 1]. This result plays a key tool in obtaining our purpose in this paper.

Theorem 3.1. Let the following hypotheses hold.

(H1) E is a nonempty set, Y is a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2, $j \in \mathbb{N}$, $g_1, \ldots, g_j : E \longrightarrow E$ and $L_1, \ldots, L_j : E \longrightarrow \mathbb{R}_+$,
(H2) $\mathcal{T} : Y^E \longrightarrow Y^E$ is an operator satisfying the inequality

$$\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| \leq \max_{i \in \{1,\ldots,j\}} L_i(x) \|\lambda(g_i(x)) - \mu(g_i(x))\|,$$

for all $\lambda, \mu \in Y^E, x \in E$,
(H3) $\Lambda : \mathbb{R}_+^E \longrightarrow \mathbb{R}_+^E$ is an operator defined through

$$\Lambda\delta(x) := \max_{i \in \{1,\ldots,j\}} L_i(x)\delta(g_i(x)) \qquad \delta \in \mathbb{R}_+^E, x \in E.$$

Suppose that the function $\theta : E \longrightarrow \mathbb{R}_+$ and the mapping $\varphi : E \longrightarrow Y$ fulfill the following two conditions:

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \leq \theta(x), \quad \lim_{l \to \infty} \Lambda^l \theta(x) = 0 \qquad (x \in E).$$

Then, for every $x \in E$, the limit $\lim_{l\to\infty} \mathcal{T}^l \varphi(x) =: \psi(x)$ exists and the mapping $\psi \in Y^E$, defined in this way, is a fixed point of $\mathcal{T}$ with

$$\|\varphi(x) - \psi(x)\| \leq sup_{l \in \mathbb{N}_0} \Lambda^l \theta(x) \qquad (x \in E).$$

From now on, for the mapping $f : V^n \longrightarrow W$, we consider the difference operator $\mathbf{D}f : V^n \times V^n \longrightarrow W$ by

$$\mathbf{D}f(x_1, x_2) := f(x_1 + 2x_2) - \sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} \sum_{p_3=0}^{n-p_1-p_2} (-3)^{p_1} 6^{p_2} 3^{p_3} f\left(\mathbb{A}^n_{(p_1,p_2,p_3)}\right)$$

where $f\left(\mathbb{A}^n_{(p_1,p_2,p_3)}\right)$ is defined in (2.1).

In the sequel, all mappings $f : V^n \longrightarrow W$ are assumed that satisfy zero condition. With this assumption, we have the next stability result for functional equation (2.2).

Theorem 3.2. Let $\beta \in \{-1, 1\}$ be fixed. Let also V be a linear space and W be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2. Suppose that $\varphi : V^n \times V^n \longrightarrow \mathbb{R}_+$ is a mapping satisfying the equality

$$\lim_{l\to\infty} \left(\frac{1}{|8|^{n\beta}}\right)^l \varphi(2^{l\beta}x_1, 2^{l\beta}x_2) = 0, \tag{3.1}$$

for all $x_1, x_2 \in V^n$. Assume also $f : V^n \longrightarrow W$ is an odd mapping in each variable and satisfies the inequality

$$\|\mathbf{D}f(x_1, x_2)\| \le \varphi(x_1, x_2), \tag{3.2}$$

for all $x_1, x_2 \in V^n$. Then, there exists a unique multi-cubic mapping $\mathcal{C} : V^n \longrightarrow W$ such that

$$\|f(x) - \mathcal{C}(x)\| \le sup_{l\in\mathbb{N}_0} \frac{1}{|8|^{n\frac{\beta+1}{2}}} \left(\frac{1}{|8|^{n\beta}}\right)^l \varphi\left(0, 2^{l\beta+\frac{\beta-1}{2}}x\right), \tag{3.3}$$

for all $x \in V^n$.

Proof . Putting $x = 0$ and $x_2 = x_1 = x$ in (3.2), we have

$$\left\| f(2x) - \left[\sum_{p_2=0}^{n}\sum_{p_3=0}^{n-p_2} \binom{n}{p_2}\binom{n-p_2}{p_3} 6^{p_2}3^{p_3}(-1)^{n-p_2-p_3}\right] f(x)\right\| \le \varphi(0, x) \tag{3.4}$$

for all $x \in V^n$ (here and the rest of proof). A calculation shows that

$$\begin{aligned}
&\sum_{p_2=0}^{n}\sum_{p_3=0}^{n-p_2} \binom{n}{p_2}\binom{n-p_2}{p_3} 6^{p_2}3^{p_3}(-1)^{n-p_2-p_3}\\
&= \sum_{p_2=0}^{n} \binom{n}{p_2} 6^{p_2} \sum_{p_3=0}^{n-p_2} \binom{n-p_2}{p_3} 3^{p_3}(-1)^{n-p_2-p_3}\\
&= \sum_{p_2=0}^{n} \binom{n}{p_2} 6^{p_2}(3-1)^{n-p_2} = (6+2)^n = 8^n.
\end{aligned} \tag{3.5}$$

Relations (3.4) and (3.5) imply that

$$\|f(2x) - 8^n f(x)\| \le \varphi(0, x). \tag{3.6}$$

Set

$$\theta(x) := \frac{1}{|8|^{n\frac{\beta+1}{2}}}\varphi\left(0, 2^{\frac{\beta-1}{2}}x\right), \qquad \mathcal{T}\xi(x) := \frac{1}{8^{n\beta}}\xi(2^\beta x),$$

for all $\xi \in W^{V^n}$. Here, we rewrite (3.6) as follows:

$$\|f(x) - \mathcal{T}f(x)\| \le \theta(x). \tag{3.7}$$

For each $\eta \in \mathbb{R}_+^{V^n}$, we define $\Lambda\eta(x) := \frac{1}{|8|^{n\beta}}\eta(r^\beta x)$. It is easy to check that Λ has the form described in (H3) with $E = V^n$, $g_1(x) := 2^\beta x$ and $L_1(x) = \frac{1}{|8|^{n\beta}}$. Furthermore, for each $\lambda, \mu \in W^{V^n}$, we obtain

$$\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| = \left\|\frac{1}{8^{n\beta}}\lambda(2^\beta x) - \frac{1}{8^{n\beta}}\mu(2^\beta x)\right\| \le L_1(x)\,\|\lambda(g_1(x)) - \mu(g_1(x))\|.$$

It follows the above relation that the hypothesis (H2) hold. One can argue by induction on $l \in \mathbb{N}$ that

$$\Lambda^l\theta(x) := \left(\frac{1}{|8|^{n\beta}}\right)^l \theta(2^{l\beta}x) = \frac{1}{|8|^{n\frac{\beta+1}{2}}}\left(\frac{1}{|8|^{n\beta}}\right)^l \varphi\left(0, 2^{l\beta+\frac{\beta-1}{2}}\right). \tag{3.8}$$

Now, (3.7) and (3.8) necessitate that all assumptions of Theorem 3.1 hold and so there exists a unique mapping $\mathcal{C} : V^n \longrightarrow W$ such that $\mathcal{C}(x) = \lim_{l\to\infty}(\mathcal{T}^l f)(x)$ and (3.3) is valid as well. In addition, by induction on l, we can prove that

$$\|\mathbf{D}(\mathcal{T}^l f)(x_1, x_2)\| \leq \left(\frac{1}{|8|^{n\beta}}\right)^l \varphi(2^{l\beta}x_1, 2^{l\beta}x_2) \tag{3.9}$$

for all $x_1, x_2 \in V^n$. Letting $l \to \infty$ in (3.9) and applying (3.1), we achieve $\mathbf{D}\mathcal{C}(x_1, x_2) = 0$ for all $x_1, x_2 \in V^n$. This means that the mapping satisfies equation (2.2) and the proof is now completed. □

In what follows, it is assumed that the non-Archimedean field has the characteristic different from 2 and $|2| < 1$. In the next corollaries which are some direct applications of Theorem 3.2, V is a non-Archimedean normed space and W is a complete non-Archimedean normed space.

Corollary 3.3. Let $\delta > 0$. Suppose that $f : V^n \longrightarrow W$ is an odd mapping in each variable and satisfies the inequality

$$\|\mathbf{D}f(x_1, x_2)\| \leq \delta,$$

for all $x_1, x_2 \in V^n$. Then, there exists a unique multi-cubic mapping $\mathcal{C} : V^n \longrightarrow W$ such that

$$\|f(x) - \mathcal{C}(x)\| \leq \delta,$$

for all $x \in V^n$.

Proof . We firstly note that $|8| < 1$. Given $\varphi(x_1, x_2) = \delta$ in the case $\beta = -1$ of Theorem 3.2, we get $\lim_{l\to\infty} |8|^{nl}\delta = 0$. Therefore, one can obtain the desired result. □

Corollary 3.4. Let $\alpha \in \mathbb{R}$ fulfills $\alpha \neq 3n$. Suppose that $f : V^n \longrightarrow W$ is an odd mapping in each variable and satisfies the inequality

$$\|\mathbf{D}f(x_1, x_2)\| \leq \sum_{k=1}^{2}\sum_{j=1}^{n} \|x_{kj}\|^\alpha,$$

for all $x_1, x_2 \in V^n$. Then, there exists a unique multi-cubic mapping $\mathcal{C} : V^n \longrightarrow W$ such that

$$\|f(x) - \mathcal{C}(x)\| \leq \begin{cases} \frac{1}{|8|^n}\sum_{j=1}^{n} \|x_{1j}\|^\alpha, & \alpha > 3n, \\ \\ \frac{1}{|2|^\alpha}\sum_{j=1}^{n} \|x_{1j}\|^\alpha, & \alpha < 3n, \end{cases}$$

for all $x = x_1 \in V^n$.

Proof . Letting $\varphi(x_1, x_2) = \sum_{k=1}^{2}\sum_{j=1}^{n} \|x_{kj}\|^\alpha$, we have $\varphi(2^l x_1, 2^l x_2) = |8|^{lp}\varphi(x_1, x_2)$. It now follows from Theorem 3.2 the first and second inequalities in the cases $\beta = 1$ and $\beta = -1$, respectively. □

Recall that a functional equation $\mathcal{F}$ is *hyperstable* if any mapping f satisfying the equation $\mathcal{F}$ approximately is a true solution of $\mathcal{F}$. Under some conditions the functional equation (2.2) can be hyperstable as follows.

Corollary 3.5. Suppose that $\alpha_{kj} > 0$ for $k \in \{1, 2\}$ and $j \in \{1, \dots, n\}$ fulfill $\sum_{k=1}^{2}\sum_{j=1}^{n} \alpha_{kj} \neq 3n$. If $f : V^n \longrightarrow W$ is an odd mapping in each variable and satisfies the inequality

$$\|\mathbf{D}f(x_1, x_2)\| \leq \prod_{k=1}^{2}\prod_{j=1}^{n} \|x_{kj}\|^{\alpha_{kj}},$$

for all $x_1, x_2 \in V^n$, then f is multi-cubic.

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