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Efficient quadrature methods for solving Hammerstein integral equations on the half-line

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Abstract

In this paper, we proposed two numerical methods to solve the nonlinear integral equations of Hammerstein type on the half-line. By using a Sinc-Nyström method based on Single-Exponential (SE) and Double-Exponential (DE) transformations, the problem is converted into a nonlinear system of equations. We provided an error analysis of the proposed schemes and showed that these methods have exponential convergence rates. Finally, several numerical examples are given to show the effectiveness of the methods.

Keywords: Hammerstein integral equations, Half-line, Sinc quadrature, Nyström method
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1 Introduction

We consider the nonlinear integral equation of Hammerstein type on the half-line given by the general form

$$u(s) - \int_0^\infty k(s,t) f(t, u(t)) dt = g(s), \qquad s \in I = [0, \infty), \tag{1.1}$$

where k, g, and f are known functions and u is a solution to be determined. More recently, some researchers have returned their interest to numerical methods for solving linear integral equations on unbounded intervals, such as [2, 3, 4, 5]. However, nonlinear integral equations on unbounded intervals are still a challenge where there is a scarcity of research that's interested in this type, as far as we know there is only the paper of Nahid and Nelakanti [1] where they have applied the Galerkin and collocation methods to solve the nonlinear Hammerstein integral equations on the half-line.

In this work we presented two numerical schemes for solving Eq.(1.1). The first one is based on the so-called single exponential transformations, which has the convergence rate $\mathcal{O}(\exp(-C\sqrt{N}))$, the second one is based on double exponential transformations, which improves the order of convergence to $\mathcal{O}(\exp(-C(N/\log N)))$. It should be noted that we have already provided these methods to solve the linear Fredholm integral equation on the infinite intervals in

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[4], so this work is an extending of our previous study on the linear case to the nonlinear case. We can write Eq.(1.1) in operator notation as

$$(\mathcal{I} - \mathcal{K}f)u = g, \tag{1.2}$$

where

$$(\mathcal{K}f)u(s) = \int_I k(s,t)f(t,u(t))dt = g(s), \quad s \in I. \tag{1.3}$$

The operator $\mathcal{K}f$ is defined on the Banach space $X = Hol(\mathcal{D}) \cap C(\overline{\mathcal{D}})$, where $\mathcal{D}$ is a simply connected domain in the complex plane $\mathbb{C}$ which satisfies $I \subset \mathcal{D}$ and $Hol(\mathcal{D})$ denotes the family of all functions v that are analytic in the domain $\mathcal{D}$. Moreover, assume that Eq.(1.1) has an analytic solution.

The layout of this paper is as follows, in section 2, we present the basic properties for the sinc quadrature rule. In section 3, Sinc-Nyström methods for Eq.(1.1) are developed. In section 4, the convergence analysis are described for the proposed methods. In section 5, several typical examples are presented to illustrate the effectiveness of our approaches, and the conclusions follow in section 6.

2 Approximation on real line

The Sinc function is defined on the whole real line by

$$\text{Sinc}(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t}, & t \neq 0, \\ 1, & t = 0. \end{cases} \tag{2.1}$$

For $h > 0$ and integer j, we define the j-th Sinc function with step size h by

$$S(j,h)(t) = \frac{\sin[\pi(t/h - j)]}{\pi(t/h - j)}.$$

The Sinc approximation for a function f on the entire interval $(-\infty, \infty)$ is defined as

$$f(s) \approx \sum_{j=-N}^{N} f(jh)S(j,h)(s), \tag{2.2}$$

and the Sinc quadrature rule can be derived by integrating both side of (2.2) as follows

$$\int_{-\infty}^{\infty} f(s)ds \approx \sum_{j=-N}^{N} f(jh) \int_{-\infty}^{\infty} S(j,h)(s)ds = h \sum_{j=-N}^{N} f(jh). \tag{2.3}$$

In order to define a convenient function space, the step domain

$$\mathcal{D}_d = \{z \in \mathbb{C} : |\Im(z)| < d\},$$

for some $d > 0$ is introduced.
Let $t = \psi(z)$ denote a conformal map of $\mathcal{D}$ into $\mathcal{D}_d$ such that $\psi(0) = -\infty$ and $\psi(\infty) = \infty$.
Let $\psi = \phi^{-1}$ denote the inverse map, we define the range of ϕ^{-1} on the real line as

$$I = \left\{\psi(\xi) = \phi^{-1}(\xi) \in \mathcal{D} : -\infty < \xi < \infty\right\}. \tag{2.4}$$

Applying the variable transformation $t = \psi(z)$, we obtain

$$\int_0^{\infty} f(s)ds = \int_{-\infty}^{\infty} f(\psi(s))\psi'(s)ds \approx h \sum_{j=-N}^{N} f(\psi(jh))\psi'(jh). \tag{2.5}$$

Let α and C be positive constants, then $L_\alpha(\mathcal{D})$ denotes the family of all functions $f \in Hol(\mathcal{D})$ where

$$|f(z)| \leq C \frac{|Q(z)|^\alpha}{[1 + |Q(z)|]^{2\alpha}}, \tag{2.6}$$

for all z in $\mathcal{D}$, where $Q(z) = e^{\phi(z)}$. Moreover, $M_\alpha(\mathcal{D})$ with $0 < \alpha \leqslant 1$ consists of all functions $f \in X$ such that

$$f(z) - \frac{f(0) + Q(z)f(\infty)}{1 + Q(z)} \in L_\alpha(\mathcal{D}). \tag{2.7}$$

The equation (2.2) can be adapted to approximate on I with the aid of "Single-Exponential transformation" (SE) given by Stenger [6] by

$$\psi_{SE_1}(t) = e^t, \quad \psi_{SE_2}(t) = \operatorname{arcsinh}(e^t),$$

and then by ones of the following transformations

$$\psi_{DE_1}(t) = \psi_{SE_1}(\frac{\pi}{2}\sinh t), \quad \psi_{DE_2}(t) = \psi_{SE_2}(\frac{\pi}{2}\sinh t), \quad \psi_{DE_3}(t) = e^{t-exp(-t)}.$$

These are called the "Double-Exponential transformation" (DE) and were originally introduced by Takahasi and Mori [7]. The following theorems shows the exponential convergence of the SE-Sinc approximation and the DE-Sinc approximation.

Theorem 2.1. (**Stenger [6]**) Assume that $f\psi' \in L_\alpha(\psi_{SE_i}(\mathcal{D}_d))$ for d with $0 < d < \pi/2$. Then, there exists a constant C, independent of N, such that

$$\left| \int_0^\infty f(t)dt - h\sum_{j=-N}^{N} f(\psi_{SE_i}(jh))\psi'_{SEi}(jh) \right| \leq C\exp(-\sqrt{2\pi d\alpha N}), \tag{2.8}$$

where $h = \sqrt{2\pi d/(\alpha N)}$.

Theorem 2.2. (**Tanaka et al. [8]**) Assume that $f\psi' \in L_\alpha(\psi_{DE_i}(\mathcal{D}_d))$ for d with $0 < d < \pi/2$. Then, there exists a constant C, independent of N, such that

$$\left| \int_0^\infty f(t)dt - h\sum_{j=-N}^{N} f(\psi_{DE_i}(jh))\psi'_{DE_i}(jh) \right| \leq C\exp\left(\frac{-2\pi dN}{\log(8dN/\alpha)}\right), \tag{2.9}$$

where $h = \frac{\log(8dN/\alpha)}{N}$.

3 Sinc-Nyström method

3.1 SE-Sinc scheme

Let $k(s,.)f(.,u(.))\psi'_{SE_i}(.) \in L_\alpha(\psi_{SE_i}(\mathcal{D}_d))$ for all $s \in I$. Then the discrete SE-Sinc operator can be defined by

$$\left(\mathcal{K}_N^{SE_i} f\right)(u(s)) = h\sum_{j=-N}^{N} k(s, t_j^{SE_i}) f\left(t_j^{SE_i}, u(t_j^{SE_i})\right)\psi'_{SE_i}(jh). \tag{3.1}$$

The Nyström method applied to (1.1) is exploited to find $u_N^{SE_i}$ such that

$$u_N^{SE_i}(s) - h\sum_{j=-N}^{N} k(s, t_j^{SE_i}) f\left(t_j^{SE_i}, u(t_j^{SE_i})\right)\psi'_{SE_i}(jh) = g(s), \tag{3.2}$$

where the quadrature points are defined by

$$t_j^{SE_i} = \psi_{SE_i}(jh), \quad j = -N, \cdots, N.$$

Solving (3.2) reduces to solving a finite dimensional nonlinear system. For any solution of (3.2) the values $u_N^{SEi}(t_j^{SE_i})$ at the quadrature points satisfy the nonlinear system

$$u_N^{SE_i}(t_l^{SE_i}) - h\sum_{j=-N}^{N} k(t_l^{SE_i}, t_j^{SE_i}) f\left(t_j^{DE_i}, u(t_j^{DE_i})\right)\psi'_{SE_i}(jh) = g(t_l^{SE_i}), \quad l = -N, \cdots, N.$$

Then the approximate solution $u_N^{SE_i}(s)$ at an arbitrary point s, can be expressed as

$$u_N^{SE_i}(s) = g(s) + h \sum_{j=-N}^{N} k(s, t_j^{SE_i}) f(t_j^{SE_i}, u(t_j^{SE_i})) \psi'_{SE_i}(jh). \tag{3.3}$$

Equation (3.2) can be written in the following discrete SE-sinc operator equation

$$\left(\mathcal{I} - \mathcal{K}_N^{SE_i} f\right) u_N^{SE_i} = g. \tag{3.4}$$

3.2 DE-Sinc scheme

Let $k(s,.)f(.,u(.))\psi'_{DE_i}(.) \in L_\alpha(\psi_{DE_i}(\mathcal{D}_d))$ for all $s \in I$. Then, the discrete DE-Sinc operator can be defined by

$$\left(\mathcal{K}_N^{DE_i} f\right)(u(s)) = h \sum_{j=-N}^{N} k(s, t_j^{DE_i}) f\left(t_j^{DE_i}, u(t_j^{DE_i})\right) \psi'_{DE_i}(jh). \tag{3.5}$$

The Nyström method applied to (1.1) is exploited to fined u_N^{DEi} such that

$$u_N^{DE_i}(s) - h \sum_{j=-N}^{N} k(s, t_j^{DE_i}) f\left(t_j^{DE_i}, u(t_j^{DE_i})\right) \psi'_{DE_i}(jh) = g(s), \tag{3.6}$$

where the quadrature points are defined by

$$t_j^{DE_i} = \psi_{DE_i}(jh), \quad j = -N \cdots N.$$

solving (3.6) reduce to solving a finite dimensional nonlinear system. For any solution of (3.6) the value $u_N^{DE_i}(t_j^{DE_i})$ at the quadrature points satisfy the nonlinear system

$$u_N^{DE_i}(t_l^{DE_i}) - h \sum_{j=-N}^{N} k(t_l^{DE_i}, t_j^{DE_i}) f\left(t_j^{DE_i}, u(t_j^{DE_i})\right) \psi'_{DE_i}(jh) = g(t_l^{DE_i}), \quad l = -N, \cdots, N.$$

Then the approximate solution $u_N^{DE_i}(s)$ at an arbitrary point s can be expressed as

$$u_N^{DE_i}(s) = g(s) + h \sum_{j=-N}^{N} k(s, t_j^{DE_i}) f(t_j^{DE_i}, u(t_j^{DE_i})) \psi'_{DE_i}(jh). \tag{3.7}$$

Equation (3.6) can be written in the following discrete DE-Sinc operator equation

$$\left(\mathcal{I} - \mathcal{K}_N^{DE_i} f\right) u_N^{DE_i} = g. \tag{3.8}$$

4 Convergence analysis

Throughout this section, we discuss the convergence of the SE and DE Sinc-Nyström methods on the semi-infinite interval $I = [0, \infty)$, we first consider the SE-case. Assume that u and g belong to the space $\mathcal{C}_l$, the space of all continuous functions on $[0, \infty)$ having a limit at infinity, which is a Banach space when equipped with the norm $\|g\|_\infty = \sup_{s \in I} |g(s)|$. Also, we suppose that (1.2) has an isolated solution $u_0 \in \mathcal{C}_l$ and $\mathcal{K}f$ possesses a continuous first and a bounded second derivative on $B(u_0, \delta)$ where

$$B(u_0, \delta) = \left\{u \in \mathcal{C}_l : \|u - u_0\|_\infty \leqq \delta, \ \delta > 0\right\}.$$

For prove the following Theorem we need to mentioned the following required conditions, let the kernel $k(.,.)$ satisfy

A_1. $k(s,t)$ is bounded and continuous for $s, t \in I$.

A_2. $k(s,t)$ is continuous in t uniformly with respect to s for all $s, t \in I$.

A_3. For each $t \in I$, $k(s,t) \to 0$ as $s \to \infty$.

We also assume that the following conditions are met on the nonlinear function $f(.,u(.))$

B_1. $f(s,u)$ is defined and continuous on $I \times \mathbb{R}$.

B_2. $f(s,u)$ is bounded for $s \in I$ uniformly for u in any bounded set.

B_3. the partial derivative $f_u(s,u) = \frac{\partial}{\partial u} f(s,u)$ exists and is continuous on $I \times \mathbb{R}$.

B_4. the second partial derivative $f_{uu}(s,u) = \frac{\partial^2}{\partial u^2} f(s,u)$ exists, continuous on $I \times \mathbb{R}$, and bounded for $s \in I$ uniformly for u in any bounded set.

Theorem 4.1. Let $A_1 - A_3$ and $B_1 - B_4$ hold, assume that $k(s,.)f(.,u(.))\psi' \in L_\alpha(\psi_{SE_i}(\mathcal{D}_d))$ with $0 < d < \pi/2$ and $u \in B(u_0, \delta)$, then

C_1. $\{\mathcal{K}_N^{SE_i} f : N \geq 1\}$ is a collectively compact family on $\mathcal{C}_l$.

C_2. $\mathcal{K}_N^{SE_i} f$ is pointwise convergent to $\mathcal{K} f$ on $\mathcal{C}_l$.

C_3. For $N \geq 1$, $\mathcal{K}_N^{SE_i} f$ possesses continuous first and bounded second Fréchet derivatives on $B(u_0,\delta)$. Moreover,

$$\|(\mathcal{K}_N^{SE_i} f)''\| \leq \lambda < \infty,$$

where λ is a constant independent of N.

Proof. We recall that the set $\{\mathcal{K}_N^{SE_i} f : N \geq 1\}$ is a collectively compact family on the Banach space $\mathcal{C}_l$ if the set $\Lambda = \{(\mathcal{K}_N^{SE_i} f)u : N \geq 1, u \in \mathcal{B}\}$, (where $\mathcal{B}$ is the unit ball in $\mathcal{C}_l$) is a relatively compact subset of $\mathcal{C}_l$, we deduce that the set $\{\mathcal{K}_N^{SE_i} f : N \geq 1\}$ is collectively compact if Λ is equicontinuous at each point $s \in I$, equiconvergent at infinity and bounded.
From (3.1) we have

$$\left| \left(\mathcal{K}_N^{SE_i} f\right)(u(s')) - \left(\mathcal{K}_N^{SE_i} f\right)(u(s)) \right| \leq h \sum_{j=-N}^{N} \left| k(s', t_j^{SE_i}) - k(s, t_j^{SE_i}) \right| \left| f(t_j^{SE_i}, u(t_j^{SE_i}))\psi'_{SE_i}(jh) \right|,$$

due to the conditions A_1, A_2, B_1 and B_2 we obtain

$$\left| \left(\mathcal{K}_N^{SE_i} f\right)(u(s')) - \left(\mathcal{K}_N^{SE_i} f\right)(u(s)) \right| \to 0 \text{ as } s' \to s, \forall s \in I, \text{ uniformly for } N \geq 1, \tag{4.1}$$

hence, we conclude that Λ is equicontinuous at each point of I.
Also from (3.1)

$$\left| \left(\mathcal{K}_N^{SE_i} f\right)(u(s)) \right| \leq h \sum_{j=-N}^{N} \left| k(s, t_j^{SE_i}) \right| \left| f\left(t_j^{SE_i}, u(t_j^{SE_i})\right) \psi'_{SE_i}(jh) \right|, \tag{4.2}$$

hence by the condition $A3$, the set $\{(\mathcal{K}_N^{SE_i} f)u \mid N \geqslant 1,\ u \in B\}$ is equiconvergent to zero at infinity.
Next, we seek to show that S is bounded. It follows from the assumption of Theorem 2.1 that

$$(\mathcal{K}_N^{SE_i} f)u(s) \to (\mathcal{K} f)u(s), \quad \text{for all } s \in I.$$

It is known that pointwise convergence on the interval $[0,\infty)$ of a family that is equicontinuous at each point of $[0,\infty)$ and equiconvergent at infinity is sufficient to guarantee uniform convergence, hence

$$\lim_{N\to\infty} \|(\mathcal{K}_N^{SE_i} f)u - (\mathcal{K} f)u\|_\infty = 0,$$

for all $u \in \mathcal{C}_l$, then

$$\sup_N \|(\mathcal{K}_N^{SE_i} f)u\|_\infty < \infty,$$

since $\{\mathcal{K}_N^{SE_i} f\}$ is a sequence of bounded operators on the Banach space $\mathcal{C}_l$, it follows from the uniform-boundedness (Banach-Steinhaus) theorem that

$$\sup_N \|\mathcal{K}_N^{SE_i} f\| < \infty,$$

thus, Λ is bounded.
So from the above-mentioned discussions, C_1 holds.
Due to the Theorem 2.1 the assumption C_2 holds immediately.
The condition B_3 implies that $\mathcal{K}f$ is Fréchet differentiable with

$$(\mathcal{K}f)'(u)x(s) = \int_I k(s,t) f_u(t,u(t))x(t)dt, \qquad s \in I, \quad x \in B(u_0,\delta),$$

and the condition B_4 leading to the existence and the boundedness of the second Fréchet derivative with

$$(\mathcal{K}f)''(u)(x,y)(s) = \int_I k(s,t) f_{uu}(t,u(t))x(t)y(t)dt, \qquad s \in I, \quad x,y \in B(u_0,\delta),$$

similar to $\left(\mathcal{K}_N^{SE_i} f\right)$, $(\mathcal{K}_N^{SE_i} f)'$ and $(\mathcal{K}_N^{SE_i} f)''$ can be defined by the SE-Sinc quadrature formula as follows

$$\left(\mathcal{K}_N^{SE_i} f\right)'(u)x(s) = h \sum_{j=-N}^{N} k(s,t_j^{SE_i}) f_u\left(t_j^{SE_i}, u(t_j^{SE_i})\right) \psi'_{SE_i}(jh) x(t_j^{SE_i}),$$

$$\left(\mathcal{K}_N^{SE_i} f\right)''(u)(x,y)(s) = h \sum_{j=-N}^{N} k(s,t_j^{SE_i}) f_{uu}\left(t_j^{SE_i}, u(t_j^{SE_i})\right) \psi'_{SE_i}(jh) x(t_j^{SE_i}) y(t_j^{SE_i}),$$

if we consider that

$$k(s,.) f_u(.,u(.))x(.)\psi'_{SE_i}(.) \in L_\alpha\left(\psi_{SE_i}(\mathcal{D}_d)\right),$$

and

$$k(s,.) f_{uu}(.,u(.))x(.)y(.)\psi'_{SE_i}(.) \in L_\alpha\left(\psi_{SE_i}(\mathcal{D}_d)\right),$$

for all $s \in I$, then by Theorem 2.1 and the conditions B_3 and B_4 it is easily to concluded C_3.

Lemma 4.2. (**Weiss [9]**) Assume that $[\mathcal{I} - (\mathcal{K}f)'(u_0)]$ is nonsingular and that the hypotheses $C_1 - C_3$ hold. Then the linear operator $\left[\mathcal{I} - (\mathcal{K}_N^{SE_i} f)'(u_0)\right]$ are non-singular for sufficiently large N and

$$\left\| \left[\mathcal{I} - (\mathcal{K}_N^{SE_i} f)'(u_0)\right]^{-1} \right\|_\infty \le \beta < \infty, \tag{4.3}$$

where β is a positive constant.

Theorem 4.3. Assume that the assumptions of Lemma 4.2 hold. Then there exists a positive integer N_1 such that, for all $N \ge N_1$, Eq.(3.4) has a unique solution $u_N^{SE_i} \in B(u_0,\delta)$. Furthermore, there exists a constant C independent of N such that

$$\left\| u_0 - u_N^{SE_i} \right\|_\infty \le C \exp(-\sqrt{2\pi d\alpha N}). \tag{4.4}$$

Proof. By subtracting (1.2) from (3.4) we obtain

$$u_0 - u_N^{SE_i} = (\mathcal{K}f)(u_0) - (\mathcal{K}_N^{SEi} f)(u_N^{SEi}),$$

by adding the term $(\mathcal{K}_N^{SE_i} f)'(u_0)(u_0 - u_N^{SE_i})$ on both sides we have

$$\left[\mathcal{I} - (\mathcal{K}_N^{SE_i} f)'(u_0)\right](u_0 - u_N^{SE_i}) = (\mathcal{K}f)(u_0) - (\mathcal{K}_N^{SE_i} f)(u_0) - \Big[(\mathcal{K}_N^{SE_i} f)(u_N^{SE_i}) - (\mathcal{K}_N^{SE_i} f)(u_0) - (\mathcal{K}_N^{SE_i} f)'(u_0)(u_N^{SE_i} - u_0)\Big].$$

By condition C_3, the term $(\mathcal{K}_N^{SE_i} f)(u_N^{SE_i}) - (\mathcal{K}_N^{SE_i} f)(u_0) - (\mathcal{K}_N^{SE_i} f)'(u_0)(u_N^{SE_i} - u_0)$ has been bounded by the term $\frac{1}{2}\lambda\|u_0 - u_N^{SE_i}\|_\infty^2$, then from Lemma 4.2 we have

$$\left\|u_0 - u_N^{SE_i}\right\|_\infty \leq \beta\left[\|(\mathcal{K}f)(u_0) - (\mathcal{K}_N^{SE_i} f)(u_0)\|_\infty + \frac{1}{2}\lambda\|u_0 - u_N^{SE_i}\|_\infty^2\right],$$

hence

$$\begin{aligned}\left\|u_0 - u_N^{SE_i}\right\|_\infty &\leq \frac{\beta\|(\mathcal{K}f)(u_0) - (\mathcal{K}_N^{SE_i} f)(u_0)\|_\infty}{(1 - \frac{\beta\lambda}{2})} \\ &\leq \frac{\beta}{1 - \frac{\beta\lambda}{2}}\|(\mathcal{K}f)(u_0) - (\mathcal{K}_N^{SE_i} f)(u_0)\|_\infty.\end{aligned}$$

Then by using Theorem 2.1 we obtain the desired result.
Concerning the convergence of the DE-Sinc Nyström method, we can define the assumptions $C_1 - C_3$ for the DE-case by replacing the SE-transformation ψ_{SE_i} with DE-transformation ψ_{DE_i}. Then we can formulate and prove the following Theorem in the same way as in the SE-case.

Theorem 4.4. Assume that the same assumptions of Lemma 4.2 are satisfied for the DE-case. Then there exists a positive integer N_1 such that, for all $N \geq N_1$, Eq.(3.8) has a unique solution $u_N^{DE_i} \in B(u_0, \delta)$. Furthermore, there exist a constant C independent of N such that

$$\|u_0 - u_N^{DE_i}\|_\infty \leq C \exp\left(\frac{-2\pi dN}{\log(8dN/\alpha)}\right). \tag{4.5}$$

5 Numerical results

In this section, we show numerical results that illustrate the theoretical results obtained previously. As we mentioned in the preceding section, the convergence of the SE-Sinc and DE-Sinc methods depends to the parameters α and d, the important parameter d value is 1.57 for both methods, and the parameter α changes by each Example.

Example 5.1. Consider the following nonlinear integral equation

$$u(s) + \int_0^\infty e^{-(s+t)}u^2(t)dt = 6e^{-s},$$

where the exact solution is given by $u(s) = 3e^{-s}$, Table 1 shows a comparison of the maximum absolute errors obtained using SE-Sinc and DE-Sinc methods with $\alpha = 1$, respectively $\alpha = 3$, and those obtained from [1].

Table 1: Maximum absolute errors for Example 5.1 .

N	SE1	SE2	DE1	DE2	DE3	[1]
8	2.17e-04	9.98e-04	6.10e-05	8.84e-09	6.36e-08	
16	3.31e-05	2.38e-05	4.79e-08	3.55e-15	1.33e-15	6.54e-03
32	2.54e-07	1.19e-07	5.03e-11	1.78e-15	1.33e-15	8.39e-04
64	1.02e-10	6.94e-11	2.22e-15	2.66e-15	2.22e-15	1.05e-04

Example 5.2. Consider the following nonlinear integral equation

$$u(s) + \int_0^\infty \frac{e^{-(s+t)}}{1 + u(t) + u^2(t)}dt = (1 - \frac{\pi}{3\sqrt{3}})e^{-s},$$

whose exact solution is $u(s) = e^{-s}$, we choose $\alpha = 1$, for the SE-Sinc method and $\alpha = 2$, for the DE-Sinc method, the obtained numerical results are given in Table 2.

Example 5.3. Consider the following nonlinear integral equation

$$u(s) + \int_0^\infty e^{-t(s+1)}u^2(t)dt = \sin(s) - \frac{2}{s^3 + 3s^2 + 7s + 5},$$

whose exact solution is $u(s) = \sin(s)$, we choose $\alpha = 5$, for the SE-Sinc method and $\alpha = 9$, for the DE-Sinc method, the numerical results for this Example are given in Table 3.

Table 2: Maximum absolute errors for Example 5.2.

N	SE1	SE2	DE1	DE2
4	1.83e-02	2.00e-03	1.46e-02	1.46e-05
8	2.33e-04	1.16e-04	1.50e-03	1.09e-08
16	1.67e-05	2.12e-06	3.15e-05	6.88e-15
32	1.18e-08	7.65e-09	1.08e-08	2.22e-16
64	9.69e-11	2.62e-12	5.15e-13	4.44e-16

Table 3: Maximum absolute errors for Example 5.3.

N	SE1	SE2	DE1	DE2	DE3
4	2.45e-02	1.32e-02	4.72e-02	7.59e-04	1.30e-03
8	4.70e-03	3.70e-03	6.50e-03	6.86e-05	2.05e-04
16	6.01e-04	7.75e-04	2.80e-03	4.40e-07	1.75e-06
32	8.86e-06	4.51e-05	1.93e-04	4.73e-11	2.36e-10
64	2.03e-07	1.70e-06	3.03e-06	3.05e-16	2.16e-16

Example 5.4. Consider the following nonlinear integral equation on the half-line

$$u(s) + \int_0^\infty e^{-(s+t)} \cos(u(t))dt = e^{-s}(1 - \sin(1)),$$

where the exact solution is given by $u(s) = e^{-s}$, we choose $\alpha = 1$, for SE-Sinc method and $\alpha = 2$, for DE-Sinc method, the numerical results for this Example are given in Table 4.

Table 4: Maximum absolute errors for Example 5.4.

N	SE1	SE1	DE1	DE2
4	9.00e-03	3.60e-03	1.35e-02	1.39e-05
8	1.65e-04	2.15e-04	1.40e-03	1.92e-08
16	4.43e-05	2.15e-04	3.23e-05	9.77e-15
32	8.32e-08	1.54e-08	1.10e-08	3.33e-16
64	1.41e-10	6.12e-12	5.08e-13	6.66e-16

Example 5.5. Consider the linear integral equation

$$u(s) + \int_0^\infty e^{-t^2-s} u(t)dt = g(s),$$

where $g(s)$ is selected so that the exact solution is $u(s) = \frac{1}{s^4+2s^2+1}$, Table 5 shows the numerical results using DE1-Sinc and DE2-Sinc methods with $\alpha = 7$.

Table 5: Maximum absolute errors for Example 5.5.

N	DE1	DE2
4	1.63e-03	9.91e-04
9	7.81e-04	1.97e-07
12	1.50e-05	4.96e-10
15	2.65e-06	1.90e-11
18	9.94e-08	5.28e-13
21	9.96e-08	1.65e-14

6 Conclusion

In this paper, we have provided two numerical methods based on Sinc-quadrature for nonlinear integral equations of Hammerstein type on the half line, the methods have been developed by means of the Sinc approximation with the Single Exponential (SE) and Double Exponential (DE) transformations. We have discussed the convergence for both schemes to prove the accuracy of our approaches then the numerical results have confirmed that the error decay exponentially.

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