Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 2347–2359
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.18349.2005

A monotone hybrid algorithm for a family of generalized nonexpansive mappings in Banach spaces

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(Communicated by Abasalt Bodaghi)

Abstract

In this paper, we propose a new monotone hybrid method for getting a common fixed point of a family of generalized nonexpansive mappings and prove a strong convergence theorem for this family in the framework of Banach spaces. Using this theorem, we obtain some new results for the class of generalized nonexpansive mappings and finitely many generalized nonexpansive mappings. Using the FMINCON optimization toolbox in MATLAB, we give a numerical example to illustrate the usability of our results.

Keywords: Monotone hybrid algorithm, Fixed point, Generalized nonexpansive mapping, Strong convergence, NST-condition
2020 MSC: 47H10, 47H09, 47J25, 47J05

1 Introduction

It is well known that many of the most important nonlinear problems of mathematics reduce to finding the fixed points of a certain operator which contractive type conditions naturally arise for many of these problems. Therefore, the methods for finding the fixed points of such mappings are fundamental subject in mathematics and so are interested by many mathematicians. Thus, many algorithms have been introduced by researchers such as Mann iteration process and Ishikawa iteration process [17, 10]. To reach the convergence in these methods, underlying space must be satisfied in suitable properties [22]. Moreover, it is well known that we can prove only weak convergence of generated sequences by the Mann iteration process even in Hilbert spaces [6]. Also, in Hilbert spaces, Ishikawa iteration process for a Lipschitz pseudocontractive mapping is convergent while the Mann process is not convergent. However, researchers use Mann process, since its formulation is simpler than the Ishikawa process.

Recently, to gain the weak or strong convergence, authors have used various iteration processes in the framework of Hilbert spaces and Banach spaces, see, [2, 3, 4, 11, 12, 13, 19, 23, 25]. Moreover, to get strong convergence, many researchers have been extensively used modified processes.

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Received: July 2019 *Accepted:* September 2021

Assume that E is a real Banach space with the dual space E^* and C is a nonempty closed convex subset of E. A self-mapping T of C is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad (x, y \in C).$$

The fixed points set of T is denoted by $F(T)$.

For a nonexpansive self-mapping T of a nonempty, closed convex subset C in a Hilbert space H, Nakajo and Takahashi [19] proposed the following modification of the Mann's iteration :

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_n = \{u \in C \ : \|y_n - u\| \leq \|x_n - u\|\}, \\ Q_n = \{u \in C \ : \langle x_n - u, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.1}$$

where P_K denotes the metric projection from H onto a closed convex subset K of H. Assuming bounded above from one for $\{\alpha_n\}$, they proved that the generated sequence $\{x_n\}$ is strongly convergent.

In 2006, the following modified Ishikawa iteration scheme has been introduced by Martinez-Yanes and Xu [18], for a nonexpansive self-mapping T of a nonempty, closed convex subset C with $F(T) \neq \emptyset$ in a Hilbert space H:

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ z_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T z_n, \\ C_n = \{u \in C \ : \|y_n - u\|^2 \leq \|x_n - u\|^2 \\ \qquad\qquad + (1 - \alpha_n)(\|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, u \rangle \geq 0)\}, \\ Q_n = \{u \in C \ : \langle x_n - u, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

assuming bounded above from one for $\{\alpha_n\}$ and $\lim_{n\to\infty} \beta_n = 1$, they proved that the generated sequence $\{x_n\}$ is strongly convergent to $P_{F(T)} x_0$.

In 2008, using the monotone hybrid method, Qin and Su [21] presented the following modification of iteration (1.1), for a nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_1 = x \in C, \ \ C_0 = Q_0 = C, \\ u_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_n = \{u \in C_{n-1} \cap Q_{n-1} \ : \|u - u_n\| \leq \|u - x_n\|\}, \\ Q_n = \{u \in C_{n-1} \cap Q_{n-1} \ : \langle x_n - u, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

for all $n \in \mathbb{N}$. Assuming suitable conditions on the sequence $\{\alpha_n\}$, they proved a strong convergence theorem.

Recently, Klin-eam, Suantai and Takahashi [15] presented a new monotone hybrid iterative method for a family of generalized nonexpansive mappings in a Banach space E:

$$\begin{cases} x_1 = x \in C, \ \ C_0 = Q_0 = C, \\ u_n = \alpha_n x_n + (1 - \alpha_n) T_n x_n, \\ C_n = \{u \in C_{n-1} \cap Q_{n-1} : \phi(u_n, u) \leq \phi(x_n, u)\}, \\ Q_n = \{u \in C_{n-1} \cap Q_{n-1} : \langle x - x_n, J x_n - J u \rangle \geq 0\}, \\ x_{n+1} = R_{C_n \cap Q_n} x, \end{cases} \tag{1.2}$$

for all $n \in \mathbb{N}$, where J is the duality mapping on E and $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\lim_{n\to\infty} (1 - \alpha_n) > 0$. They proved that strong convergence of the sequence $\{x_n\}$ generated by (1.2) under the condition that the family $\{T_n\}_{n=1}^\infty$ satisfies NST-condition.

In this paper, employing the idea of Klin-eam, Suantai and Takahashi [15], we present a new hybrid algorithm. The ultimate goals of the paper are getting a common fixed point of a countable family of generalized nonexpansive mappings and proving a strong convergence theorem in a Banach space. We obtain some new results for a generalized nonexpansive mapping and finitely manay generalized nonexpansive mappings.

2 Preliminaries

Suppose that E^* is the dual of a real Banach space E. The strong convergence and the weak convergence of a sequence $\{x_k\}$ to x in E will be denoted by $x_k \to x$ and $x_k \rightharpoonup x$, respectively.

A Banach space E is strictly convex if $\|\frac{x+y}{2}\| < 1$, whenever $x, y \in S(E)$, $x \neq y$ and $S(E)$ is the unite sphere centered at the origin of E. The modulus of convexity of E is defined by

$$\delta_E(\epsilon) = \inf\{1 - \frac{1}{2}\|(x+y)\| : \ \|x\|, \|y\| \leq 1, \ \|x-y\| \geq \epsilon\}$$

for all $\epsilon \in [0,2]$. Also, E is said to be uniformly convex if $\delta_E(0) = 0$ and $\delta_E(\epsilon) > 0$ for all $0 < \epsilon \leq 2$. The Banach space E is called smooth if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}, \tag{2.1}$$

exists for all $x, y \in S(E)$. The modulus of smoothness of E is defined by

$$\rho_E(t) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 \ : \ x \in S(E), \ \|y\| \leq t\}.$$

If $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$, then E is called uniformly smooth. Also if for every sequence $\{x_n\}$ in E that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ eventuate $x_n \to x$, then E satisfies in the Kadec–Klee property. It is worth noting that uniformly convexity of E implies that it satisfies in the Kadec–Klee property. Also, uniformly convexity of E implies that E^* is uniformly smooth and vise versa [1, 24].

The mapping J from E to 2^{E^*} defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \ \|x^*\| = \|x\|\} \quad \forall \ x \in E.$$

is called the normalized duality mapping. If E is uniformly convex and uniformly smooth, then J is uniformly norm-to-norm continuous on bounded sets of E. Many properties of J have been given in [1, 24].

Assume that E is a smooth Banach space, we define the function $\phi : E \times E \to \mathbb{R}$ by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2,$$

for all $x, y \in E$. Observe that, in a framework of Hilbert spaces, $\phi(x, y) = \|x - y\|^2$. It is clear that for all $x, y, z \in E$,

(A1) $(\|y\| - \|x\|)^2 \leq \phi(x, y) \leq (\|y\| + \|x\|)^2$,
(A2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$,
(A3) $\phi(x, y) = \langle x, Jx - Jy \rangle + \langle x - y, Jy \rangle \leq \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$.

A mapping $T : C \to E$ is called generalized nonexpansive whenever $F(T) \neq \emptyset$ and $\phi(Tx, p) \leq \phi(x, p)$ for all $x \in C$ and $p \in F(T)$, where C is a closed subset of a Banach space E. Also a mapping T in E is called closed if $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

Assume that D is a nonempty subset of a Banach space E. A mapping $R : E \to D$ is called sunny [9] if

$$R(Rx + t(x - Rx)) = Rx,$$

for all $x \in E$ and all $t \geq 0$. It is also called a retraction if $Rx = x$ for all $x \in D$. The retraction R is called a sunny nonexpansive retraction from E onto D if it is a retraction which is also sunny and nonexpansive. Let D be a nonempty subset of a smooth Banach space E. If there exists a sunny generalized nonexpansive retraction R from E onto D, then D is said to be a sunny generalized nonexpansive retract of E. More information on these retractions can be found in [7].

3 NST-CONDITION

Assume that E is a real Banach space and C is a closed subset of E. Suppose that $\{T_n\}$ and $\mathcal{F}$ are two families of the generalized nonexpansive mappings of C into E such that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{F}) \neq \emptyset$, where $F(\mathcal{F})$ is the set of all common fixed points of $\mathcal{F}$.

The sequence $\{T_n\}$ satisfies in NST-condition [20] with $\mathcal{F}$ if

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0 \Rightarrow \lim_{n\to\infty} \|x_n - T x_n\| = 0$$

for all $T \in \mathcal{F}$ and all bounded sequence $\{x_n\}$ in C.

If $\mathcal{F}$ possess one element, i.e., $\mathcal{F} = \{T\}$, then $\{T_n\}$ satisfies the NST-condition with $\{T\}$. If we put $T_n = T$ for all $\mathbb{N}$, then $\{T_n\}$ satisfies the NST-condition with $\{T\}$.

Recently, Klin- eam et al. [15] introduced two countable families of generalized nonexpansive mappings satisfying the NST condition (see Lemmas 3.1 and 3.2 of [15]).

Now, we introduce a new countable family of generalized nonexpansive mappings which satisfies in NST condition.

Lemma 3.1. Assume that E is a uniformly smooth and uniformly convex Banach space, C is a subset of E and $S_1, S_2, \cdots, S_N$ are generalized nonexpansive mappings of C into E such that $\bigcap_{k=1}^{N} F(S_k) \neq \emptyset$. Assume that $\{\lambda_{1n}\}, \{\lambda_{2n}\}, \cdots \{\lambda_{2N}\}$ are sequences in $[0, 1]$ satisfying:

(i) $\sum_{k=1}^{N} \lambda_{kn} = 1$ for all $n \in \mathbb{N}$;
(ii) $\liminf_{n\to\infty} \lambda_{in}\lambda_{jn} > 0$, for all $i, j \in \{1, 2, \cdots, N\}$ with $i < j$.

Suppose that for each $n \in \mathbb{N}$, the mapping $T_n : C \to E$ is defined by

$$T_n x = \sum_{k=1}^{N} \lambda_{kn} S_k x, \quad \forall x \in C.$$

Then, the countable family of generalized nonexpansive mappings $\{T_n\}$ satisfies NST-condition with $\mathcal{F} = \{S_1, S_2, \cdots, S_N\}$.

Proof . At first, we show that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{F})$ and T_n are generalized nonexpansive mappings for all $n \in \mathbb{N}$. We know that

$$F(\mathcal{F}) = \bigcap_{k=1}^{N} F(S_k) \subset \bigcap_{n=1}^{\infty} F(T_n).$$

Put $p \in \bigcap_{k=1}^{N} F(S_k)$. Applying Lemma 2.5 of [16] (or Proposition 4.2 of [8]), we get

$$\begin{aligned}
\phi(T_n x, p) &= \phi\Big(\sum_{k=1}^{N} \lambda_{kn} S_k x, p\Big) \\
&= \Big\|\sum_{k=1}^{N} \lambda_{kn} S_k x\Big\|^2 - 2\Big\langle \sum_{k=1}^{N} \lambda_{kn} S_k, Jp\Big\rangle + \|p\|^2 \\
&\leq \sum_{k=1}^{N} \lambda_{kn} \|S_k x\|^2 - 2\sum_{k=1}^{N} \lambda_{kn} \langle S_k x, Jp\rangle + \|p\|^2 \\
&= \sum_{n=1}^{N} \lambda_{nk} \phi(S_k x, p) \\
&\leq \sum_{n=1}^{N} \lambda_{nk} \phi(x, p) \\
&= \phi(x, p).
\end{aligned}$$

So, for all $q \in F(T_n)$ we obtain

$$\begin{aligned}
\phi(q,p) = \phi(T_n q,p) &= \phi\Big(\sum_{k=1}^{N} \lambda_{kn} S_k q, p\Big) \\
&= \Big\|\sum_{k=1}^{N} \lambda_{kn} S_k q\Big\|^2 - 2\Big\langle \sum_{k=1}^{N} \lambda_{kn} S_k, Jp\Big\rangle + \|p\|^2 \\
&\le \sum_{k=1}^{N} \lambda_{kn} \|S_k q\|^2 - 2\sum_{k=1}^{N} \lambda_{kn} \langle S_k q, Jp\rangle + \|p\|^2 \\
&= \sum_{n=1}^{N} \lambda_{nk} \phi(S_k q,p) \\
&\le \sum_{n=1}^{N} \lambda_{nk} \phi(q,p) \\
&= \phi(q,p),
\end{aligned}$$

that is,

$$\phi\Big(\sum_{k=1}^{N} \lambda_{kn} S_k q, p\Big) = \sum_{n=1}^{N} \lambda_{nk} \phi(S_k q,p) = \phi(q,p).$$

Using Lemma 3.1 of [9], we get $q = T_n q = S_1 q = S_2 q = \cdots = S_N q$. Then $F(T_n) \subset \bigcap_{k=1}^{N} F(S_k)$ for all $n \in \mathbb{N}$. Therefore $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{F})$.

Now, we prove that $\{T_n\}$ satisfies in NST-condition with $\{S_1, S_2, \cdots, S_N\}$. For this purpose, presume that $\{x_n\}$ is an arbitrary bounded sequence in C such that $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$. It follows from Lemma 2.5 of [16] that there exists a continuous, strictly increasing and convex function $g : [0, 2r) \to [0,\infty)$ such that $g(0) = 0$ and

$$\begin{aligned}
\phi(T_n x_n,p) &= \phi\Big(\sum_{k=1}^{N} \lambda_{kn} S_k x_n, p\Big) \\
&= \Big\|\sum_{k=1}^{N} \lambda_{kn} S_k x_n\Big\|^2 - 2\Big\langle \sum_{k=1}^{N} \lambda_{kn} S_k x_n, Jp\Big\rangle + \|p\|^2 \\
&\le \sum_{k=1}^{N} \lambda_{kn} \|S_k x_n\|^2 - 2\sum_{k=1}^{N} \lambda_{kn} \langle S_k x_n, Jp\rangle + \|p\|^2 \\
&\quad - \lambda_{in}\lambda_{jn} g\Big(\|S_i x_n - S_j x_n\|\Big) \\
&= \sum_{n=1}^{N} \lambda_{nk} \phi(S_k x_n,p) - \lambda_{in}\lambda_{jn} g\Big(\|S_i x_n - S_j x_n\|\Big) \\
&\le \sum_{n=1}^{N} \lambda_{nk} \phi(x_n,p) - \lambda_{in}\lambda_{jn} g\Big(\|S_i x_n - S_j x_n\|\Big) \\
&= \phi(x_n,p) - \lambda_{in}\lambda_{jn} g\Big(\|S_i x_n - S_j x_n\|\Big),
\end{aligned}$$

for all $p \in \bigcap_{n=1}^{\infty} F(T_n)$. Therefore

$$\lambda_{in}\lambda_{jn} g(\|S_i x_n - S_j x_n\|) \le \phi(x_n,p) - \phi(T_n x_n,p). \tag{3.1}$$

Assume that $\{\|S_1 x_{n_k} - S_2 x_{n_k}\|\}$ is an arbitrary subsequence of $\{\|S_1 x_n - S_2 x_n\|\}$. Boundedness of $\{x_{n_k}\}$ implies that there exists a subsequence $\{x_{n'_j}\}$ of $\{x_{n_k}\}$ such that

$$\lim_{j\to\infty} \phi(x_{n'_j},p) = \limsup_{k\to\infty} \phi(x_{n_k},p) = a.$$

Using properties (A2) and (A3) of ϕ, we conclude that

$$\begin{aligned}\phi(x_{n'_j},p) &= \phi(x_{n'_j},T_{n'_j}x_{n'_j}) + \phi(T_{n'_j}x_{n'_j},p) + 2\langle x_{n'_j} - T_{n'_j}x_{n'_j}, JT_{n'_j}x_{n'_j} - Jp\rangle \\ &\leq \phi(T_{n'_j}x_{n'_j},p) + \|x_{n'_j}\|\|Jx_{n'_j} - JT_{n'_j}x_{n'_j}\| \\ &\quad + \|T_{n'_j}x_{n'_j} - x_{n'_j}\|\|T_{n'_j}x_{n'_j}\| + 2\|x_{n'_j} - T_{n'_j}x_{n'_j}\|\|JT_{n'_j}x_{n'_j} - Jp\|.\end{aligned} \tag{3.2}$$

Utilizing the uniformly norm to norm continuity of J on bounded sets, we can derive from $\lim_{n\to\infty}\|x_n - T_nx_n\| = 0$ that $\lim_{n\to\infty}\|Jx_n - JT_nx_n\| = 0$. Taking the limit inferier on both sides of (3.2), we obtain

$$a = \liminf_{j\to\infty}\phi(x_{n'_j},p) \leq \liminf_{j\to\infty}\phi(T_{n'_j}x_{n'_j},p).$$

Moreover, for all $n \in \mathbb{N}$, T_n is generalized nonexpansive, thus

$$\limsup_{j\to\infty}\phi(T_{n'_j}x_{n'_j},p) \leq \limsup_{j\to\infty}\phi(x_{n'_j},p) = a.$$

Thus, we can obtain from (3.1) that

$$\lim_{n\to\infty} g(\|S_1x_{n'_j} - S_2x_{n'_j}\|) = 0,$$

since $\liminf_{n\to\infty}\lambda_{1n}\lambda_{2n} > 0$. Therefore, $\lim_{n\to\infty}\|S_1x_{n'_j} - S_2x_{n'_j}\| = 0$ and hence $\lim_{n\to\infty}\|S_1x_n - S_2x_n\| = 0$. Using the similar method, we can prove that $\lim_{n\to\infty}\|S_1x_n - S_jx_n\| = 0$ for all $j = 3, 4, \cdots, N$. Since

$$\begin{aligned}\|x_n - S_1x_n\| &\leq \|x_n - T_nx_n\| + \|T_nx_n - S_1x_n\| \\ &\leq \|x_n - T_nx_n\| + \sum_{k=2}^{N}\lambda_{kn}\|S_1x_n - S_kx_n\|,\end{aligned}$$

we yield that $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$. Similarly, we have $\lim_{n\to\infty}\|x_n - S_jx_n\| = 0$ for all $j = 2, 3, \cdots, N$. □

4 Main Results

Now, using monotone hybrid method, we establish a strong convergence theorem for a family of non-self generalized nonexpansive mappings in Banach spaces.

Throughout this section, we assume that E is a uniformly smooth and uniformly convex Banach space, C is a nonempty closed convex subset of E and JC is closed convex. Also, we suppose that for each $n \in \mathbb{N}$, the mapping $T_n : C \to E$ is generalized nonexpansive and $\{T_n\}$ is a countable family of such mappings. Also we suppose that $\mathcal{F}$ is a family of closed generalized nonexpansive mappings from C into E such that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{F}) \neq \emptyset$. Furthermore, we assume that $\{T_n\}$ satisfies in NST-condition with $\mathcal{F}$. Moreover, sunny generalized nonexpansive retraction from E onto D will be denoted by R_D, where D is a nonempty subset of E.

Theorem 4.1. Suppose that for each $n \in \mathbb{N}$, x_n is generated as follows:

$$\begin{cases} x_1 = x \in C, \ \ C_0 = Q_0 = C, \\ u_n = \theta_nx_n + (1-\theta_n)T_nx_n, \\ y_n = \beta_nu_n + (1-\beta_n)T_nx_n, \\ C_n = \{u \in C_{n-1}\cap Q_{n-1} : \phi(y_n,u) \leq \phi(x_n,u)\}, \\ Q_n = \{u \in C_{n-1}\cap Q_{n-1} : \langle x - x_n, Jx_n - Ju\rangle \geq 0\}, \\ x_{n+1} = R_{C_n\cap Q_n}x, \end{cases}$$

where the sequences $\{\theta_n\}$ and $\{\beta_n\}$ are in $[0,1]$ such that $\liminf_{n\to\infty}(1-\theta_n) > 0$ and $\lim_{n\to\infty}\beta_n = 1$. Then, $\{x_n\}$ is strongly convergent to $R_{F(\mathcal{F})}x$.

Proof . At first, we prove that for each $n \in \mathbb{N}$, JC_n and JQ_n are closed convex. Using the definition of C_n and Q_n, we can easily obtain the closedness of JC_n and the closedness and convexity of JQ_n for all $n \in \mathbb{N}$. The condition of C_n implies that

$$\|x_n\|^2 - \|y_n\|^2 - 2\langle x_n - y_n, Ju\rangle \geq 0,$$

for all $u \in JC_n$, so $u \in JC_n$ is convex. Let $p \in F(\mathcal{F})$. Since $\{T_n\}$ are generalized nonexpansive mappings for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned} \phi(u_n, p) &= \phi(\theta_n x_n + (1-\theta_n)T_n x_n, p) \\ &\leq \theta_n\|x_n\|^2 + (1-\theta_n)\|T_n x_n\|^2 - 2\theta_n\langle x_n, Jp\rangle - 2(1-\theta_n)\langle T_n x_n, Jp\rangle + \|p\|^2 \\ &\leq \theta_n\phi(x_n, p) + (1-\theta_n)\phi(x_n, p) \\ &= \phi(x_n, p) \end{aligned} \tag{4.1}$$

and therefore

$$\begin{aligned} \phi(y_n, p) &= \phi(\beta_n u_n + (1-\beta_n)T_n x_n, p) \\ &= \|\beta_n u_n + (1-\beta_n)T_n x_n\|^2 - 2\langle \beta_n u_n + (1-\beta_n)T_n x_n, Jp\rangle + \|p\|^2 \\ &\leq \beta_n\|u_n\|^2 + (1-\beta_n)\|T_n x_n\|^2 - 2\beta_n\langle u_n, Jp\rangle - 2(1-\beta_n)\langle T_n x_n, Jp\rangle + \|p\|^2 \\ &= \beta_n\phi(u_n, p) + (1-\beta_n)\phi(T_n x_n, p) \\ &\leq \beta_n\phi(x_n, p) + (1-\beta_n)\phi(x_n, p) \\ &= \phi(x_n, p), \end{aligned} \tag{4.2}$$

thus $p \in C_n$ for all $n \in \mathbb{N}$. Hence,

$$F(\mathcal{F}) \subset C_n.$$

Now, we will prove that $F(\mathcal{F}) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, $J(C_n \cap Q_n) = JC_n \cap JQ_n$ is closed convex, since J is one-to-one . Using Lemma 2.10 of [26], we can conclude that $C_n \cap Q_n$ is a sunny generalized nonexpansive retract of E.

It is easy to see that $F(\mathcal{F}) \subset C = C_0 \cap Q_0$. Assume that $F(\mathcal{F}) \subset C_{n-1} \cap Q_{n-1}$ for some $n \in \mathbb{N}$. Since $x_n = R_{C_{n-1}\cap Q_{n-1}}x$, utilizing Proposition 4.2 of [8], we get

$$\langle x - x_n, Jx_n - Ju\rangle \geq 0, \tag{4.3}$$

for all $u \in C_{n-1} \cap Q_{n-1}$. Using the induction assumption, we can derive that for all $u \in C_{n-1} \cap Q_{n-1}$ the inequality (4.3) is satisfied. Furthermore, the definition of Q_n implies that $F(\mathcal{F}) \subset Q_n$. Therefore, $F(\mathcal{F}) \subset C_n \cap Q_n$. So $\{x_n\}$ is well-defined.

On the other hand, by the definition of Q_n we can conclude that $x_n = R_{Q_n}x$. Using Proposition 4.2 of [8], we obtain

$$\phi(x, x_n) = \phi(x, R_{Q_n}x) \leq \phi(x, u) - \phi(R_{Q_n}x, u) \leq \phi(x, u), \tag{4.4}$$

for all $u \in F(\mathcal{F}) \subset Q_n$, i.e., $\{\phi(x, x_n)\}$ is bounded. Thus $\{x_n\}$, $\{u_n\}$ and $\{T_n x_n\}$ are also bounded.

Using the definition of R_{Q_n}, we get

$$\phi(x, x_n) \leq \phi(x, x_{n+1}), \tag{4.5}$$

for all $n \in \mathbb{N}$, because of $x_{n+1} = R_{C_n\cap Q_n}x \in C_n \cap Q_n \subset Q_n$ and $x_n = R_{Q_n}x$. So, $\lim_{n\to\infty}\{\phi(x, x_n)\}$ exists. Using Proposition 4.2 of [8] and $x_n = R_{Q_n}x$, we derive that for every positive integer k and for each $n \in \mathbb{N}$,

$$\begin{aligned} \phi(x_n, x_{n+1}) &= \phi(R_{Q_n}x, x_{n+k}) \\ &\leq \phi(x, x_{n+k}) - \phi(x, R_{Q_n}x) \\ &= \phi(x, x_{n+k}) - \phi(x, x_n), \end{aligned} \tag{4.6}$$

thus

$$\lim_{n\to\infty} \phi(x_n, x_{n+k}) = 0. \tag{4.7}$$

Utilizing Corollary 4.19 of [5], there exists a strictly increasing, convex and continuous function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ and for $m, n \in \mathbb{N}$ with $m > n$,

$$g(\|x_n - x_m\|) \leq \phi(x_n, x_m) \leq \phi(x, x_m) - \phi(x, x_n).$$

The properties of g implies that $\{x_n\}$ is a Cauchy sequence. Hence there exists $v \in C$ so that $x_n \to v$. Due to $x_{n+1} = R_{C_n \cap Q_n} x \in C_n$ and the definition of C_n, we obtain

$$\phi(x_n, x_{n+1}) - \phi(y_n, x_{n+1}) \geq 0, \quad \forall n \in \mathbb{N}, \tag{4.8}$$

Using (4.7) and (4.8), we conclude that

$$\lim_{n\to\infty} \phi(y_n, x_{n+1}) = \phi(x_n, x_{n+1}) = 0. \tag{4.9}$$

Utilizing uniformly convexity and smoothness of E the equality, (4.9) and Proposition 2 of [14], we have

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = \lim_{n\to\infty} \|y_n - x_{n+1}\| = 0. \tag{4.10}$$

So, we obtain

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Since $\{u_n\}$ and $\{T_n x_n\}$ are bounded and $\beta_n \to 1$, we deduce that

$$\|u_n - y_n\| = (1-\beta_n)\|u_n - T_n x_n\| \to 0$$

and hence

$$\|u_n - x_{n+1}\| \leq \|u_n - y_n\| + \|y_n - x_{n+1}\| \to 0.$$

Moreover,

$$\begin{aligned} \|x_{n+1} - u_n\| &= \|x_{n+1} - \theta_n u_n - (1-\theta_n) T_n x_n\| \\ &= \|(1-\theta_n)(x_{n+1} - T_n x_n) - \theta_n (x_n - x_{n+1})\| \\ &\geq (1-\theta_n)\|x_{n+1} - T_n x_n\| - \theta_n \|x_n - x_{n+1}\|. \end{aligned} \tag{4.11}$$

This yields that

$$\|x_{n+1} - T_n x_n\| \leq \frac{1}{1-\theta_n}(\|x_{n+1} - u_n\| + \theta_n \|x_n - x_{n+1}\|). \tag{4.12}$$

Due to $\liminf_{n\to\infty}(1-\theta_n) > 0$, we can conclude from (4.10), (4.11) and (4.12) that

$$\lim_{n\to\infty} \|x_{n+1} - T_n x_n\| = 0. \tag{4.13}$$

From

$$\|x_n - T_n x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_n x_n\|.$$

and (4.10) and (4.13), we obtain

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0.$$

So

$$\lim_{n\to\infty} \|x_n - T x_n\| = 0, \quad \forall T \in \mathcal{F},$$

because $\{T_n\}$ satisfies the NST-condition with $\mathcal{F}$. Moreover, $x_n \to v$ and T is closed, so we can deduce that v is a fixed point of T. Utilizing Proposition 4.2 of [8], we get

$$\phi(x, R_{F(\mathcal{F})} x) \leq \phi(x, R_{F(\mathcal{F})} x) + \phi(R_{F(\mathcal{F})} x, v) \leq \phi(x, v).$$

Using Proposition 4.2 of [8], $x_{n+1} = R_{C_n \cap Q_n} x$ and $v \in F(\mathcal{F}) \subset C_n \cap Q_n$, we can derive that

$$\phi(x, x_{n+1}) \leq \phi(x, x_{n+1}) + \phi(x_{n+1}, R_{F(\mathcal{F})} x) \leq \phi(x, R_{F(\mathcal{F})} x).$$

Therefore $\phi(x, v) \leq \phi(x, R_{F(\mathcal{F})} x)$, due to $x_n \to v$. Thus $\phi(x, v) = \phi(x, R_{F(\mathcal{F})} x)$. Hence, since $R_{F(\mathcal{F})} x$ is unique, we conclude that $v = R_{F(\mathcal{F})} x$. $\square$

Corollary 4.2. Assume that $T : C \to E$ is a closed generalized nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that for each $n \in \mathbb{N}$, x_n is generated as follows:

$$\begin{cases} x_1 = x \in C, \;\; C_0 = Q_0 = C, \\ u_n = \theta_n x_n + (1-\theta_n)Tx_n, \\ y_n = \beta_n u_n + (1-\beta_n)Tx_n, \\ C_n = \{u \in C_{n-1} \cap Q_{n-1} : \phi(y_n, u) \leq \phi(x_n, u)\}, \\ Q_n = \{u \in C_{n-1} \cap Q_{n-1} : \langle x - x_n, Jx_n - Ju \rangle \geq 0\}, \\ x_{n+1} = R_{C_n \cap Q_n} x, \end{cases}$$

where the sequences $\{\theta_n\}$ and $\{\beta_n\}$ are in $[0, 1]$ such that $\liminf_{n\to\infty}(1-\theta_n) > 0$ and $\lim_{n\to\infty} \beta_n = 1$. Then, $\{x_n\}$ is strongly convergent to $R_{F(T)}x$.

Proof . By letting $T_n = T$ for all $n \in \mathbb{N}$, the result can be obtained from Theorem 4.1. □

Corollary 4.3. Assume that $T : C \longrightarrow E$ is a closed generalized nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that for each $n \in \mathbb{N}$, x_n is generated as follows:

$$\begin{cases} x_1 = x \in C, \;\; C_0 = Q_0 = C, \\ u_n = \theta_n x_n + (1-\theta_n)(\gamma_n x_n + (1-\gamma_n)Tx_n), \\ y_n = \beta_n u_n + (1-\beta_n)(\gamma_n x_n + (1-\gamma_n)Tx_n), \\ C_n = \{u \in C_{n-1} \cap Q_{n-1} : \phi(y_n, u) \leq \phi(x_n, u)\}, \\ Q_n = \{u \in C_{n-1} \cap Q_{n-1} : \langle x - x_n, Jx_n - Ju \rangle \geq 0\}, \\ x_{n+1} = R_{C_n \cap Q_n} x, \end{cases}$$

where the sequences $\{\theta_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are in $[0, 1]$ such that $\liminf_{n\to\infty}(1-\theta_n) > 0$, $\lim_{n\to\infty} \beta_n = 1$ and $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$. Then, $\{x_n\}$ is strongly convergent to $R_{F(T)}x$.

Proof . Defining $T_n x = \gamma_n x + (1-\gamma_n)Tx$ for all $n \in \mathbb{N}$ and $x \in C$ and using Lemma 3.1 of [15], the result can be obtained from Theorem 4.1. □

Corollary 4.4. Assume that $S_1, S_2 : C \longrightarrow E$ are closed generalized nonexpansive mappings with $F(S_1) \cap F(S_2) \neq \emptyset$. Suppose that for each $n \in \mathbb{N}$, x_n is generated as follows:

$$\begin{cases} x_1 = x \in C, \;\; C_0 = Q_0 = C, \\ u_n = \theta_n x_n + (1-\theta_n)(\gamma_n S_1 x_n + \lambda_n S_2 x_n), \\ y_n = \beta_n u_n + (1-\beta_n)(\gamma_n S_1 x_n + \lambda_n S_2 x_n), \\ C_n = \{u \in C_{n-1} \cap Q_{n-1} : \phi(y_n, u) \leq \phi(x_n, u)\}, \\ Q_n = \{u \in C_{n-1} \cap Q_{n-1} : \langle x - x_n, Jx_n - Ju \rangle \geq 0\}, \\ x_{n+1} = R_{C_n \cap Q_n} x, \end{cases}$$

where the sequences $\{\theta_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\lambda_n\}$ are in $[0, 1]$ and the following conditions hold:

(i) $\liminf_{n\to\infty}(1-\theta_n) > 0$;
(ii) $\lim_{n\to\infty} \beta_n = 1$;
(iii) $\gamma_n + \lambda_n = 1$;
(iv) $\liminf_{n\to\infty} \gamma_n \lambda_n > 0$.

Then, $\{x_n\}$ is strongly convergent to $R_{F(S_1)\cap F(S_2)}x$.

Proof . By defining $T_n x = \gamma_n S_1 x + \lambda_n S_2 x$ for all $n \in \mathbb{N}$ and $x \in C$ and utilizing Lemma 3.2 of [15], the desired result can be obtained from Theorem 4.1. □

Corollary 4.5. Assume that $S_1, S_2, \cdots, S_N : C \longrightarrow E$ are closed generalized nonexpansive mappings with $\bigcap_{k=1}^{N} F(S_k) \neq \emptyset$. Suppose that for each $n \in \mathbb{N}$, x_n is generated as follows:

$$\begin{cases} x_1 = x \in C, \;\; C_0 = Q_0 = C, \\ u_n = \theta_n x_n + (1-\theta_n)\Big(\sum_{k=1}^{N} \lambda_{kn} S_k x_n\Big), \\ y_n = \beta_n u_n + (1-\beta_n)\Big(\sum_{k=1}^{N} \lambda_{kn} S_k x_n\Big), \\ C_n = \{u \in C_{n-1} \cap Q_{n-1} : \phi(y_n, u) \leq \phi(x_n, u)\}, \\ Q_n = \{u \in C_{n-1} \cap Q_{n-1} : \langle x - x_n, Jx_n - Ju\rangle \geq 0\}, \\ x_{n+1} = R_{C_n \cap Q_n} x, \end{cases}$$

where the sequences $\{\theta_n\}, \{\beta_n\}, \{\lambda_{1n}\}, \{\lambda_{2n}\}, \cdots, \{\lambda_{1N}\}$ are in $[0, 1]$ and the following conditions hold:

(i) $\liminf_{n\to\infty}(1-\theta_n) > 0$;
(ii) $\lim_{n\to\infty} \beta_n = 1$;
(iii) $\sum_{k=1}^{N} \lambda_{kn} = 1$;
(iv) $\liminf_{n\to\infty} \lambda_{in}\lambda_{jn} > 0$ for all $i, j \in 1, 2, \cdots, N$ with $i < j$.

Then, $\{x_n\}$ is strongly convergent to $R_{\bigcap_{k=1}^{N} F(S_k)} x$.

Proof . By defining $T_n x = \sum_{k=1}^{N} \lambda_{kn} S_k x$ for all $n \in \mathbb{N}$ and $x \in C$ and using Lemma 3.1, the result can be obtained from Theorem 4.1. □

Remark 4.6. By letting $\beta_n = 1$ for all $n \in \mathbb{N}$ in Theorem 4.1, we can deduce Theorem 4.3 of Klin-eam et al. [15]. This means the main result of Klin-eam et al. [15] is a special case of our Theorem 4.1.

The following theorem is a result of our main theorem in the framework of Hilbert spaces.

Theorem 4.7. Assume that C is a nonempty, closed and convex subset of a Hilbert space H. Suppose that $\{T_n\}$ and $\mathcal{F}$ are a family of nonexpansive mappings of C into H such that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{F}) \neq \emptyset$. Assume that NST-condition with $\mathcal{F}$ holds for $\{T_n\}$. Suppose that for each $n \in \mathbb{N}$, x_n is generated as follows:

$$\begin{cases} x_1 = x \in C, \;\; C_0 = Q_0 = C, \\ u_n = \theta_n x_n + (1-\theta_n) T_n x_n, \\ y_n = \beta_n u_n + (1-\beta_n) T_n x_n, \\ C_n = \{u \in C_{n-1} \cap Q_{n-1} : \|y_n - u\| \leq \|x_n - u\|\}, \\ Q_n = \{u \in C_{n-1} \cap Q_{n-1} : \langle x - x_n, x_n - u\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \end{cases}$$

where the sequences $\{\theta_n\}$ and $\{\beta_n\}$ are in $[0, 1]$ such that $\liminf_{n\to\infty}(1-\theta_n) > 0$ and $\lim_{n\to\infty} \beta_n = 1$. Then, $\{x_n\}$ is strongly convergent to $P_{F(\mathcal{F})} x$, where $P_{F(\mathcal{F})}$ is the metric projection from C onto $F(\mathcal{F})$.

Proof . Since, in a Hilbert space, $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$ and J is the identity mapping and a nonexpansive mapping $T : C \to H$ with a fixed point is also a generalized nonexpansive mapping, Theorem 4.1 gives the desired result. □

5 Numerical example

Now, to illustrate our algorithm which is given in Theorem 4.1, we give a numerical example.

Example 5.1. Assume that $E=\mathbb{R}$ and $C=[-2,2]$. Define $S_1,S_2:C\to E$ by $S_1x=\frac{1}{6}x, S_2x=\frac{1}{37}x$ for all $x\in C$, then $F(S_1)=F(S_2)=\{0\}$ and

$$\phi(0,S_1x)=\phi\left(0,\frac{1}{26}x\right)=\left|0-\frac{1}{26}x\right|^2\leq|x|^2=\phi(0,x),$$

for all $x\in C$. So, S_1 is closed generalized nonexpansive mapping. It is readily seen that S_2 is also a closed generalized nonexpansive mapping. Define $\theta_n=\frac{1}{7}+\frac{1}{4+n}$, $\beta_n=1-\frac{1}{8+n}$, $\gamma_n=\frac{2}{3}-\frac{1}{3+n}$ and $\lambda_n=\frac{1}{3}+\frac{1}{3+n}$, hence $\{\theta_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ satisfy the conditions Theorem 4.1. Define $T_nx=\gamma_nS_1x+\lambda_nS_2x$ for all $n\in\mathbb{N}$ and $x\in C$. By Lemma 3.2

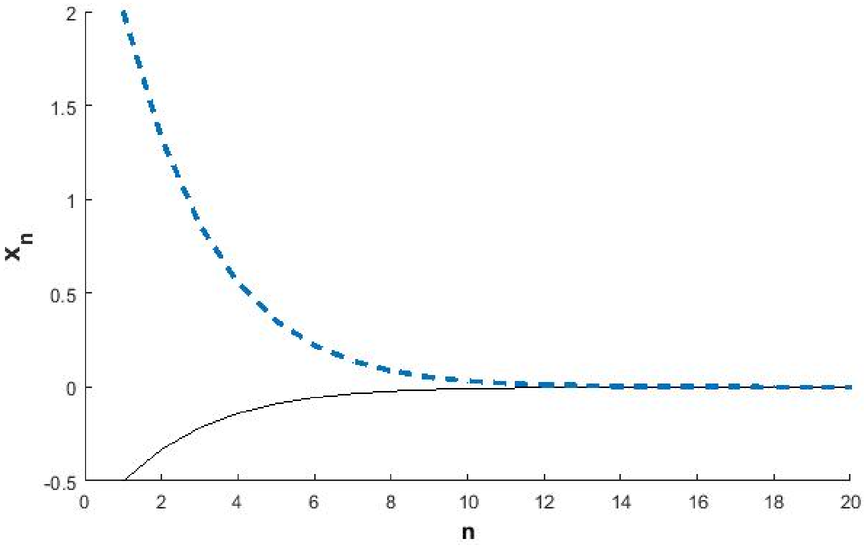

Figure 1: The convergence behavior of the generated sequence $\{x_n\}$ with our algorithm for starting points $x_1=2$ and $x_1=-0.5$.

of [15], $\{T_n\}$ satisfies the NST-condition with $\mathcal{F}=\{S_1,S_2\}$. Therefore, under the above hypotheses in Theorem 4.1, for each $n\in\mathbb{N}$, x_n is generated by the following algorithm:

$$\begin{cases} x_1=x\in C, \;\; C_0=Q_0=C,\\ u_n=\theta_nx_n+(1-\theta_n)T_nx_n,\\ y_n=\beta_nu_n+(1-\beta_n)T_nx_n,\\ C_n=\{u\in C_{n-1}\cap Q_{n-1}:|u_n-u|\leq|x_n-u|\},\\ Q_n=\{u\in C_{n-1}\cap Q_{n-1}:(x-x_n)(x_n-u)\geq 0\},\\ x_{n+1}=P_{C_n\cap Q_n}x \;\; or \;\; |x_{n+1}-x|=\min\limits_{z\in C_n\cap Q_n}|z-x|. \end{cases}$$

Obtained results for the above algorithm show that the sequence $\{x_n\}$ converges strongly to 0, see Figure 1. Moreover, Table1 is a numerical result for the sequence $\{x_n\}$ with different starting points $x_1=2$ and $x_1=-0.5$ to satisfy condition $|x_{n+1}-x_n|\leq 10^{-5}$. For $x_1=2$, we obtain the approximate solution after 22 iterations with CPU time 1.665 s and for $x_1=-0.5$, we obtain the approximate solution after 20 iterations with CPU time 1.366 s. We have solved the optimization subproblems in this example with the solver FMINCON from optimization toolbox in MATLAB software.

Acknowledgment

We are grateful to the referees for a number of valuable comments and suggestions regarding a original version of our paper. Also, We would like to thank Marand Branch, Islamic Azad University for the Financial support of this research, which is based on a research project contract.

Table1	Numerical Results for x_n	
n	$x = 2$	$x = -0.5$
1	2	-0.5
2	1.3269	-0.3317
3	0.8638	-0.2159
4	0.5545	-0.1386
5	0.3522	-0.0881
6	0.2219	-0.0555
7	0.1388	-0.0347
$\vdots$	$\vdots$	$\vdots$
Stop	22	20
CPU time	1.665	1.366

Table 1: Numerical results for the generated sequence $\{x_n\}$ with our algorithm for starting points $x_1 = 2$ and $x_1 = -0.5$.

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