Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 3105–3114
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.24577.2775

Hankel determinant of a subclass of analytic and bi-univalent functions defined by means of subordination and q-differentiation

Ayotunde Olajide Lasode*, Timothy Oloyede Opoola

Department of Mathematics, University of Ilorin, PMB 1515, Ilorin, Nigeira

(Communicated by Ali Jabbari)

Abstract

In this present article, the q-derivative operator and the subordination principle are use to define a class of functions that are analytic and bi-univalent in the open unit disk. Our aim for this class is to obtain the upper bound for the second Hankel determinant for functions in this new subclass of analytic and bi-univalent functions.

Keywords: Bi-univalent function, Ma-Minda function, Hankel determinant, q-differentiation, subordination
2020 MSC: Primary 30C45, Secondary 30C50

1 Introduction, Definitions and Preliminaries

Here we let $\mathcal{A}$ represent the class of analytic functions having series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ f(0) = 0, \ f'(0) = 1 \text{ and } z \in \pounds := \{z \in \mathbb{C} : |z| < 1\}. \tag{1.1}$$

Also let $\mathcal{S} \subset \mathcal{A}$ denote the subclass of *normalised analytic and univalent functions* in $\pounds$.

Let $h(z)$ and $H(z)$ be analytic functions. $h(z)$ is said to be *subordinate* to $H(z)$, represented as $h(z) \prec H(z)$ $(z \in \pounds)$, if there is an analytic function $\omega(z)$ ($\omega(0) = 0$ and $|\omega(z)| < 1$) such that $h(z) = H(\omega(z))$. If $H(z)$ is univalent in $\pounds$, then

$$h \prec H \iff h(0) = H(0) \quad \text{and} \quad h(\pounds) \subseteq H(\pounds).$$

See [19]. Let $\mu(z)$ be an analytic function with positive real part in $\pounds$ such that $\mu(0) = 1$, $\mu'(0) > 0$, and $\mu(\pounds)$ is starlike with respect to 1 and symmetric with respect to the real axis. Thus $\mu(z)$ has series representation

$$\mu(z) = 1 + \sum_{n \geqslant 1} B_n z^n \quad (B_1 > 0, \ z \in \pounds). \tag{1.2}$$

Recently, the concept of quantum calculus (or q-calculus) has inspired many researchers in geometric function theory. It was first introduced by Jackson [8, 9] and since then many researchers (for instance [4, 6, 10, 14, 15]) have

*Corresponding author
Email addresses: lasode_ayo@yahoo.com (Ayotunde Olajide Lasode), opoola.to@unilorin.edu.ng (Timothy Oloyede Opoola)

Received: September 2021 *Accepted:* February 2022

used it in various ways to define and establish some properties for some classes and subclasses of univalent functions and bi-univalent functions. In [1, 8, 9, 12] the q-derivative $(0 < q < 1)$ of functions $f \in \mathcal{A}$ was defined by

$$\left.\begin{array}{lll} D_q f(0) = f'(0) & when & z = 0 \ (\textit{if it exists}) \\ D_q f(z) = \frac{f(qz)-f(z)}{z(q-1)} & when & z \neq 0 \\ D_q^2 f(z) = D_q(D_q f(z)). & & \end{array}\right\} \tag{1.3}$$

We remark that from (1.1) and (1.3), we can formulate that

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} \quad \text{and} \quad z D_q^2 f(z) = \sum_{n=2}^{\infty} [n-1]_q [n]_q a_n z^{n-1} \tag{1.4}$$

$[n]_q = \frac{1-q^n}{1-q}$ and note that if $q \longrightarrow 1^-$, then $[n]_q \longrightarrow n$. For more information see [1, 2, 12].

It is well-known by the Koebe one-quarter theorem (see [7]) that the range of every function $f \in \mathcal{S}$ covers the disk $|w| < 0.25$, thus, every $f \in \mathcal{S}$ in the form (1.1) has inverse f^{-1} where

$$f^{-1}(f(z)) = z \quad (z \in \pounds)$$

and

$$f(f^{-1}(w)) = w \quad (w : |w| < r_0(f);\ r_0(f) \geq 0.25)\,.$$

The inverse $f^{-1}(w)$ has series representation

$$F(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots\,. \tag{1.5}$$

In [16], Lewin introduced the class $\mathcal{B}$ of analytic and bi-univalent functions in $\pounds$ and proved that the bound for the second coefficient of every $f \in \mathcal{B}$ satisfies the inequality $|a_2| < 1.51$. A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\pounds$ if both $f(z)$ and $f^{-1}(w)$ are univalent in $\pounds$. See [14, 20, 24, 25] for more discussions, comprehensive history and some properties of bi-univalent functions. Note that the class $\mathcal{B}$ is non-void. Instances of functions $f \in \mathcal{B}$ are

$$f(z) = z, \quad f(z) = -\log(1-z), \quad f(z) = z/(1-z), \quad f(z) = \log[(1+z)/(1-z)]^{1/2}.$$

Note that the familiar *Koebe function* $\kappa(z) = \sum\limits_{n=1}^{\infty} n z^n$, its rotation function $\kappa(\theta; z) = \sum\limits_{n=1}^{\infty} n e^{i(n-1)\theta} z^n$, $f(z) = z/(1-z^2)$ and $f(z) = z - z^2/2$ are non-members of $\mathcal{B}$.

Represented by $\mathcal{H}_{m,n}(f)$ is the m^{th} Hankel determinant whose elements are the coefficients of function $f \in \mathcal{S}$ given by (1.1). The determinant $\mathcal{H}_{m,n}(f)$ was defined by Pommerenke [22, 23] as

$$\mathcal{H}_{m,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+m-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+m} \\ \vdots & \vdots & & \vdots \\ a_{n+m-1} & a_{n+m} & \cdots & a_{n+2m-2} \end{vmatrix} \quad (m, n \in \mathbb{N}). \tag{1.6}$$

This determinant has been considered by several authors for various classes and subclasses of $\mathcal{B}$ mostly for $a_1 = 1, m = 2, n = 1$ and $a_1 = 1, m = 2, n = 2$, for instance see [3, 15]. It is easy to see that for $f \in \mathcal{A}$,

$$\mathcal{H}_{2,1}(f) = \begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix} = (a_3 - a_2^2) \quad \text{and} \quad \mathcal{H}_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = (a_2 a_4 - a_3^2). \tag{1.7}$$

$\mathcal{H}_{2,1}(f)$ and $\mathcal{H}_{2,2}(f)$ are well-known as Fekete-Szegö and second Hankel determinant functionals, respectively. In particular, Noor [21] established the rate at which $\mathcal{H}_{m,n}(f)$ grows as $n \longrightarrow \infty$ for f in (1.1). In [22], Pommerenke highlighted some of the applications of Hankel determinants which includes the study of singularities and the solution of power series with integral coefficients of analytic functions. Junod [11] established that problems of orthogonal polynomials can be solved by the use of Hankel determinants while Layman [13] explained some fundamental notations and properties of Hankel matrices and determinants.

Using the above discussions, the following subclass of analytic and bi-univalent functions was defined in [14] as follows.

Definition 1.1. Let $\tau \in \mathbb{C} - \{0\}$, $0 \leqslant \lambda \leqslant 1$ and μ be as defined in (1.2). A function $f \in \mathcal{B}$ is said to be in the class $\mathcal{B}_q(\tau, \lambda, \mu)$ if the subordinations

$$1 + \frac{1}{\tau}[D_q f(z) + \lambda z D_q^2 f(z) - 1] \prec \mu(z) \quad (z \in \pounds) \tag{1.8}$$

and

$$1 + \frac{1}{\tau}[D_q F(w) + \lambda w D_q^2 F(w) - 1] \prec \mu(w) \quad (w \in \pounds) \tag{1.9}$$

hold where $F(w) = f^{-1}(w)$ is defined by (1.5). Note that the class $\mathcal{B}(\tau, \lambda, \mu) = \lim_{q \uparrow 1} \mathcal{B}_q(\tau, \lambda, \mu)$ was studied in [25].

In this work, an upper bound for the second Hankel determinant $\mathcal{H}_{2,2}(f)$ for functions $f \in \mathcal{B}_q(\tau, \lambda, \mu)$ is investigated.

2 Relevant Lemmas

To establish our results, we shall need the following lemmas.

Let $\mathcal{P}$ be the class of analytic functions with positive real parts in $\pounds$ such that $p(0) = 1$.

Lemma 2.1 ([5, 7]). If $p(z) = 1 + \sum_{n \geqslant 1} p_n z^n \in \mathcal{P}$, then $|p_n| \leqslant 2,\ n \in \mathbb{N}$.

Lemma 2.2 ([17, 18]). If $p(z) = 1 + \sum_{n \geqslant 1} p_n z^n \in \mathcal{P}$, then

$$2p_2 = p_1^2 + (4 - p_1^2)x$$

and

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 x - (4 - p_1^2)p_1 x^2 + 2(4 - p_1^2)\left(1 - |x|^2\right) z$$

for some x, z with $|x|, |z| \leqslant 1$.

Proposition 2.3. Let the functions $b(z) = 1 + \sum_{n \geqslant 1} b_n z^n$, $c(z) = 1 + \sum_{n \geqslant 1} c_n z^n$ belong to the class $\mathcal{P}$ and $c_1 = -b_1$, then

$$\left.\begin{array}{l} 2b_2 = b_1^2 + x(4 - b_1^2) \\ 2c_2 = c_1^2 + y(4 - c_1^2) \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} b_2 - c_2 = \frac{(4-b_1^2)(x-y)}{2} \\ b_2 + c_2 = b_1^2 + \frac{(4-b_1^2)(x+y)}{2} \end{array}\right.$$

and

$$\left.\begin{array}{l} 4b_3 = b_1^3 + 2(4 - b_1^2)b_1 x - (4 - b_1^2)b_1 x^2 + 2(4 - b_1^2)\left(1 - |x|^2\right) z \\ 4c_3 = c_1^3 + 2(4 - c_1^2)c_1 y - (4 - c_1^2)c_1 y^2 + 2(4 - c_1^2)\left(1 - |y|^2\right) w \end{array}\right\}$$

$$\Longrightarrow b_3 - c_3 = \frac{b_1^3}{2} + \frac{b_1(4 - b_1^2)(x + y)}{2} - \frac{b_1(4 - b_1^2)(x^2 + y^2)}{4} + \frac{(4 - b_1^2)[(1 - |x|^2)z - (1 - |y|^2)w]}{2}$$

for some w, x, y, z with $|w|, |x|, |y|, |z| \leqslant 1$ and $|b_1|, |c_1| \in [0, 2]$.

The coefficients a_2 and a_3 were found by Lasode and Opoola in [14]. Now, we obtain the coefficient a_4.

Lemma 2.4. Let $f \in \mathcal{B}_q(\tau, \lambda, \mu)$, then there exists the analytic functions $b(z) = 1 + \sum_{n \geqslant 1} b_n z^n$, $c(z) = 1 + \sum_{n \geqslant 1} c_n z^n \in \mathcal{P}$ such that

$$a_2^2 = \frac{\tau^2 B_1^3 (b_2 + c_2)}{4\{\tau B_1^2 [3]_q Q_2 + [2]_q^2 Q_1^2 (B_1 - B_2)\}} \tag{2.1}$$

$$a_3 = \frac{\tau^2 B_1^2 b_1^2}{4[2]_q^2 Q_1^2} + \frac{\tau B_1 (b_2 - c_2)}{4[3]_q Q_2} \tag{2.2}$$

$$a_4 = \frac{\tau \left\{\frac{1}{2}(B_1 - 2B_2 + B_3) b_1^3 - [(B_1 - B_2)(b_2 + c_2)] b_1 + B_1 (b_3 - c_3)\right\}}{4[4]_q Q_3} + \frac{5B_1^2 \tau^2 b_1 (b_2 - c_2)}{16[2]_q [3]_q Q_1 Q_2} \tag{2.3}$$

where

$$Q_n = (1 + [n]_q \lambda) \geqslant 1, \quad n \in \mathbb{N}, \tag{2.4}$$

$\tau \in \mathbb{C} - \{0\}$, $0 \leqslant \lambda \leqslant 1$ and B_n are coefficients of $\mu(z)$ in (1.2).

Proof . Let $f(z) \in \mathcal{B}$ and $F(w) = f^{-1}(w)$, then there are analytic functions $u(z)$ and $v(w)$ ($u(0) = 0 = v(0)$, $|u(z)|, |v(w)| < 1$) with $w, z \in \pounds$, such that

$$1 + \frac{1}{\tau}[D_q f(z) + \lambda z D_q^2 f(z) - 1] = \mu(u(z)) \tag{2.5}$$

and

$$1 + \frac{1}{\tau}[D_q F(w) + \lambda w D_q^2 F(w) - 1] = \mu(v(w)). \tag{2.6}$$

Now by using (2.4), LHS of (2.5) simplifies to

$$1 + \frac{1}{\tau}[D_q f(z) + \lambda z D_q^2 f(z) - 1] = 1 + \frac{[2]_q Q_1 a_2}{\tau} z + \frac{[3]_q Q_2 a_3}{\tau} z^2 + \frac{[4]_q Q_3 a_4}{\tau} z^3 + \cdots \tag{2.7}$$

and by (1.5) the LHS of (2.6) simplifies to

$$\begin{aligned} 1 + \frac{1}{\tau}&[D_q F(w) + \lambda w D_q^2 F(w) - 1] \\ &= 1 - \frac{[2]_q Q_1 a_2}{\tau} w + \frac{[3]_q Q_2 (2a_2^2 - a_3)}{\tau} w^2 - \frac{[4]_q Q_3 (5a_2^3 - 5a_2 a_3 + a_4)}{\tau} w^3 + \cdots . \end{aligned} \tag{2.8}$$

Define the analytic functions $b(z)$ and $c(w)$ as follows

$$b(z) = \frac{1 + u(z)}{1 - u(z)} \quad \text{and} \quad c(w) = \frac{1 + v(w)}{1 - v(w)},$$

then

$$u(z) = \left(\frac{b(z) - 1}{b(z) + 1}\right) = \frac{1}{2}\left[b_1 z + \left(b_2 - \frac{b_1^2}{2}\right) z^2 + \left(\frac{b_1^3}{2^2} - b_1 b_2 + b_3\right) z^3 + \cdots\right] \tag{2.9}$$

and

$$v(w) = \left(\frac{c(w) - 1}{c(w) + 1}\right) = \frac{1}{2}\left[c_1 w + \left(c_2 - \frac{c_1^2}{2}\right) w^2 + \left(\frac{c_1^3}{2^2} - c_1 c_2 + c_3\right) w^3 + \cdots\right]. \tag{2.10}$$

Now we have from RHS of (2.5) that

$$\begin{aligned} \mu(u(z)) = 1 + \frac{1}{2}B_1 b_1 z + \frac{1}{2}\left[B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{2}B_2 b_1^2\right] z^2 \\ + \frac{1}{2}\left[B_1\left(\frac{b_1^3}{2^2} - b_1 b_2 + b_3\right) + B_2 b_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_3 b_1^3\right] z^3 + \cdots \end{aligned} \tag{2.11}$$

and from RHS of (2.6) that

$$\begin{aligned} \mu(v(w)) = 1 + \frac{1}{2}B_1 c_1 w + \frac{1}{2}\left[B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{2}B_2 c_1^2\right] w^2 \\ + \frac{1}{2}\left[B_1\left(\frac{c_1^3}{2^2} - c_1 c_2 + c_3\right) + B_2 c_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_3 c_1^3\right] w^3 + \cdots . \end{aligned} \tag{2.12}$$

Comparing coefficients in (2.7) and (2.11) gives

$$\frac{[2]_q Q_1 a_2}{\tau} = \frac{B_1 b_1}{2} \tag{2.13}$$

and

$$\frac{[4]_q Q_3 a_4}{\tau} = \frac{1}{2}\left[B_1\left(\frac{b_1^3}{2^2} - b_1 b_2 + b_3\right) + B_2 b_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_3 b_1^3\right]. \tag{2.14}$$

From (2.8) and (2.12), we obtain

$$-\frac{[2]_q Q_1 a_2}{\tau} = \frac{B_1 c_1}{2} \tag{2.15}$$

and

$$-\frac{[4]_q Q_3(5a_2^3-5a_2a_3+a_4)}{\tau}=\frac{1}{2}\left[B_1\left(\frac{c_1^3}{2^2}-c_1c_2+c_3\right)+B_2c_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_3c_1^3\right]. \tag{2.16}$$

Adding (2.13) and (2.15) and simplifying give

$$\Longrightarrow b_1=-c_1,\quad b_1^2=c_1^2 \quad \text{and} \quad b_1^3=-c_1^3. \tag{2.17}$$

Subtracting (2.14) from (2.16) and using (2.17) leads to

$$\frac{-[4]_q Q_3\{5(a_2^3-a_2a_3)+2a_4\}}{\tau}$$
$$=\frac{1}{2}\left\{-\frac{1}{2}\left(B_1-2B_2+B_3\right)b_1^3+[(B_1-B_2)(b_2+c_2)]b_1-B_1(b_3-c_3)\right\} \tag{2.18}$$

and

$$2a_4=\frac{\tau\left\{\frac{1}{2}\left(B_1-2B_2+B_3\right)b_1^3-[(B_1-B_2)(b_2+c_2)]b_1+B_1(b_3-c_3)\right\}}{2[4]_qQ_3}-5(a_2^3-a_2a_3). \tag{2.19}$$

Consider using a_2 in (2.13) and a_3 in Lemma 2.4 and simplifying leads to (2.3). □

3 Main Results

Unless otherwise mentioned in what follows, we assume that $\tau\in\mathbb{C}-\{0\}$, $0\leqslant\lambda\leqslant 1$, μ is defined by (1.2) and $f\in\mathcal{B}$. With these, we establish the second Hankel determinant of functions $f\in\mathcal{B}_q(\tau,\lambda,\mu)$.

Theorem 3.1. Let $\mathcal{B}_q(\tau,\lambda,\mu)$, then

$$|\mathcal{H}_{2,2}(f)|\leqslant\begin{cases}\Phi(2) & for \quad \phi_1\geqslant 0,\quad \phi_2\geqslant 0\\ \max\left\{\frac{B_1^2|\tau|^2}{[3]_q^2Q_2^2},\ \Phi(2)\right\} & for \quad \phi_1>0,\quad \phi_2<0\\ \frac{B_1^2|\tau|^2}{[3]_q^2Q_2^2} & for \quad \phi_1\leqslant 0,\quad \phi_2\leqslant 0\\ \max\{\Phi(t_0),\ \Phi(2)\} & for \quad \phi_1<0,\quad \phi_2>0\end{cases}$$

where

$$\begin{aligned}\phi_1 \equiv&\phi_1(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)\\ =&B_1|\tau|^2\{(2[2]_q^3[3]_q^2|B_1-2B_2+B_3|Q_1^3Q_2^2)+(2[3]_q^2[4]_qQ_2^2Q_3B_1^3|\tau|^2)-(4[2]_q^3[3]_q^2Q_1^3Q_2^2B_1)\\ &+(2[2]_q^4[4]_qB_1Q_1^4Q_3)-(9[2]_q^2[3]_q[4]_qB_1^2|\tau|)Q_1^2Q_2Q_3\}\\ \phi_2 \equiv&\phi_1(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)\\ =&B_1|\tau|^2\{(4[2]_q[3]_q^2|B_1-B_2|Q_1Q_2^2)+(6[2]_q[3]_q^2B_1Q_1Q_2^2)+(9[3]_q[4]_qB_1^2|\tau|Q_2Q_3)\\ &-(4[2]_q^2[4]_qB_1Q_1^2Q_3)\}\\ \Phi(t_0)=&\frac{B_1^2|\tau|^2}{[3]_q^2Q_2^2}-\frac{3[\phi_2(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)]^2}{32[3]_q^2[4]_qQ_2^2Q_3\phi_1(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)}\end{aligned}$$

and

$$\Phi(2)=\frac{B_1^2|\tau|^2}{[3]_q^2Q_2^2}+\frac{B_1|\tau|^2\left\{(2[2]_q^3[3]_q^2|B_1-2B_2+B_3|Q_1^3Q_2^2)+(2[3]_q^2[4]_qQ_2^2Q_3B_1^3|\tau|^2)-(4[2]_q^3[3]_q^2Q_1^3Q_2^2B_1)+(2[2]_q^4[4]_qB_1Q_1^4Q_3)-(9[2]_q^2[3]_q[4]_qB_1^2|\tau|)Q_1^2Q_2Q_3\right\}}{2[2]_q^4[3]_q^2[4]_qQ_1^4Q_2^2Q_3}$$

Proof . Using Lemma 2.4 in (1.7) leads to

$$\begin{aligned} a_2a_4-a_3^2 =& \frac{B_1b_1^4\tau^2(B_1-2B_2+B_3)}{16[2]_q[4]_qQ_1Q_3}-\frac{B_1b_1^2\tau^2(B_1-B_2)(b_2+c_2)}{8[2]_q[4]_qQ_1Q_3}+\frac{B_1^2b_1\tau^2(b_3-c_3)}{8[2]_q[4]_qQ_1Q_3}\\ &+\frac{5B_1^3b_1^2\tau^3(b_2-c_2)}{32[2]_q^2[3]_qQ_1^2Q_2}-\frac{B_1^4b_1^4\tau^4}{16[2]_q^4Q_1^4}-\frac{B_1^3b_1^2\tau^3(b_2-c_2)}{8[2]_q^2[3]_qQ_1^2Q_2}-\frac{B_1^2\tau^2(b_2-c_2)^2}{16[3]_q^2Q_2^2} \end{aligned} \tag{3.1}$$

so that by using Proposition 2.3 in (3.1) leads to

$$\begin{aligned} a_2a_4-a_3^2 =& \frac{B_1b_1^4\tau^2(B_1-2B_2+B_3)}{16[2]_q[4]_qQ_1Q_3}-\frac{B_1b_1^4\tau^2(B_1-B_2)}{8[2]_q[4]_qQ_1Q_3}-\frac{B_1b_1^2\tau^2(B_1-B_2)(4-b_1^2)(x+y)}{16[2]_q[4]_qQ_1Q_3}\\ &+\frac{B_1^2b_1^4\tau^2}{16[2]_q[4]_qQ_1Q_3}+\frac{B_1^2b_1^2\tau^2(4-b_1^2)(x+y)}{16[2]_q[4]_qQ_1Q_3}-\frac{B_1^2b_1^2\tau^2(4-b_1^2)(x^2+y^2)}{32[2]_q[4]_qQ_1Q_3}\\ &+\frac{B_1^2b_1\tau^2(4-b_1^2)[(1-|x|^2)z-(1-|y|^2)w]}{16[2]_q[4]_qQ_1Q_3}+\frac{5B_1^3b_1^2\tau^3(4-b_1^2)(x-y)}{64[2]_q^2[3]_qQ_1^2Q_2}\\ &-\frac{B_1^4b_1^4\tau^4}{16[2]_q^4Q_1^4}-\frac{B_1^3b_1^2\tau^3(4-b_1^2)(x-y)}{16[2]_q^2[3]_qQ_1^2Q_2}-\frac{B_1^2\tau^2(4-b_1^2)^2(x-y)^2}{64[3]_q^2Q_2^2}. \end{aligned} \tag{3.2}$$

Recall that for functions $b(z)\in\mathcal{P}$, $|b_1|\leqslant 2$ (Lemma 2.1). Now letting $t=b_1$, we may assume without any restriction that $t\in[0,2]$. Thus, using triangle inequality with $\alpha=|x|\leqslant 1$ and $\beta=|y|\leqslant 1$ in (3.2) we obtain

$$\begin{aligned} &|a_2a_4-a_3^2|\leqslant\\ &\left\{\frac{B_1|\tau|^2|B_1-2B_2+B_3|t^4}{16[2]_q[4]_qQ_1Q_3}+\frac{B_1|\tau|^2|B_1-B_2|t^4}{8[2]_q[4]_qQ_1Q_3}+\frac{B_1^2|\tau|^2t^4}{16[2]_q[4]_qQ_1Q_3}+\frac{B_1^2|\tau|^2(4-t^2)t}{8[2]_q[4]_qQ_1Q_3}+\frac{B_1^4|\tau|^4t^4}{16[2]_q^4Q_1^4}\right\}\\ &+\left\{\frac{B_1|\tau|^2|B_1-B_2|(4-t^2)t^2}{16[2]_q[4]_qQ_1Q_3}+\frac{B_1^2|\tau|^2(4-t^2)t^2}{16[2]_q[4]_qQ_1Q_3}+\frac{5B_1^3|\tau|^3(4-t^2)t^2}{64[2]_q^2[3]_qQ_1^2Q_2}+\frac{B_1^3|\tau|^3(4-t^2)t^2}{16[2]_q^2[3]_qQ_1^2Q_2}\right\}(\alpha+\beta)\\ &+\left\{\frac{B_1^2|\tau|^2(4-t^2)t^2}{32[2]_q[4]_qQ_1Q_3}-\frac{B_1^2|\tau|^2(4-t^2)t}{16[2]_q[4]_qQ_1Q_3}\right\}(\alpha^2+\beta^2)\\ &+\frac{B_1^2|\tau|^2(4-t^2)^2}{64[3]_q^2Q_2^2}(\alpha+\beta)^2 \end{aligned}$$

which by equivalence we have

$$|a_2a_4-a_3^2|\leqslant V_1(t)+V_2(t)(\alpha+\beta)+V_3(t)(\alpha^2+\beta^2)+V_4(t)(\alpha+\beta)^2=F(\alpha,\beta) \tag{3.3}$$

where

$$\begin{aligned} V_1(t) =& \left\{\frac{B_1|\tau|^2|B_1-2B_2+B_3|t^4}{16[2]_q[4]_qQ_1Q_3}+\frac{B_1|\tau|^2|B_1-B_2|t^4}{8[2]_q[4]_qQ_1Q_3}+\frac{B_1^2|\tau|^2t^4}{16[2]_q[4]_qQ_1Q_3}+\frac{B_1^2|\tau|^2(4-t^2)t}{8[2]_q[4]_qQ_1Q_3}\right.\\ &\left.+\frac{B_1^4|\tau|^4t^4}{16[2]_q^4Q_1^4}\right\}\geqslant 0\\ V_2(t) =& \left\{\frac{B_1|\tau|^2|B_1-B_2|(4-t^2)t^2}{16[2]_q[4]_qQ_1Q_3}+\frac{B_1^2|\tau|^2(4-t^2)t^2}{16[2]_q[4]_qQ_1Q_3}+\frac{5B_1^3|\tau|^3(4-t^2)t^2}{64[2]_q^2[3]_qQ_1^2Q_2}+\frac{B_1^3|\tau|^3(4-t^2)t^2}{16[2]_q^2[3]_qQ_1^2Q_2}\right\}\geqslant 0\\ V_3(t) =& \left\{\frac{B_1^2|\tau|^2(4-t^2)(t-2)t}{32[2]_q[4]_qQ_1Q_3}\right\}\leqslant 0\\ V_4(t) =& \frac{B_1^2|\tau|^2(4-t^2)^2}{64[3]_q^2Q_2^2}\geqslant 0 \end{aligned}$$

Next is to maximize the function $F(\alpha,\beta)$ in the closed square $\mathbb{S}=\{(\alpha,\beta):(\alpha,\beta)\in[0,1]\times[0,1]\}$. Since the coefficients of the function $F(\alpha,\beta)$ have dependent variable t, then we need to maximize $F(\alpha,\beta)$ in the cases $t=0$, $t=2$ and $t\in(0,2)$.

Case 1: For $t=0$, then from (3.3),

$$F(\alpha,\beta)=\frac{B_1^2|\tau|^2(\alpha+\beta)^2}{4[3]_q^2Q_2^2}$$

and since maximum of $F(\alpha,\beta)$ occurs at $\alpha=1=\beta$, then

$$\max\{F(\alpha,\beta):(\alpha,\beta)\in[0,1]\times[0,1]\}=F(1,1)=\frac{B_1^2|\tau|^2}{[3]_q^2Q_2^2}. \tag{3.4}$$

Case 2: For $t=2$, then from (3.3)

$$F(\alpha,\beta)=\frac{B_1^4|\tau|^4}{[2]_q^4Q_1^4}+\frac{B_1|\tau|^2\{|B_1-2B_2+B_3|+2|B_1-B_2|+B_1\}}{[2]_q[4]_qQ_1Q_3}$$

which is purely a constant function.

Case 3: For $t\in(0,2)$, let $\alpha+\beta=m$ and $\alpha\beta=n$, then (3.3) can be expressed as:

$$F(\alpha,\beta)=V_1(t)+V_2(t)m+[V_3(t)+V_4(t)]m^2-2V_3(t)n=G(m,n),\ m\in[0,2] \text{ and } n\in[0,1]. \tag{3.5}$$

Now to investigate the maximum of

$$G(m,n):(m,n)\in\mathbb{T}=[0,2]\times[0,1] \tag{3.6}$$

consider the partial derivatives

$$\frac{\partial G}{\partial m}=V_2(t)+2[V_3(t)+V_4(t)]m=0$$
$$\frac{\partial G}{\partial n}=-2V_3(t)=0$$

It is clear from the above partial derivatives that the function $G(m,n)$ has no critical point in $\mathbb{T}$, hence, $F(\alpha,\beta)$ has no critical point in the square $\mathbb{S}$. Thus, the function $F(\alpha,\beta)$ can not take a local maximum value in the interior of the square $\mathbb{S}$.

Now to investigate the maximum of $F(\alpha,\beta)$ on the boundary of the square $\mathbb{S}$.

Case 3a: Let $\alpha=0$, $\beta\in[0,1]$ (or similarly, $\beta=0$, $\alpha\in[0,1]$), then from (3.5),

$$F(0,\beta)=V_1(t)+V_2(t)\beta+[V_3(t)+V_4(t)]\beta^2=\psi_1(\beta), \tag{3.7}$$

then

$$\psi_1'(\beta)=V_2(t)+2[V_3(t)+V_4(t)]\beta.$$

Now, since $[V_3(t)+V_4(t)]\geqslant 0$, then

$$\psi_1'(\beta)=V_2(t)+2[V_3(t)+V_4(t)]\beta>0,$$

therefore, the function $\psi_1(\beta)$ is an increasing function for all $\beta\in[0,1]$ and that the maximum occurs at $\beta=1$. Thus, from (3.7),

$$\max\{F(0,\beta):\beta\in[0,1]\}=F(0,1)=V_1(t)+V_2(t)+V_3(t)+V_4(t). \tag{3.8}$$

Case 3b: Let $\alpha=1$, $\beta\in[0,1]$ (or similarly, $\beta=1$, $\alpha\in[0,1]$) then from (3.5),

$$F(1,\beta)=V_1(t)+V_2(t)+V_3(t)+V_4(t)+[V_2(t)+2V_4(t)]\beta+[V_3(t)+V_4(t)]\beta^2=\psi_2(\beta), \tag{3.9}$$

then

$$\psi_2'(\beta)=[V_2(t)+2V_4(t)]+2[V_3(t)+V_4(t)]\beta.$$

Now, since $[V_3(t)+V_4(t)]\geqslant 0$, then $\psi_2'(\beta)=[V_2(t)+2V_4(t)]+2[V_3(t)+V_4(t)]\beta>0$. Therefore, the function $\psi_2(\beta)$ is an increasing function for all $\beta\in[0,1]$ and the maximum occurs at $\beta=1$. Thus, from (3.9),

$$\max\{F(1,\beta):\beta\in[0,1]\}=F(1,1)=V_1(t)+2[V_2(t)+V_3(t)]+4V_4(t). \tag{3.10}$$

Hence, for every $t\in(0,2)$, it can be concluded from (3.8) and (3.10) that

$$V_1(t)+2[V_2(t)+V_3(t)]+4V_4(t)>V_1(t)+V_2(t)+V_3(t)+V_4(t).$$

Therefore,

$$\max\{F(\alpha,\beta):\alpha\in[0,1],\beta\in[0,1]\}=V_1(t)+2[V_2(t)+V_3(t)]+4V_4(t). \tag{3.11}$$

Now in conclusion, since $\psi_1(1)\leqslant\psi_2(1)$ for $t\in[0,2]$ then

$$\max\{F(\alpha,\beta)\}=F(1,1)$$

occurs on the boundary of square $\mathbb{S}$. Thus, the maximum of F occurs at $\alpha=1=\beta$ on the closed square $\mathbb{S}$.

Also to consider the maximum point at (3.11), let $\Phi:(0,2)\longrightarrow\mathbb{R}$ be defined as:

$$\Phi(t)=\max\{F(\alpha,\beta)\}=F(1,1)=V_1(t)+2[V_2(t)+V_3(t)]+4V_4(t). \tag{3.12}$$

Substituting the values of $V_1(t),V_2(t),V_3(t)$ and $V_4(t)$ into (3.12) and simplifying give

$$\Phi(t)=\frac{B_1^2|\tau|^2}{[3]_q^2Q_2^2}+\left\{\frac{B_1|\tau|^2|B_1-2B_2+B_3|}{16[2]_q[4]_qQ_1Q_3}+\frac{B_1^4|\tau|^4}{16[2]_q^4Q_1^4}-\frac{B_1^2|\tau|^2}{8[2]_q[4]_qQ_1Q_3}+\frac{B_1^2|\tau|^2}{16[3]_q^2Q_2^2}-\frac{9B_1^3|\tau|^3}{32[2]_q^2[3]_qQ_1^2Q_2}\right\}t^4$$
$$+\left\{\frac{B_1|\tau|^2|B_1-B_2|}{2[2]_q[4]_qQ_1Q_3}+\frac{3B_1^2|\tau|^2}{4[2]_q[4]_qQ_1Q_3}+\frac{9B_1^3|\tau|^3}{8[2]_q^2[3]_qQ_1^2Q_2}-\frac{B_1^2|\tau|^2}{2[3]_q^2Q_2^2}\right\}t^2$$

or

$$\Phi(t)=\frac{B_1^2|\tau|^2}{[3]_q^2Q_2^2}+\frac{B_1|\tau|^2\left\{\begin{array}{l}(2[2]_q^3[3]_q^2|B_1-2B_2+B_3|Q_1^3Q_2^2)+(2[3]_q^2[4]_qB_1^3|\tau|^2Q_2^2Q_3)\\ -(4[2]_q^3[3]_q^2B_1Q_1^3Q_2^2)+(2[2]_q^4[4]_qB_1Q_1^4Q_3)-(9[2]_q^2[3]_q[4]_qB_1^2|\tau|Q_1^2Q_2Q_3)\end{array}\right\}}{32[2]_q^4[3]_q^2[4]_qQ_1^4Q_2^2Q_3}t^4$$
$$+\frac{B_1|\tau|^2\left\{\begin{array}{l}(4[2]_q[3]_q^2|B_1-B_2|Q_1Q_2^2)+(6[2]_q[3]_q^2B_1Q_1Q_2^2)\\ +(9[3]_q[4]_qB_1^2|\tau|Q_2Q_3)-(4[2]_q^2[4]_qB_1Q_1^2Q_3)\end{array}\right\}}{8[2]_q^2[3]_q^2[4]_qQ_1^2Q_2^2Q_3}t^2 \tag{3.13}$$

which for convenience can be written as

$$\Phi(t)=\frac{B_1^2|\tau|^2}{[3]_q^2Q_2^2}+\frac{\phi_1(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)}{32[2]_q^4[3]_q^2[4]_qQ_1^4Q_2^2Q_3}t^4+\frac{\phi_2(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)}{8[2]_q^2[3]_q^2[4]_qQ_1^2Q_2^2Q_3}t^2 \tag{3.14}$$

where $\phi_1(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)$ and $\phi_2(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)$ are respectively the numerators of the second and third fractions in (3.13).

Now to investigate the maximum value of $\Phi(t)$ in the interval $(0,2)$, then from (3.13),

$$\Phi'(t)=\frac{4\phi_1(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)}{32[2]_q^4[3]_q^2[4]_qQ_1^4Q_2^2Q_3}t^3+\frac{2\phi_2(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)}{8[2]_q^2[3]_q^2[4]_qQ_1^2Q_2^2Q_3}t$$

Let us examine the different results of $\phi_1(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)$ and $\phi_2(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)$ as it follows from (3.13).

•**Result 1:** If $\phi_1(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)\geqslant 0$ and $\phi_2(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)\geqslant 0$ then $\Phi'(t)\geqslant 0$. Hence the function $\Phi(t)$ is an increasing function, which means the maximum point must be on the boundary of $t\in[0,2]$, that is at $t=2$, therefore from (3.13)

$$\max\{F(\alpha,\beta):\alpha\in[0,1],\beta\in[0,1]\}=\Phi(2)$$
$$=\frac{B_1^2|\tau|^2}{[3]_q^2Q_2^2}+\frac{\phi_1(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)}{2[2]_q^4[3]_q^2[4]_qQ_1^4Q_2^2Q_3}+\frac{\phi_2(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)}{2[2]_q^2[3]_q^2[4]_qQ_1^2Q_2^2Q_3}. \tag{3.15}$$

•**Result 2:** If $\phi_1(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)>0$ and $\phi_2(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)<0$, then for

$$\Phi'(t)=\frac{4\phi_1(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)}{32[2]_q^4[3]_q^2[4]_qQ_1^4Q_2^2Q_3}t^3+\frac{2\phi_2(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)}{8[2]_q^2[3]_q^2[4]_qQ_1^2Q_2^2Q_3}t=0$$

implies

$$t_0 = \sqrt{\frac{-2[2]_q^2 Q_1^2 \phi_2(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)}{\phi_1(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)}} \tag{3.16}$$

is a critical point of $\Phi(t)$ and since

$$\Phi''(t_0) = -\frac{\phi_2(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)}{2[2]_q^2[3]_q^2[4]_q Q_1^2 Q_2^2 Q_3} > 0,$$

hence for $\Phi''(t_0) > 0$, the maximum value of function $\Phi(t)$ occurs at $t = t_0$ and (3.13) simplifies to

$$\Phi(t_0) = \frac{B_1^2|\tau|^2}{[3]_q^2 Q_2^2} - \frac{3[\phi_2(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)]^2}{32[3]_q^2[4]_q Q_2^2 Q_3 \phi_1(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)}.$$

In this case,

$$\Phi(t_0) < \frac{B_1^2|\tau|^2}{[3]_q^2 Q_2^2}$$

and

$$\max\{F(\alpha,\beta) : \alpha \in [0,1], \beta \in [0,1]\} = \max\left\{\frac{B_1^2|\tau|^2}{[3]_q^2 Q_2^2},\ \Phi(2)\right\} \tag{3.17}$$

•Result 3: If $\phi_1(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3) \leqslant 0$ and $\phi_2(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3) \leqslant 0$, then $\Phi'(t) \leqslant 0$. The function $\Phi(t)$ is a decreasing function which means maximum point must be on the boundary of $t \in [0,2]$, that is at $t = 0$, therefore, from (3.13),

$$\max\{F(\alpha,\beta) : \alpha \in [0,1], \beta \in [0,1]\} = \Phi(0) = \frac{B_1^2|\tau|^2}{[3]_q^2 Q_2^2} \tag{3.18}$$

•Result 4: If $\phi_1(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3) < 0$ and $\phi_2(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3) > 0$, then for

$$\Phi'(t) = \frac{4\phi_1(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)}{32[2]_q^4[3]_q^2[4]_q Q_1^4 Q_2^2 Q_3} t^3 + \frac{2\phi_2(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)}{8[2]_q^2[3]_q^2[4]_q Q_1^2 Q_2^2 Q_3} t = 0$$

implies

$$t_1 = \sqrt{\frac{-2[2]_q^2 Q_1^2 \phi_2(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)}{\phi_1(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)}} \tag{3.19}$$

is a critical point of $\Phi(t)$ and since

$$\Phi''(t_1) = -\frac{\phi_2(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)}{2[2]_q^2[3]_q^2[4]_q Q_1^2 Q_2^2 Q_3} < 0$$

hence for $\Phi''(t_1) < 0$, the maximum value of function $\Phi(t)$ occurs at $t = t_1$ and from (3.13),

$$\Phi(t_0) = \frac{B_1^2|\tau|^2}{[3]_q^2 Q_2^2} - \frac{3[\phi_2(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)]^2}{32[3]_q^2[4]_q Q_2^2 Q_3 \phi_1(q,\tau,Q_1,Q_2,Q_3,B_1,B_2,B_3)}.$$

In this case,

$$\Phi(t_0) > \frac{B_1^2|\tau|^2}{[3]_q^2 Q_2^2}$$

and

$$\max\{F(\alpha,\beta) : \alpha \in [0,1], \beta \in [0,1]\} = \max\left\{\frac{B_1^2|\tau|^2}{[3]_q^2 Q_2^2},\ \Phi(2)\right\}. \tag{3.20}$$

Hence from (3.15), (3.17), (3.18) and (3.20) the proof is complete. □

Acknowledgments

The authors sincerely appreciate the efforts of the reviewers and the editor for their valuable suggestions that added values to this paper.

References

[1] A. Aral, V. Gupta and R.P. Agarwal, *Applications of q-calculus in operator theory,* Springer, New York, 2013.

[2] M.H. Annaby and Z.S. Mansour, *q-Fractional calculus and equations,* Springer Science+Business Media, New York, 2012.

[3] R.O. Ayinla and T.O. Opoola, *The Fekete Szegö functional and second Hankel determinant for a certain subclass of analytic functions,* Appl. Math. **10** (2019), 1071–1078.

[4] E. Deniz, M. Kamali and S. Korkmaz, *A certain subclass of bi-univalent functions associated with Bell numbers and q-Srivastava Attiya operator,* AIMS Math. **5** (2020), 7259—7271.

[5] O.A. Fadipe-Joseph, W.T. Ademosu and G. Murugusundaramoorthy, *Coefficient bounds for a new class of univalent functions involving Sălăgean operator and the modified sigmoid function,* Int. J. Nonlinear Anal. Appl. **9** (2018), 59–69.

[6] M.H. Golmohamadi, S. Najafzadeh and M.R. Foroutan, *q-Analogue of Liu-Srivastava operator on meromorphic functions based on subordination,* Int. J. Nonlinear Anal. Appl. **11** (2020), 219–228.

[7] A.W. Goodman, *Univalent functions vol. I,* Mariner Publishing Company Inc, Tampa, Florida, 1983.

[8] F.H. Jackson, *On q-functions and a certain difference operator,* Trans. Royal Soc. Edin. **46** (1908), 64–72.

[9] F.H. Jackson, *On q-difference,* Amer. J. Math. **32** (1910), 305–314.

[10] A.S. Juma, M.S. Abdulhussain and S.N. Al-khafaji, *Certain subclass of p-valent meromorphic Bazilevič functions defined by fractional q-calculus operators,* Int. J. Nonlinear Anal. Appl. **9** (2018), 223–230.

[11] A. Junod, *Hankel determinants and orthogonal polynomials,* Expo. Math. **21** (2003), 63–74.

[12] V. Kac and P. Cheung, *Quantum calculus,* Springer, New York, 2002.

[13] J.W. Layman, *The Hankel transform and some of its properties,* J. Integer Seq. **4** (2001), 1–11.

[14] A.O. Lasode and T.O. Opoola, *On a generalized class of bi-univalent functions defined by subordination and q-derivative operator,* Open J. Math. Anal. **5** (2021), 46–52.

[15] A.O. Lasode and T.O. Opoola, *Fekete-Szegö estimates and second Hankel determinant for a generalized subfamily of analytic functions defined by q-differential operator,* Gulf J. Math. **11** (2021), no. 2, 36–43.

[16] M. Lewin, *On a coefficient problem for bi-univalent functions,* Proc. Amer. Math. Soc. **18** (1967), 63–68.

[17] R.J. Libera and E.J. Zlotkiewicz, *Early coefficients of the inverse of a regular convex function,* Proc. Amer. Math. Soc. **85** (1982), 225–230.

[18] R.J. Libera and E.J. Zlotkiewicz, *Coefficient bounds for the inverse of a function with derivative in $\mathcal{P}$,* Proc. Amer. Math. Soc. **87** (1983), 251–257.

[19] W.C. Ma and D. Minda, *A unified treatment of some special classes of univalent functions*, Proc. Conf. Complex Anal., International Press, 1992, p. 157-–169.

[20] E.P. Mazi and T.O. Opoola,*A newly defined subclass of bi-univalent functions satisfying subordinate conditions,* Mathematica **84** (2019), 146–155.

[21] K.I. Noor, *Hankel determinant problem for the class of functions with bounded boundary rotation,* Rev. Roum. Math. Pures. Et. Appl. **28** (1983), 731–739.

[22] C. Pommerenke, *On the coefficients and Hankel determinants of univalent functions,* Proc. Lond. Math. Soc. **41** (1966), 111–122.

[23] C. Pommerenke, *On the Hankel determinants of univalent functions,* Mathematika **14** (1967), 108–112.

[24] S. Salehian and A. Motamednezhad, *New subclasses of meromorphic bi-univalent functions by associated with subordinate,* Int. J. Nonlinear Anal. Appl. **12** (2021), 61–74.

[25] H.M. Srivastava and D. Bansal, *Coefficient estimates for a subclass of analytic and bi-univalent functions,* J. Egyptian Math. Soc. **23** (2015), 242–246.