Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 2287–2295
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.18956.2044

On optimization problems of the difference of non-negative valued affine IR functions and their dual problems

Somayeh Mirzadeh[a,*], Samaneh Bahrami[b]

[a] Department of Mathematics, University of Hormozgan, P.O. Box, 3995, Bandar Abbas, Iran
[b] Department of Mathematics, Shahid Bahonar University of Kerman, P.O. Box, 76169133 Kerman, Iran

(Communicated by Mohammad Bagher Ghaemi)

Abstract

The aim of this paper is to present dual optimality conditions for the difference of two non-negative valued affine increasing and radiant (IR) functions. We first give a characterization of dual optimality conditions for the difference of two non-negative valued increasing and radiant (IR) functions. Our approach is based on the Toland-Singer formula.

Keywords: global optimization, abstract concavity, increasing and radiant function, superdifferential, upper support set, affine function, Toland-Singer formula
2020 MSC: 90C46, 26A48

1 Introduction

Abstract convexity (concavity) has found many applications in the study of mathematical analysis and optimization problems (see [1, 2, 3, 5, 7, 8, 9]). Functions which can be represented as upper envelopes of subsets of a set H of sufficiently simple (*elementary*) functions, are studied in this theory (for more details see [8, 10]). It is well-known that some classes of increasing functions are abstract convex (concave). For example, the class of increasing and convex-along-rays (ICAR) functions (see [8, 9]) and the class of increasing and radiant (IR) functions are abstract convex (concave) (see [4, 6]).

One of the most important global optimization problems is that of minimizing a DC-functions (difference of two convex functions). In a general case, DC-functions can be replaced by DAC-functions (difference of two abstract convex (concave) functions).

Consider the problem

$$\text{minimize } [g(x) - f(x)] \text{ subject to } x \in X, \tag{1.1}$$

where g and f are abstract convex (concave) with respect to H (or H-convex (concave)). Now, consider the problem

$$\text{minimize } [f^*(h) - g^*(h)] \text{ subject to } h \in H, \tag{1.2}$$

*Corresponding author
Email addresses: mirzadeh@hormozgan.ac.ir (Somayeh Mirzadeh), s_bahrami53@yahoo.com (Samaneh Bahrami)

Received: October 2019 *Accepted:* June 2021

where f^* and g^* are the Fenchel-Moreau H-conjugate of f and g, respectively (see Definition 2.1, below). This problem is called the dual problem with respect to (1.1). The Toland-Singer formula (see Theorem 3.1, below) allows us to establish links between solutions of the problem (1.1) and the problem (1.2). These problems were studied by many scholars, and much attention has been paid to characterizing the optimality conditions for these problems (see [9, 10]). For example, A. M. Rubinov and B. M. Glover has studied dual optimality conditions for the difference of increasing and convex-along-rays functions in [9]. In particular, the minimizing of the difference of two increasing and co-radiant (ICR) functions and also the minimizing of the difference of two non-positive IR functions (see, for example, [1, 2, 5]).

In this paper, we replace g and f by two non-negative valued affine increasing and radiant (IR) functions (a non-negative valued affine IR function is the shift of a non-negative valued IR function on a constant) and we establish relationships between solutions of $h := g - f$ and its conjugate. We outline a dual approach to the study of the global optimization problem for this class of functions. Our approach is based on a particular case of the Toland-Singer formula. The motivations for using non-negative valued affine IR functions in this paper are given in the Remark 4.1 (below).

The layout of the paper is as follows. In Section 2, we collect definitions, notations and preliminary results which will be used later. A characterization of dual optimality conditions for the difference of two non-negative valued increasing and radiant (IR) functions is given in section 3. Finally, in Section 4, we examine the dual optimality conditions for the difference of non-negative valued affine IR functions.

2 Preliminaries

Let X be a topological vector space. We assume that X is equipped with a closed convex pointed cone S (the latter means that $S \cap (-S) = \{0\}$). We say that $x \le y$ or $y \ge x$ if and only if $y - x \in S$.

A function $f : X \longrightarrow [-\infty, +\infty]$ is called radiant if $f(\gamma x) \le \gamma f(x)$ for all $x \in X$ and all $\gamma \in (0, 1]$. It is easy to see that f is radiant if and only if $f(\gamma x) \ge \gamma f(x)$ for all $x \in X$ and all $\gamma \ge 1$. The function f is called increasing if $x \le y \Longrightarrow f(x) \le f(y)$.

In this paper, we study IR (increasing and radiant) functions f such that

$$0 \in \operatorname{dom} f := \{x \in X \,:\, -\infty < f(x) < +\infty\}.$$

Remark 2.1. Let $f : X \longrightarrow [0, +\infty]$ be IR functions. Then it is clear that $f(x) = 0$ for all $x \in -S$.

Definition 2.1. Let X be a non-empty set, H be a non-empty set of functions $h : X \longrightarrow [-\infty, +\infty]$ defined on X and $f : X \longrightarrow [-\infty, +\infty]$ be a function.

(1) The upper support set of f with respect to H is defined by

$$\operatorname{supp}^+(f, H) := \{h \in H \,: h(x) \ge f(x), \forall\, x \in X\}.$$

(2) The function f is called abstract concave with respect to H (or H-concave) if there exists a subset U of H such that:

$$f(x) = \inf_{h \in U} h(x), \quad (x \in X).$$

(3) We define the Fenchel-Moreau H-conjugate $f^* : H \longrightarrow [-\infty, +\infty]$ of f by

$$f^*(h) := \inf_{x \in X}\{h(x) - f(x)\}, \quad (h \in H).$$

The set H in Definition 2.1 is called the set of elementary functions. Let U be a set of elementary functions defined on a set X. The function u_c of the form

$$u_c(x) := u(x) + c, \quad (x \in X),$$

with $u \in U$, $c \in \mathbb{R}$ is called an U-affine function. The set of all U-affine functions is denoted by H_U.

Now, consider the functions $t : X \times X \times (0, +\infty) \longrightarrow [0, +\infty]$ defined by

$$t(x, y, \alpha) := \inf\{\lambda : 0 \leq \lambda \leq \alpha, \lambda y \geq x\}, \quad \forall\, x, y \in X, \forall\, \alpha > 0, \tag{2.1}$$

(with the convention $\inf \emptyset = +\infty$).

The functions t was introduced and examined in [6]. In the following, we present some properties of the function t which were obtained in [6].

Proposition 2.1. For every $x,\ y,\ x',\ y' \in X;\ \gamma \in (0, 1];\ \mu\ , \alpha, \alpha' \in (0, +\infty)$, one has

$$t(\mu x, y, \alpha) = \mu t(x, y, \frac{\alpha}{\mu}), \tag{2.2}$$

$$t(x, \mu y, \alpha) = \frac{1}{\mu} t(x, y, \mu\alpha), \tag{2.3}$$

$$x \leq x' \implies t(x, y, \alpha) \leq t(x', y, \alpha), \tag{2.4}$$

$$y \leq y' \implies t(x, y', \alpha) \leq t(x, y, \alpha), \tag{2.5}$$

$$\alpha \leq \alpha' \implies t(x, y, \alpha') \leq t(x, y, \alpha), \tag{2.6}$$

$$t(\gamma x, y, \alpha) \leq \gamma t(x, y, \alpha), \tag{2.7}$$

$$t(x, \gamma y, \alpha) \geq \frac{1}{\gamma} t(x, y, \alpha), \tag{2.8}$$

$$t(x, y, \alpha) = 0 \iff x \in -S, \tag{2.9}$$

$$t(x, x, 1) = 1, \forall x \in X \setminus (-S), \tag{2.10}$$

$$y \in -S \iff t(x, y, \alpha) = \begin{cases} 0, & \text{if } x \in -S, \\ +\infty, & \text{if } x \notin -S. \end{cases} \tag{2.11}$$

Now, for each $y \in X$ and $\alpha > 0$ we consider the functions $t_{(y,\alpha)} : X \longrightarrow [0, +\infty]$ defined by $t_{(y,\alpha)}(x) := t(x, y, \alpha)$ for all $x \in X$. Let $T := \{t_{(y,\alpha)}\ : y \in X,\ \alpha > 0\}$. It is easy to check that T is set of non-negative IR functions.

Theorem 2.1. ([6]). Let $f : X \longrightarrow [0, +\infty]$ be a function.Then the following assertions are equivalent:

(*i*) f is IR.

(*ii*) $\lambda f(y) \leq f(x)$ for all $x,\ y \in X$ and all $\lambda \geq 1$ such that $\lambda y \leq x$.

(*iii*) $t(x, y, \alpha) f(\alpha y) \geq \alpha f(x)$ for all $x,\ y \in X$ and all $\alpha > 0$, with the convention $0 \times (+\infty) = +\infty$.

Theorem 2.2. ([6]). Let $f : X \longrightarrow [0, +\infty]$ be a function. Then f is IR if and only if there exists a set $A \subseteq T$ such that

$$f(x) = \inf_{t_{(y,\alpha)} \in A} t_{(y,\alpha)}(x), \quad (x \in X).$$

In this case, one can take $A := \{t_{(y,\alpha)} \in T\ :\ f(\alpha y) \leq \alpha\}$. Hence, f is IR if and only if f is T-concave.

Proposition 2.2. ([6]). Let $f : X \longrightarrow [0, +\infty]$ be an IR function. Then

$$\operatorname{supp}^+(f, T) = \{t_{(y,\alpha)} \in T\ :\ f(\alpha y) \leq \alpha\}.$$

Let H be a set of elementary functions $h : X \longrightarrow [-\infty, +\infty]$. Recall (see [8]) that the superdifferential of the function $f : X \longrightarrow [-\infty, +\infty]$ at a point $x_0 \in X$ with respect to H (or H-superdifferential) is defined as follows

$$\partial_H^+ f(x_0) := \{h \in H\ : h(x_0) \in \mathbb{R},\ f(x) - f(x_0) \leq h(x) - h(x_0), \forall\, x \in X\}.$$

So, the T-superdifferential of a non-negative IR function $f : X \longrightarrow [0, +\infty]$ at a point $x_0 \in X$ such that $f(x_0) \neq +\infty$ is defined as follows

$$\partial_T^+ f(x_0) := \{t_{(y,\alpha)} \in T\ : t_{(y,\alpha)}(x_0) \in \mathbb{R},\ f(x) - f(x_0) \leq t_{(y,\alpha)}(x) - t_{(y,\alpha)}(x_0), \forall\, x \in X\}.$$

In the following, we give a characterization for the T-superdifferential of a non-negative IR function $f : X \longrightarrow [0,+\infty]$ at a point $x_0 \in X$, which will be used.

Theorem 2.3. Let $f : X \longrightarrow [0,+\infty]$ be an IR function and $x_0 \in X$ be such that $f(x_0) \neq 0, +\infty$. Then

$$\partial_T^+ f(x_0) = \{t_{(y,\alpha)} \in T : f(x_0) \geq t_{(y,\alpha)}(x_0),\ \alpha - t_{(y,\alpha)}(x_0) \geq f(\alpha y) - f(x_0)\}.$$

Moreover, if $x_0 \in X$ is such that $f(x_0) \neq +\infty$, 0, then $\partial_T^+ f(x_0) \neq \emptyset$.

Proof . Let $D := \{t_{(y,\alpha)} \in T : f(x_0) \geq t_{(y,\alpha)}(x_0),\ \alpha - t_{(y,\alpha)}(x_0) \geq f(\alpha y) - f(x_0)\}$, and $t_{(y,\alpha)} \in D$ be arbitrary. Let $x \in X$ be arbitrary. If $t_{(y,\alpha)}(x) = +\infty$, then $f(x) - f(x_0) \leq t_{(y,\alpha)}(x) - t_{(y,\alpha)}(x_0)$. Let $t_{(y,\alpha)}(x) < +\infty$. Since $0 \leq \frac{t_{(y,\alpha)}(x)}{\alpha} \leq 1$, and $f(x_0) - t_{(y,\alpha)}(x_0) \geq 0$, it follows that

$$\frac{t_{(y,\alpha)}(x)}{\alpha}(\alpha - f(\alpha y)) \geq \frac{t_{(y,\alpha)}(x)}{\alpha}(t_{(y,\alpha)}(x_0) - f(x_0)) \geq t_{(y,\alpha)}(x_0) - f(x_0) \tag{2.12}$$

By Theorem 2.1 (iii), we have $\frac{t_{(y,\alpha)}(x)}{\alpha} f(\alpha y) \geq f(x)$, so in view of (2.12) we get $f(x) - f(x_0) \leq t_{(y,\alpha)}(x) - t_{(y,\alpha)}(x_0)$. It follows that $t_{(y,\alpha)} \in \partial_T^+ f(x_0)$. That is $D \subseteq \partial_T^+ f(x_0)$. For the converse, let $t_{(y,\alpha)} \in \partial_T^+ f(x_0)$ be arbitrary. By the definition of $\partial_T^+ f(x_0)$, one has

$$f(x) - f(x_0) \leq t_{(y,\alpha)}(x) - t_{(y,\alpha)}(x_0) \tag{2.13}$$

Thus, if $x = 0$, by (2.9), we obtain $f(x_0) \geq t_{(y,\alpha)}(x_0)$. Now, in (2.13) put $x = \alpha y$. Since $t_{(y,\alpha)}(\alpha y) \leq \alpha$, we get $\alpha - t_{(y,\alpha)}(x_0) \geq f(\alpha y) - f(x_0)$. So, $t_{(y,\alpha)} \in D$. It follows that $\partial_T^+ f(x_0) \subseteq D$, and hence $\partial_T^+ f(x_0) = D$.

Now, let $x_0 \in X$ be such that $f(x_0) \neq 0, +\infty$. Let $y = \frac{x_0}{f(x_0)}$, and $\alpha = f(x_0)$. Then, $f(\alpha y) = f(x_0) = \alpha$, and $f(x_0) \geq t_{(y,\alpha)}(x_0)$. So,

$$f(\alpha y) - f(x_0) = \alpha - f(x_0) \leq \alpha - t_{(y,\alpha)}(x_0).$$

It follows that $t_{(y,\alpha)} \in D$. Hence, $\partial_T^+ f(x_0) \neq \emptyset$, and this completes the proof. □

3 Characterization of Dual Optimality Conditions for the Difference of Non-negative IR Functions

Let $f,\ g : X \longrightarrow [0,+\infty)$ be IR functions. Consider the following extremal problem

$$g(x) - f(x) \to min \ \text{ subject to } x \in X. \tag{3.1}$$

We assume that $\inf_{x\in X}\{g(x) - f(x)\} > -\infty$. Now, consider the following problem

$$f^*(t) - g^*(t) \to min \ \text{ subject to } t \in T. \tag{3.2}$$

The problem defined by (3.2) is called the dual problem with respect to (3.1). We assume that $(+\infty) - (+\infty) = +\infty$. In the following, we give a particular case of the Toland-Singer formula which will be used later (see [8, 10, 11]).

Theorem 3.1. Let X be a set and U be a set of functions $u : X \longrightarrow [-\infty,+\infty)$ defined on X such that $\operatorname{dom} u \neq \emptyset$. Let $f,\ g : X \longrightarrow \mathbb{R}$ be H_U-concave functions. Then

$$\inf_{x\in X}\{g(x) - f(x)\} = \inf_{u\in U}\{f^*(u) - g^*(u)\},$$

(*with the convention* $+\infty - (+\infty) = +\infty$), where H_U is called the set of U-affine functions (abstract affine functions).

The following result can be obtained directly from Theorem 3.1.

Corollary 3.1. Let $f,\ g : X \longrightarrow [0,+\infty)$ be IR functions. Then

$$\inf_{x\in X}\{g(x) - f(x)\} = \inf_{t\in T}\{f^*(t) - g^*(t)\},$$

(*with the convention* $+\infty - (+\infty) = +\infty$).

Lemma 3.1. Let $f : X \longrightarrow [0, +\infty)$ be an IR function. Let $\varepsilon > 0$ and $x_0 \in X$ be such that $f(x_0) \leq \varepsilon$. Then

$$f^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) = 0.$$

Proof . We have $\varepsilon \geq f(x_0) = f(\varepsilon \frac{x_0}{\varepsilon})$. In view of Proposition 2.2, we conclude that $t_{(\frac{x_0}{\varepsilon},\varepsilon)} \in \text{supp}^+(f, T)$. Therefore, $t_{(\frac{x_0}{\varepsilon},\varepsilon)}(x) \geq f(x)$ for all $x \in X$. Hence

$$f^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) \geq 0. \tag{3.3}$$

On the other hand, in view of (2.9) and by the definition of f^* one has

$$f^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) \leq t_{(\frac{x_0}{\varepsilon},\varepsilon)}(0) - f(0) \leq 0.$$

This, together with (3.3) implies that $f^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) = 0$. □

Example 3.1. Consider function $f : \mathbb{R} \to [0, +\infty)$ defined as follows:

$$f(x) := \begin{cases} x^2, & x > 0, \\ 0, & x \leq 0, \end{cases}$$

where $x \in \mathbb{R}$. It is clear that f is a non-negative IR function. Put $\varepsilon := 2$ and $x_0 := 1 \in \mathbb{R}$, then $f(x_0) = f(1) = 1 < 2 = \varepsilon$. So,

$$t_{(\frac{x_0}{\varepsilon},\varepsilon)}(x) = t_{(\frac{1}{2},2)}(x) = \inf\{\lambda : 0 \leq \lambda \leq \ 2, \lambda \geq 2x\} = \begin{cases} 0, & x \leq 0, \\ 2x, & 0 < x \leq 1, \\ +\infty, & x > 1, \end{cases}$$

Thus,

$$f^*(t_{(\frac{1}{2},2)}) = \inf_{x\in\mathbb{R}}\{t_{(\frac{1}{2},2)}(x) - f(x)\} = \inf_{x\leq 0}\{t_{(\frac{1}{2},2)}(x) - f(x)\} = 0 - 0 = 0.$$

Lemma 3.2. Let $f : X \longrightarrow [0, +\infty)$ be an IR function. Let $t_{(y,\alpha)} \in T$ and $x_0 \in X$ be arbitrary. Then the following assertions are equivalent:

(i) $t_{(y,\alpha)} \in \partial_T^+ f(x_0)$.

(ii) $f^*(t_{(y,\alpha)}) = t_{(y,\alpha)}(x_0) - f(x_0)$. Moreover, if $f(x_0) \geq \varepsilon$ for some $\varepsilon > 0$, then $t_{(\frac{x_0}{\varepsilon},\varepsilon)} \in \partial_T^+ f(x_0)$ and $f^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) = \varepsilon - f(x_0)$.

Proof . By the definition of f^*, since $t_{(y,\alpha)}(x_0) - f(x_0) \geq f^*(t_{(y,\alpha)})$, we have

$$\begin{aligned} t_{(y,\alpha)} \in \partial_T^+ f(x_0) &\Longleftrightarrow f(x) - f(x_0) \leq t_{(y,\alpha)}(x) - t_{(y,\alpha)}(x_0), \forall\, x \in X \\ &\Longleftrightarrow t_{(y,\alpha)}(x_0) - f(x_0) \leq t_{(y,\alpha)}(x) - f(x), \forall\, x \in X \\ &\Longleftrightarrow t_{(y,\alpha)}(x_0) - f(x_0) \leq f^*(t_{(y,\alpha)}) \\ &\Longleftrightarrow t_{(y,\alpha)}(x_0) - f(x_0) = f^*(t_{(y,\alpha)}). \end{aligned}$$

Now, let $f(x_0) \geq \varepsilon > 0$. It follows from Remark 2.1 that $x_0 \notin -S$, and hence by (2.3) and (2.10) we deduce that $t_{(\frac{x_0}{\varepsilon},\varepsilon)}(x_0) = \varepsilon$. Thus, $t_{(\frac{x_0}{\varepsilon},\varepsilon)}(x_0) \leq f(x_0)$ and $\varepsilon - t_{(\frac{x_0}{\varepsilon},\varepsilon)}(x_0) = 0 = f(\varepsilon\frac{x_0}{\varepsilon}) - f(x_0)$. It follows from Theorem 2.2 that $t_{(\frac{x_0}{\varepsilon},\varepsilon)} \in \partial_T^+ f(x_0)$. Therefore, by the implication ((i) $\Longleftrightarrow$ (ii)) we have $f^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) = t_{(\frac{x_0}{\varepsilon},\varepsilon)}(x_0) - f(x_0) = \varepsilon - f(x_0)$, which completes the proof. □

Proposition 3.1. Let $f,\, g : X \longrightarrow [0, +\infty)$ be IR functions. Let $x_0 \in X$ and $\varepsilon > 0$ be such that $\varepsilon \leq \min\{f(x_0), g(x_0)\}$. Then, $t_{(\frac{x_0}{\varepsilon},\varepsilon)} \in \partial_T^+ f(x_0) \cap \partial_T^+ g(x_0)$. Moreover,

$$f^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) - g^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) = g(x_0) - f(x_0).$$

Proof . Since $f(x_0) \geq \varepsilon$ and $g(x_0) \geq \varepsilon$, then by a similar argument as in the proof of Lemma 3.2, we obtain $t_{(\frac{x_0}{\varepsilon},\varepsilon)} \in \partial_T^+ f(x_0) \cap \partial_T^+ g(x_0)$. So, by Lemma 3.2 one has $f^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) = \varepsilon - f(x_0)$ and $g^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) = \varepsilon - g(x_0)$. This implies that $f^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) - g^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) = g(x_0) - f(x_0)$. $\square$

Proposition 3.2. Let $f,\ g : X \longrightarrow [0,+\infty)$ be IR functions. Let $x_0 \in X$ and $\varepsilon > 0$ be such that $\varepsilon \geq \max\{f(x_0), g(x_0)\}$. Then, $t_{(\frac{x_0}{\varepsilon},\varepsilon)} \in T$ is a global minimizer of the problem (3.2) if and only if $\inf_{x\in X}\{g(x) - f(x)\} = 0$.

Proof . We have $f(x_0) \leq \varepsilon$ and $g(x_0) \leq \varepsilon$. By Lemma 3.1, we get $f^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) = 0$ and $g^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) = 0$. Thus, $f^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) - g^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) = 0$. By Corollary 3.1, this implies that $t_{(\frac{x_0}{\varepsilon},\varepsilon)}$ is a global minimizer of the problem (3.2) if and only if

$$0 = f^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) - g^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) = \inf_{t\in T}\{f^*(t) - g^*(t)\} = \inf_{x\in X}\{g(x) - f(x)\} = 0,$$

which completes the proof. $\square$

Proposition 3.3. Let $f,\ g : X \longrightarrow [0,+\infty)$ be IR functions such that $\inf_{x\in X}\{g(x) - f(x)\} \neq 0$. Let $t_{(\frac{x_0}{\varepsilon},\varepsilon)} \in T$ be a global minimizer of the problem (3.2) for some $x_0 \in X$ and some $\varepsilon > 0$. Then, $\varepsilon \leq \min\{f(x_0), g(x_0)\}$.

Proof . Since $\inf_{x\in X}\{g(x) - f(x)\} \neq 0$, then by Proposition 3.2, we get $\varepsilon < \max\{f(x_0), g(x_0)\}$. Now, assume if possible that $\varepsilon > \min\{f(x_0), g(x_0)\}$. Then, in view of

$$\varepsilon < \max\{f(x_0), g(x_0)\}$$

we have the following two possible cases:
Case 1: Suppose that $f(x_0) < \varepsilon < g(x_0)$. Since $f(x_0) < \varepsilon$, by Lemma 3.1, we get

$$f^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) = 0.$$

On the other hand, $g(x_0) > \varepsilon$, thus by a similar argument as in the proof of Lemma 3.2 one has

$$g^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) = \varepsilon - g(x_0).$$

Therefore, we have

$$\inf_{t\in T}\{f^*(t) - g^*(t)\} = f^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) - g^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) = g(x_0) - \varepsilon.$$

By Corollary 3.1 we conclude that

$$\inf_{x\in X}\{g(x) - f(x)\} = g(x_0) - \varepsilon.$$

This implies that

$$g(x) - f(x) \geq g(x_0) - \varepsilon,\ \forall\, x \in X. \tag{3.4}$$

Put $x = 0$ in (3.4), we get $g(x_0) \leq \varepsilon$, which is a contradiction.
Case 2: Assume that $g(x_0) < \varepsilon < f(x_0)$. By a similar argument as the above we have $g^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) = 0$ and $f^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) = \varepsilon - f(x_0)$. Thus, by the hypothesis and Corollary 3.1, we obtain

$$g(x) - f(x) \geq \varepsilon - f(x_0),\ \forall\, x \in X. \tag{3.5}$$

Put $x = x_0$ in (3.5), we get $g(x_0) \geq \varepsilon$. This is a contradiction. Hence, $\varepsilon \leq \min\{f(x_0), g(x_0)\}$. $\square$

The following result establish a link between solutions of the problem (3.1) and the problem (3.2).

Proposition 3.4. Let $f,\ g : X \longrightarrow [0,+\infty)$ be IR functions such that $\inf_{x\in X}\{g(x) - f(x)\} \neq 0$. Let $x_0 \in X$. Then, $t_{(\frac{x_0}{\varepsilon},\varepsilon)} \in T$ is a global minimizer of the problem (3.2) if and only if x_0 is a global minimizer of the problem (3.1) and $\varepsilon \leq \min\{f(x_0), g(x_0)\}$.

Proof . Let $t_{(\frac{x_0}{\varepsilon},\varepsilon)}$ be a global minimizer of the problem (3.2). By Proposition 3.3, we have $\varepsilon \leq \min\{f(x_0), g(x_0)\}$. It follows from Proposition 3.1 that $f^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) - g^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) = g(x_0) - f(x_0)$. On the other hand, we have $\inf_{x\in X}\{g(x) - f(x)\} = \inf_{t\in T}\{f^*(t) - g^*(t)\}$. Therefore, $\inf_{x\in X}\{g(x) - f(x)\} = g(x_0) - f(x_0)$, that is, x_0 is a global minimizer of the problem (3.1).

Conversely, let $x_0 \in X$ be a global minimizer of the problem (3.1) and let $\varepsilon \leq \min\{f(x_0), g(x_0)\}$. In view of Proposition 3.1, we conclude that $f^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) - g^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) = g(x_0) - f(x_0)$. Thus,

$$\inf_{t\in T}\{f^*(t) - g^*(t)\} = \inf_{x\in X}\{g(x) - f(x)\} = f^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}) - g^*(t_{(\frac{x_0}{\varepsilon},\varepsilon)}).$$

Hence, $t_{(\frac{x_0}{\varepsilon},\varepsilon)}$ is a global minimizer of the problem (3.2). □

Example 3.2. Consider two IR functions f, $g : \mathbb{R}_+ \longrightarrow [0, +\infty)$ defined as follows

$$f(x) := x^{\frac{3}{2}} \quad \text{and} \quad g(x) := x^2, \; \forall x \geq 0.$$

It is clear that f and g are non-negative IR functions. It is not difficult to show that $f^*(t_{(y,\varepsilon)}) = \varepsilon - (\varepsilon y)^{\frac{3}{2}}$ and $g^*(t_{(y,\varepsilon)}) = \varepsilon - (\varepsilon y)^2$. Moreover, by a simple calculation, one can obtain that for every $0 < \varepsilon \leq \frac{27}{64}$, $t_{(y,\varepsilon)}$ is a global minimizer of $f^* - g^*$ if and only if $\varepsilon y = \frac{9}{16}$. By the previous proposition, we have $x_0 = \varepsilon y = \frac{9}{16}$ is a global minimizer of $g - f$.

4 On Dual Optimality Conditions for Optimization Problems of the Difference of Non-negative Valued Affine Increasing and Radiant Functions

In this section, we outline a dual approach for the study of the global optimization problems involving functions that can be represented as the difference of non-negative affine IR functions.
For each IR function $f_0 : X \longrightarrow [0, +\infty]$ and $c_0 \in \mathbb{R}$, the function $f = f_0 + c_0$ is called non-negative affine IR. Since $f_0(0) = 0$, then $c_0 = f(0)$. For each $t_{(y,\alpha)} \in T$ and $c \in \mathbb{R}$, consider the shift $t_{(y,\alpha),c}$ of $t_{(y,\alpha)}$ on the constant c:

$$t_{(y,\alpha),c}(x) := t_{(y,\alpha)}(x) + c, \quad (x \in X).$$

Recall that the function $t_{(y,\alpha),c}$ is called T-affine and the set of all T-affine functions will be denoted by H_T, ($H_T := \{t_{(y,\alpha),c} : t_{(y,\alpha)} \in T, \, c \in \mathbb{R}\}$).

Remark 4.1. In a forthcoming study, our main goal is to investigate the optimization of the difference of extended real valued affine IR functions. It is worth noting that for studying optimality conditions for the global minimum of this class of functions, we first need to study non-negative and non-positive valued IR functions. The essential results of optimization problems of the difference of non-positive valued IR functions have been studied in [5]. In this paper, we outline a dual approach to the study of the global optimization problem for non-negative valued affine IR functions. By a similar argument as in *[1, Remark 3.1]*, it can be seen that indeed, we cannot get the essential results of optimization problems of non-negative valued IR functions from those for non-positive valued IR functions.

Proposition 4.1. Let $f : X \longrightarrow (-\infty, +\infty]$ be a non-negative affine IR function. Then, f is an H_T-concave function.

Proof . Since f is a non-negative affine IR function, then $f = f_0 + c_0$, where $f_0 : X \longrightarrow [0, +\infty]$ is an IR function and $c_0 = f(0)$. By Theorem 2.1, f_0 is an T-concave function, so we have

$$f_0(x) = \inf_{t_{(y,\alpha)}\in\Delta} t_{(y,\alpha)}(x), \quad (x \in X), \tag{4.1}$$

where $\Delta = \{t_{(y,\alpha)} \in T \,:\, f_0(\alpha y) \leq \alpha\}$. In view of (4.1), one has

$$f(x) = f_0(x) + c_0 = \inf_{t_{(y,\alpha)}\in\Delta}[t_{(y,\alpha)}(x) + c_0] = \inf_{t_{(y,\alpha)}\in\Delta} t_{(y,\alpha),c_0}(x)$$

for all $x \in X$. Hence, f is an H_T-concave function. □

For a non-negative affine IR function $f : X \longrightarrow (-\infty, +\infty]$, the Fenchel-Moreau T-conjugate f^* of f is defined as follows

$$f^*(t) = \inf_{x\in X}\{t(x) - f(x)\}, \quad (t \in T).$$

Now, let $f = f_0 + c_0$, where $f_0 : X \longrightarrow [0, +\infty]$ is an IR function and $c_0 \in \mathbb{R}$. Then

$$f^*(t) = f_0^*(t) - c_0, \quad (t \in T). \tag{4.2}$$

Indeed, for every $t \in T$, we have

$$\begin{aligned} f^*(t) &= \inf_{x\in X}\{t(x) - f(x)\} \\ &= \inf_{x\in X}\{t(x) - f_0(x) - c_0\} \\ &= \inf_{x\in X}\{t(x) - f_0(x)\} - c_0 \\ &= f_0^*(t) - c_0. \end{aligned}$$

Lemma 4.1. Let $f,\, g : X \longrightarrow (-\infty, +\infty]$ be non-negative affine IR functions. Then, $t_{(y,\alpha)} \in T$ is a global minimizer of $f^* - g^*$ if and only if $t_{(y,\alpha)}$ is a global minimizer of $f_0^* - g_0^*$, where $f_0 = f - f(0)$ and $g_0 = g - g(0)$. Moreover, $x_0 \in X$ is a global minimizer of $g - f$ if and only if $x_0 \in X$ is a global minimizer of $g_0 - f_0$.

Proof . In view of (4.2) we conclude that

$$\begin{aligned} t_{(y,\alpha)} \in T \text{ is a global minimizer of } f^* - g^* &\Longleftrightarrow \inf_{t\in T}\{f^*(t) - g^*(t)\} = f^*(t_{(y,\alpha)}) - g^*(t_{(y,\alpha)}) \\ &\Longleftrightarrow \inf_{t\in T}\{f_0^*(t) - f(0) - g_0^*(t) + g(0)\} \\ &= f_0^*(t_{(y,\alpha)}) - f(0) - g_0^*(t_{(y,\alpha)}) + g(0) \\ &\Longleftrightarrow \inf_{t\in T}\{f_0^*(t) - g_0^*(t)\} = f_0^*(t_{(y,\alpha)}) - g_0^*(t_{(y,\alpha)}) \\ &\Longleftrightarrow t_{(y,\alpha)} \text{ is a global minimizer of } f_0^* - g_0^*. \end{aligned}$$

$$\begin{aligned} x_0 \in X \text{ is a global minimizer of } g - f &\Longleftrightarrow \inf_{x\in X}\{g(x) - f(x)\} = g(x_0) - f(x_0) \\ &\Longleftrightarrow \inf_{x\in X}\{g_0(x) + g(0) - f_0(x) - f(0)\} \\ &= g_0(x_0) + g(0) - f_0(x_0) - f(0) \\ &\Longleftrightarrow \inf_{x\in X}\{g_0(x) - f_0(x)\} = g_0(x_0) - f_0(x_0) \\ &\Longleftrightarrow x_0 \in X \text{ is a global minimizer of } g_0 - f_0. \end{aligned}$$

□

Proposition 4.2. Let $f,\, g : X \longrightarrow \mathbb{R}$ be non-negative affine IR functions such that $\inf_{x\in X}\{g(x) - f(x)\} \neq g(0) - f(0)$. Let $x_0 \in X$. Then, $t_{(\frac{x_0}{\varepsilon},\varepsilon)} \in T$ is a global minimizer of $f^* - g^*$ if and only if x_0 is a global minimizer of $g - f$ and $\varepsilon \leq \min\{f(x_0) - f(0), g(x_0) - g(0)\}$.

Proof . We have $f = f_0 + f(0)$ and $g = g_0 + g(0)$, where $f_0,\, g_0 : X \longrightarrow [0, +\infty)$ are IR functions. Since $\inf_{x\in X}\{g(x) - f(x)\} \neq g(0) - f(0)$, then $\inf_{x\in X}\{g_0(x) - f_0(x)\} \neq 0$. It follows from Proposition 3.4 and Lemma 4.1 that x_0 is a global minimizer of $g - f$ and $\varepsilon \leq \min\{f(x_0) - f(0), g(x_0) - g(0)\} \Longleftrightarrow x_0$ is a global minimizer of $g_0 - f_0$ and $\varepsilon \leq \min\{f_0(x_0), g_0(x_0)\} \Longleftrightarrow t_{(\frac{x_0}{\varepsilon},\varepsilon)} \in T$ is a global minimizer of $f_0^* - g_0^* \Longleftrightarrow t_{(\frac{x_0}{\varepsilon},\varepsilon)}$ is a global minimizer of $f^* - g^*$. This completes the proof. □

Example 4.1. Consider two non-negative affine IR functions $f,\, g : X \longrightarrow \mathbb{R}$ defined as follows

$$f(x) := \begin{cases} x - 2, & x > 0, \\ -2, & x \leq 0. \end{cases} \quad \textit{and} \quad g(x) := \begin{cases} x^{\frac{3}{2}} - 1, & x > 0, \\ -1, & x \leq 0, \end{cases}$$

For every $t_{(y,\alpha)} \in T$, we conclude that

$$f^*(k_{(y,\alpha)}) = \begin{cases} 2 + \alpha - \alpha y, & y > 1, \\ 2, & y \leq 1. \end{cases} \quad \textit{and} \quad g^*(t_{(y,\alpha)}) = \begin{cases} 1 + \alpha - (\alpha y)^{\frac{3}{2}}, & y > \alpha^{-\frac{1}{3}}, \\ 1, & y \leq \alpha^{-\frac{1}{3}}, \end{cases}$$

By a simple calculation, one can obtain that for every $0 < \alpha \leq \frac{8}{27}$, $t_{(y,\alpha)}$ is a global minimizer of $f^* - g^*$ such that $y = \frac{4}{9\alpha}$. Now, by Proposition 4.2, we have $x_0 = \alpha y = \frac{4}{9}$ is a global minimizer of $g - f$.

References

[1] M. H. Daryaei and H. Mohebi, *Global minimization of the difference of strictly non-positive valued affine ICR functions*, J. Glob. Optim. **61** (2015), no. 2, 311–323.

[2] A.R. Doagooei and H. Mohebi, *Optimization of the difference of ICR functions*, Nonlinear Anal. **71** (2009), no. 10, 4493–4499.

[3] J. B. Hiriart, *From convex to nonconvex minimization: necessary and sufficient conditions for global optimality,* Nonsmooth Optimization and Related Topics, 1989, 219–240.

[4] H. Mohebi, *Abstract convexity of radiant functions with applications,* J. Glob. Optim. **55** (2013), no. 3, 521–538.

[5] H. Mohebi and M. Esmaeili, *Optimization of the difference of increasing and radiant functions*, Appl. Math. Sci. **6** (2012), no. 67, 3339–3345.

[6] H. Mohebi and S. Mirzadeh, *Abstract convexity of extended real valued increasing and radiant functions,* Filomat **26** (2012), no. 5, 1002–1022.

[7] A. Nedić, A. Ozdaglar and A.M. Rubinov, *Abstract convexity for nonconvex optimization duality,* Optim. **56** (2007), no. 5-6, 655–674.

[8] A.M. Rubinov, *Abstract convexity and global optimization,* Kluwer Academic Publishers, Dordrecht-Boston-London, 2000.

[9] A.M. Rubinov and B.M. Glover, *Increasing convex-along-rays functions with application to global optimization,* J. Optim. Theory Appl. **102** (1999), no. 3, 615-642.

[10] I. Singer, *Abstract convex analysis,* Wiley-Interscience, New York, 1997.

[11] J.F. Toland, *Duality in nonconvex optimization*, J. Math. Anal. Appl. **66** (1978), no. 2, 399–415.