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Discrete alpha-power Weibull distribution: Properties and application

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Abstract

A three-parameter discrete analogue of the Alpha-power Weibull distribution (DAPW) is provided in this study. It has established some of its basic distributional and statistical properties. The probability mass function's form, moments, skewness, kurtosis, probability generating function, characteristic function, stress-strength reliability, and order statistics are all examples of this. The unknown parameters are estimated using the maximum likelihood and moments approaches. The bias and mean square error of the maximum likelihood are demonstrated via a simulated exercise. Two datasets are used to demonstrate the model's adaptability.

Keywords: Characterization, Maximum likelihood estimator, Survival function, Quantile, Reliability, Failure rate, Second rate of failure.
2020 MSC: Primary 62F20, 62E10, 62F99

1 Introduction

Many models have been developed to describe lifetime data using continuous lifetime distributions in many disciplines of life testing experiment and reliability analysis, for example, Kapur and Lamberson [13], Lawless [17], Sinha [34], Gnedenko and Ushakou [8]. Estimating the life length of a device using continuous distributions poses derivational challenges as closed forms for integration may not exist. There is a lot of technology is used to determine how long someone lives. Even more so for a continuous procedure including a continuous measurement of longevity, a discrete model, with recordings made at periodic time points, may be more appropriate.

Roy and Gupta [29] Recently highlighted the function that studies single-variable as well as multivariate treatments in discrete distributions. Roy [30] defined the bivariate geometric distribution and found correlations among various reliability measures . Roy [31] and [32] investigated analogues of the Rayleigh and Normal distributions as a novel discrete alternative to the Rayleigh and Normal distributions, respectively. Krishna and Pundir [14] suggested discrete BXII (Burr type XII) and Pareto distributions with two parameters. Jazi et al. [25] presented a discrete inverse Weibull distribution recently. Para and Jan [11,12,13] discussed the discrete Burr type III distribution, a discrete version of the three-parameter Burr type XII distribution, and the Lomax distribution as new discrete distributions to model counts of kidney cysts using steroids, as well as the discrete inverse Weibull minimax distribution. Gomez-Deniz and Calderin-Ojeda [9] discussed the discrete Lindely distribution. Nekoukhou et al. [23] looked at a discrete variant of the

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generalized Exponential distribution, which is a type of Geometric distribution. The discrete Lindely distribution was proposed by Bakouch et al [12]. Abebe [1] discussed a discrete Lindely distribution having applications in biological research. Munindra et al. [22] considered the derivatives of a discrete Lindley quasi distribution as the goal for his research distribution with two parameters. The discrete modified Weibull distribution was proposed by Nooghabi et al. [33]. In the collective risk model, Gomez-Deniz and Calderin-Ojeda [10] examined the compound DGL/Erlang distribution. El-Morshedey et al. [6] proposed a new Exponentiated discrete Lindely distribution with two parameters. A discrete variant of Logistic distribution was also introduced by Chakraborty and Chakraborty [5].

The Weibull distribution is a well-known distribution that has been employed in duration of life data analysis. There have been numerous changes to the Weibull distribution in recent years. Flaih et al. [7], Bebbington et al. [4], Ahmed and Iqpal [2], Pal et al. [28], Wagner Barreto-Souza et al. [35], and Aryal and Tsokos [3] are just a few examples.

Nassar et al. [24] presented the Alpha-power Weibull distribution, a generalization case based on Mahdavu & Kundu's approach of Alpha-Power Transformation (APT). It can model monotone as well as non-monotone failure rate functions. It also offers a different perspective on several known life distributions. These positive characteristics of the Alpha-power Weibull distribution prompt us to consider its discrete analogue.

We propose a discrete Alpha-power Weibull (DAPW) with the same reliance as its continuous cousin and the same data-analysis features.

2 Definition and Basic Properties

Roy [30] proposed a discretization method based on the model's reliability function.

$$P(X = x) = S(x) - S(x+1) \qquad when\ x = 0, 1, 2, \ldots\ldots \tag{2.1}$$

Roy [30] used this method to discretize Geometric distributions, with S(x) being the Exponential random variable's survival function.

Using the discretization method, the discrete Alpha-Power Weibull distribution can be defined as a non-negative integer valued distribution with PMF, p(x).

$$p(x) = \frac{\alpha}{\alpha - 1}\left[\left(1 - \alpha^{-\theta^{x^{\beta}}}\right) - \left(1 - \alpha^{-\theta^{(x+1)^{\beta}}}\right)\right] \quad x = 0, 1, 2, \ldots \tag{2.2}$$

Where $\theta = e^{-\lambda}$, $0 < \theta < 1$, $\alpha, \beta > 0\ and\ \alpha \neq 1$.
We denote this distribution as DAPW $(\alpha,\ \theta, \beta)$

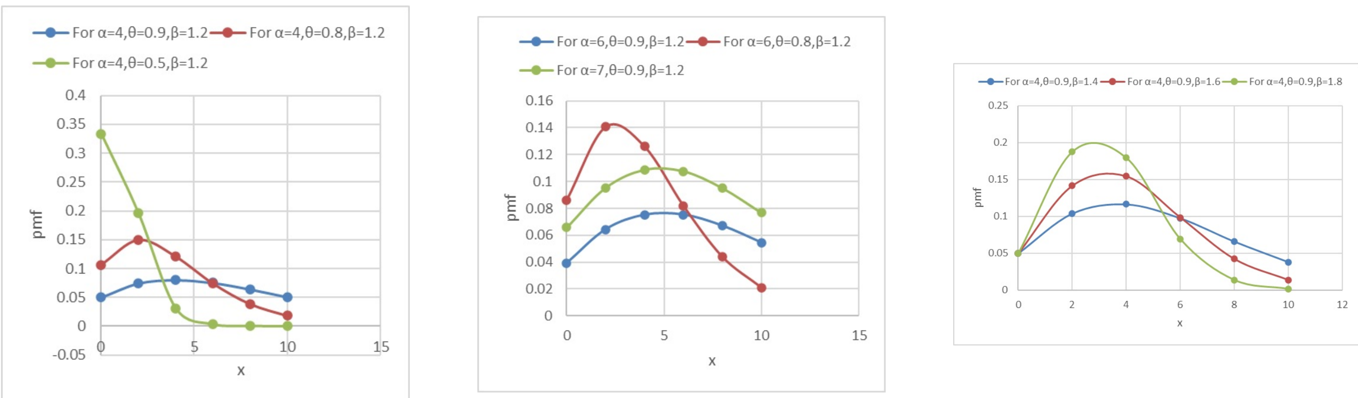

Figure 1: PMF of DAPW $(\alpha,\ \theta, \beta)$ for various values of $(\alpha,\ \theta, \beta)$.

The DAPW $(\alpha,\ \theta,\beta)$ cumulative distribution function CDF is provided by

$$F\left(x,\alpha,\ \theta,\beta\right)\ )=1-S\left(x,\alpha,\ \theta,\beta\right)\ )+P\left(X=x\right)=1-\frac{\alpha}{\alpha-1}\left(1-\alpha^{-\theta^{(x+1)^{\beta}}}\right) \tag{2.3}$$

Where $\theta=e^{-\lambda},\quad 0<\theta<1\ ,\ \ \alpha,\beta>0\ and\ \alpha\neq 1$.
The closed form of the quantile function for the discrete alpha-power Weibull distribution with three parameters is obtained by inverting (2.2) as follows:

$$x_p=\left\lceil\left(\frac{1}{\ln\theta}\ln[\frac{1}{\ln\alpha}\ln\left(\frac{p\alpha-p+1}{\alpha}\right)]\right)^{\frac{1}{\beta}}-1\right\rceil \tag{2.4}$$

Where $\theta=e^{-\lambda},\quad 0<\theta<1\ ,\ \ \alpha,\beta>0\ and\ \alpha\neq 1$.
Where $\lceil n\rceil$ stands for the largest integer value which is larger than or equal to n. Hence the median can be obtained by putting $p=\frac{1}{2}$ in (4)

$$\text{Med}\left(X\right)=\left\lceil\left(\frac{1}{\ln\theta}\ln[\frac{1}{\ln\alpha}\ln\left(\frac{\alpha+1}{2\alpha}\right)]\right)^{\frac{1}{\beta}}-1\right\rceil$$

Where $\lceil n\rceil$ denotes the greatest integer value which is greater than or equal to n. The survival function of the DAPW $(x,\alpha,\ \theta,\beta)$ distribution is given by the following formula.

$$S\left(x\right)=P\left(dX\geq x\right)=\frac{\alpha}{\alpha-1}\left(1-\alpha^{-\theta^{x^{\beta}}}\right)x=0,1,2,\ldots\ldots \tag{2.5}$$

Where $\theta=e^{-\lambda},\quad 0<\theta<1\ and\ \alpha,\beta>0$.
The relevant $h\left(x,\alpha,\ \theta,\beta\right)$ is given by the following formula

$$h\left(x,\theta_1,\ \theta_2\right)=\frac{P\left(X=x\right)}{S\left(x\right)}=\frac{(1-\alpha^{-\theta^{x^{\beta}}})-(1-\alpha^{-\theta^{(x+1)^{\beta}}})}{(1-\alpha^{-\theta^{x^{\beta}}})}=1-\left(1-\alpha^{-\theta^{(x+1)^{\beta}}}\right) \tag{2.6}$$

Hazard rate $h\left(x,\alpha,\ \theta,\beta\right)$ is a 2^{nd} failure rate function for discrete distributions.

$$h^{*}\left(x,\theta_1,\ \theta_2\right)=\log\frac{S\left(x\right)}{S\left(x+1\right)}=\log\frac{(1-\alpha^{-\theta^{x^{\beta}}})}{(1-\alpha^{-\theta^{(x+1)^{\beta}}})} \tag{2.7}$$

Where $\theta=e^{-\lambda},\quad 0<\theta<1\ ,\ \alpha,\beta>0\ and\ \alpha\neq 1$.

3 Different Properties

3.1 Moments and dispersion index

The r^{th} moment μ_r^{-} of a Discrete Alpha-Power Weibull distribution DAPW $(\alpha,\ \theta,\beta)$ about the origin is obtained as follows

$$\mu_r^{-}=E[X^r]=\sum_{x=0}^{\infty}x^rP(X=x)$$

$$\mu_r^{-}=\frac{\alpha}{\alpha-1}\sum_{x=0}^{\infty}x^r\left[\left(1-\alpha^{-\theta^{x^{\beta}}}\right)-\left(1-\alpha^{-\theta^{(x+1)^{\beta}}}\right)\right]$$

The moment generating function (MGF) $M_X(t)$ of a DAPW $(\alpha,\ \theta,\beta)$ distribution is computed as follows

$$M_X\left(t\right)=E\left[e^{tx}\right]=\sum_{x=0}^{\infty}e^{tx}P\left(X=x\right)=\frac{\alpha}{\alpha-1}\sum_{x=0}^{\infty}e^{tx}\left[\left(1-\alpha^{-\theta^{x^{\beta}}}\right)-\left(1-\alpha^{-\theta^{(x+1)^{\beta}}}\right)\right] \tag{3.1}$$

Another use of the MGF calculates the r^{th} moment about the origin. Using this method, you can also get the related moments, means, variance, skewness, and kurtosis (3.1).
DAPW $(\alpha,\ \theta,\beta)$ mean (μ) distribution is as follows

$$\mu_1^{-} = \mu = E\left[X\right] = \frac{\alpha}{\alpha-1}\sum_{x=0}^{\infty} x\left[\left(1-\alpha^{-\theta^{x^{\beta}}}\right)-\left(1-\alpha^{-\theta^{(x+1)^{\beta}}}\right)\right]$$

The second moment is given by the following

$$\mu_2^{-} = E\left[X^2\right] = \frac{\alpha}{\alpha-1}\sum_{x=0}^{\infty} x^2\left(1-\alpha^{-\theta^{x^{\beta}}}\right)-\left(1-\alpha^{-\theta^{(x+1)^{\beta}}}\right)$$

Following that, the variance (σ^2) is calculated as follows:

$$\begin{aligned}\operatorname{var}(X) = \sigma^2 = \mu_2^{-} - \mu^2 = E\left[X^2\right] - \left(E\left[X\right]\right)^2 \\ = \frac{\alpha}{\alpha-1}\sum_{x=0}^{\infty} x^2[\left(1-\alpha^{-\theta^{x^{\beta}}}\right)-\left(1-\alpha^{-\theta^{(x+1)^{\beta}}}\right)- \\ \left(\frac{\alpha}{\alpha-1}\sum_{x=0}^{\infty} x[\left(1-\alpha^{-\theta^{x^{\beta}}}\right)-\left(1-\alpha^{-\theta^{(x+1)^{\beta}}}\right)\right)^2\end{aligned} \tag{3.2}$$

The $3^{\text{rd}}$and 4^{th} moments are, respectively by the following

$$\mu_3^{-} = E\left[X^3\right] = \frac{\alpha}{\alpha-1}\sum_{x=0}^{\infty} x^3[\left(1-\alpha^{-\theta^{x^{\beta}}}\right)-\left(1-\alpha^{-\theta^{(x+1)^{\beta}}}\right)$$

And

$$\mu_4^{-} = E\left[X^4\right] = \frac{\alpha}{\alpha-1}\sum_{x=0}^{\infty} x^4\left(1-\alpha^{-\theta^{x^{\beta}}}\right)-\left(1-\alpha^{-\theta^{(x+1)^{\beta}}}\right)$$

The measure of skewness α_3 of DAPW $(\alpha,\ \theta,\beta)$ distribution is then obtained as follows

$$\begin{aligned}\alpha_3 = \frac{\mu_3^{-} - 2\mu_2^{-}\mu + \mu^2}{\sigma^3} = \frac{1}{\sigma^3}\left[\frac{\alpha}{\alpha-1}\sum_{x=0}^{\infty} x^2\left[\left(1-\alpha^{-\theta^{x^{\beta}}}\right)-\left(1-\alpha^{-\theta^{(x+1)^{\beta}}}\right)\right]\right] \\ -2\mu\frac{\alpha}{\alpha-1}\sum_{x=0}^{\infty} x^2\left[\left(1-\alpha^{-\theta^{x^{\beta}}}\right)-\left(1-\alpha^{-\theta^{(x+1)^{\beta}}}\right)\right]+\frac{\mu^3}{\sigma^3}.\end{aligned} \tag{3.3}$$

The measure of kurtosis α_4 of DAPW $(\alpha,\ \theta,\beta)$ distribution is given by the following

$$\begin{aligned}\alpha_4 = \frac{\mu_4^{-} - 4\mu_3^{-}\mu + 6\mu_2^{-}\mu^2 - 3\mu^4}{\sigma^4} = \frac{1}{\sigma^4}\left\{[\frac{\alpha}{\alpha-1}\sum_{x=0}^{\infty} x^4\left[\left(1-\alpha^{-\theta^{x^{\beta}}}\right)-\left(1-\alpha^{-\theta^{(x+1)^{\beta}}}\right)\right]-\right. \\ 4\mu[\frac{\alpha}{\alpha-1}\sum_{x=0}^{\infty} x^3\left[\left(1-\alpha^{-\theta^{x^{\beta}}}\right)-\left(1-\alpha^{-\theta^{(x+1)^{\beta}}}\right)\right] \\ \left.+6\mu^2\left[\frac{\alpha}{\alpha-1}\sum_{x=0}^{\infty} x^2[\left(1-\alpha^{-\theta^{x^{\beta}}}\right)-\left(1-\alpha^{-\theta^{(x+1)^{\beta}}}\right)\right]\right\}-\frac{3\mu^4}{\sigma^4}\end{aligned} \tag{3.4}$$

G(t), the probability generating function (PGF) of DAPW $(\alpha,\ \theta,\beta)$ distribution, is calculated as follows

$$G\left(t\right) = E\left[t^x\right] = \sum_{x=0}^{\infty} t^x P\left(X=x\right) = \frac{\alpha}{\alpha-1}\sum_{x=0}^{\infty} t^x\left[\left(1-\alpha^{-\theta^{x^{\beta}}}\right)-\left(1-\alpha^{-\theta^{(x+1)^{\beta}}}\right)\right] \tag{3.5}$$

Although obtaining a closed form expression for PGF is difficult, we can compute it numerically. In general, the following gives the r^{th}factorial moment:

$$\mu_{[r]} = G^r\left(1\right) = \frac{\alpha}{\alpha-1}\sum_{x=0}^{\infty} x\left(x-1\right)\ldots\ldots\ldots\left(x-r+1\right)\left[\left(1-\alpha^{-\theta^{x^{\beta}}}\right)-\left(1-\alpha^{-\theta^{(x+1)^{\beta}}}\right)\right].$$

The mean μ , can be calculated by taking the first derivative of the pgf, at t=1, as shown below.

$$\mu=\mu_{[1]}=\grave{G}\left(1\right)=\frac{\alpha}{\alpha-1}\sum_{x=0}^{\infty}x[\left(1-\alpha^{-\theta^{x^{\beta}}}\right)-\left(1-\alpha^{-\theta^{(x+1)^{\beta}}}\right).$$

Take the second derivative of the pmgf, at t=1 to calculate the 2^{nd} factorial moment.

$$\mu_{[2]}=G^{r}(1)=\frac{\alpha}{\alpha-1}\sum_{x=0}^{\infty}x(x-1)\left[\left(1-\alpha^{-\theta^{x^{\beta}}}\right)-\left(1-\alpha^{-\theta^{(x+1)^{\beta}}}\right)\right].$$

The variance, the variance (σ^2) of DAPW ($\alpha,\ \theta,\beta$)distribution is given by the following

$$\operatorname{var}(X)=\sigma^{2}=\tilde{G}(1)+G^{'}(1)-\left(\tilde{G}(1)\right)^{2}=\frac{\alpha}{\alpha-1}\sum_{x=0}^{\infty}x^{2}\left[\left(1-\alpha^{-\theta^{x^{\beta}}}\right)-\left(1-\alpha^{-\theta^{(x+1)^{\beta}}}\right)\right]$$
$$-\left(\frac{\alpha}{\alpha-1}\sum_{x=0}^{\infty}x\left[\left(1-\alpha^{-\theta^{x^{\beta}}}\right)-\left(1-\alpha^{-\theta^{(x+1)^{\beta}}}\right)\right]\right)^{2} \tag{3.6}$$

Characteristic function: The DAPW ($x,\alpha,\ \theta,\beta$) distribution 'characteristic function (CF), $\phi_X\left(w\right)$ takes the form

$$\phi_X\left(w\right)=E\left[e^{iwx}\right]=\sum_{x=0}^{\infty}e^{iwx}P\left(X=x\right)=\sum_{x=0}^{\infty}e^{iw}\left[\left(1-\alpha^{-\theta^{x^{\beta}}}\right)-\left(1-\alpha^{-\theta^{(x+1)^{\beta}}}\right)\right] \tag{3.7}$$

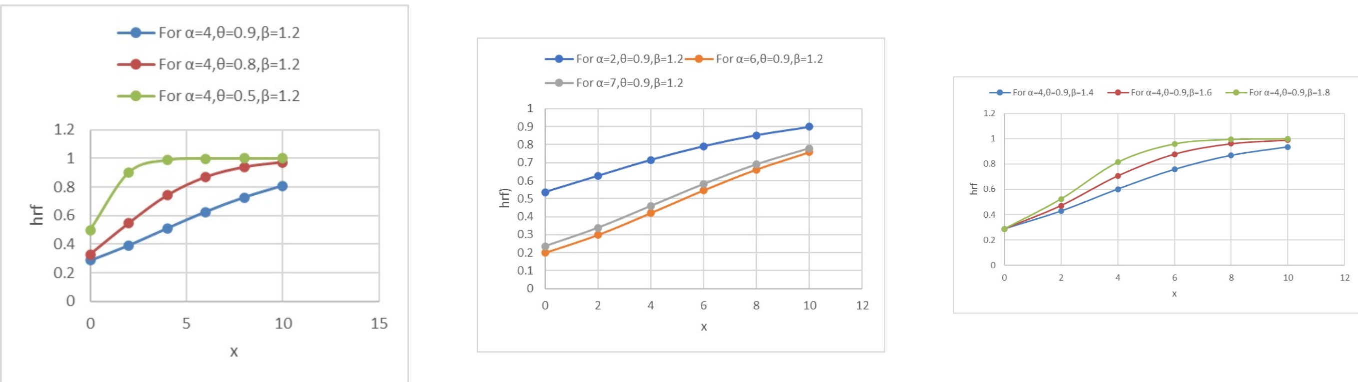

Figure 2: The HRF plots of DAPW ($\alpha,\ \theta,\beta$) for various values of ($\alpha,\ \theta,\beta$)..

3.2 Stress-Strength (S-S*)*analysis*

Stress-strength models are particularly important in reliability literature, engineering applications, and so on. The parameters $R=P(X>Y)$,in reliability studies where it is assumed that force X and stress Y. This model is used in engineering problems to compare the capabilities of two workers or the performances of two companies' products, among other things (Kotz et.al. [15],Marwa[34,35]).
The S-S* analysis is critical in technical systems. When strength falls below the stress, the likelihood of failure increases. In this case,the value of mean reliability (R*) as follows:

$$R^{*}=p\left[X_{s}\leq X_{s^{*}}\right]=\sum_{x=0}^{\infty}f_{X_{s}}\left(x\right)R_{X_{s^{*}}} \tag{3.8}$$

If $X_s\sim$ DAPW ($\alpha_1,\ \theta_1,\beta_1$)and $X_{s^*}\sim$ DAPW ($\alpha_2,\ \theta_2,\beta_2$) then R^* can be written as follows

$$R^{*}=\frac{\alpha_{1}\alpha_{2}}{\left(\alpha_{1}-1\right)\left(\alpha_{2}-1\right)}\sum_{x=0}^{\infty}\left(1-\alpha_{2}{}^{-\theta_{2}{}^{x^{\beta_{2}}}}\right)\left[\left(1-\alpha_{1}{}^{-\theta_{1}{}^{x^{\beta_{1}}}}\right)-\left(1-\alpha_{1}{}^{-\theta_{1}{}^{(x+1)^{\beta_{1}}}}\right)\right] \tag{3.9}$$

Table 1: Mean (variance) of DAPW $(x, \alpha,\ \theta, \beta)$ for various values of $\alpha,\ \theta$ *and* β

α/θ	0.1	1.5	2	3.5	5
			$\beta = 0.5$		
.1	0.05077(0.1566)	0.22050(0.67833)	0.24899(0.7631)	0.30946(0.9399)	0.35101(1.0589)
.25	0.22871(1.46959)	0.90031(5.97586)	1.00761(6.6712)	1.23175(8.0935)	1.38320(9.0305)
.5	1.24009(22.0136)	4.30675(82.9752)	4.77367(91.859)	5.73583(109.66)	6.37698(121.12)
.75	5.77380(186.823)	15.3292(511.751)	16.5691(548.79)	18.9845(617.49)	20.4945(658.31)
.9	13.9443(522.09)	17.9319(769.630)	17.8924(784.71)	17.5276(804.66)	17.1126(810.39)
			$\beta = 1$		
.1	0.03164(0.03701)	0.13269(0.14507)	0.14928(0.1611)	0.18422(0.1933)	0.20801(0.2140)
.25	0.10908(0.1567)	0.38954(0.50593)	0.43175(0.5491)	0.51813(0.6298)	0.57518(0.6774)
.5	0.39736(0.82712)	1.13873(2.22133)	1.24080(2.3714)	1.44468(2.6387)	1.57613(2.7883)
.75	1.39653(5.33593)	3.35496(13.2131)	3.61402(14.028)	4.12715(15.464)	4.45534(16.259)
.9	4.53597(40.6239)	9.97849(98.8000)	10.6923(104.77)	12.1038(115.29)	13.0052(121.09)
			$\beta = 2$		
.1	0.02879(0.02801)	0.11932(0.10533)	0.13407(0.1163)	0.16502(0.1381)	0.18602(0.1518)
.25	0.08748(0.08183)	0.29394(0.21704)	0.32362(0.2297)	0.38328(0.2500)	0.42191(0.2596)
.5	0.25795(0.22783)	0.62797(0.39337)	0.67330(0.4005)	0.76055(0.4068)	0.81455(0.4061)
.75	0.65659(0.56478)	1.25092(0.84908)	1.32101(0.8612)	1.45556(0.8718)	1.53883(0.8703)
.9	1.41057(1.44196)	2.39324(2.18376)	2.50906(2.21536)	2.73140(2.24189)	2.86899(2.23647)

3.3 Mean Residual Lifetime (MRL) and Mean Past Lifetime (MPL)

To analyze the ageing behavior of the components, several reliability and survival analysis measures are offered. MPL, say $\varsigma(i)$ is a useful tool for modelling and analyzing burn-in and maintenance plans, is one of these measurements. The MRL is defined as follows in a discrete environment.

$$\varsigma(i) = E\left[X - \frac{i}{X} \geq i\right] = \frac{1}{R(j)} \sum_{j=i+1}^{l} R(j)\,,\ i \in \mathbb{N}_0 \tag{3.10}$$

if the RV $X \sim DAPW(x, \alpha,\ \theta, \beta)$, then the MPL is:

$$\varsigma(i) = \frac{1}{\left(1 - \alpha^{-\theta^{i^\beta}}\right)} \sum_{j=i+1}^{l} \left(1 - \alpha^{-\theta^{j^\beta}}\right) \tag{3.11}$$

3.4 Order statistics

Let $X_1, X_2, \ldots, X_n$ represent a r.s. from A DAPW $(x, \alpha,\ \theta, \beta)$ distribution, and let $X_{1:n}, X_{2:n}, \ldots,\ X_{n:n}$ represent their corresponding (OS). Then, for x , the CDF of the i^{th} OS stated as follows:

$$\begin{aligned} F_{i:n}(x, \alpha,\ \theta, \beta) &= \sum_{k=i}^{n} \binom{n}{k} \left[F_i(x, \alpha,\ \theta, \beta)\right]^k \left[F_i(x, \alpha,\ \theta, \beta)\right]^{n-k} \\ &= \sum_{k=i}^{n} \sum_{m=0}^{k} (-1)^m \binom{k}{m} \frac{\alpha^{n+m-k}}{(\alpha-1)^{n+m-k}} \left(1 - \alpha^{-\theta^{(x+1)^\beta}}\right)^{n+m-k} \end{aligned} \tag{3.12}$$

Moreover, the PMF of the k^{th} O.S. stated as

$$\begin{aligned} f_{k:n}(x, \alpha,\ \theta, \beta) = \sum_{m=0}^{k-1} \Theta_m^{(n,k-1)} \frac{\alpha^{n+m-k+1}}{(\alpha-1)^{n+m-k+1}} \left(1 - \alpha^{-\theta^{(x+1)^\beta}}\right)^{n+m-k} \\ \left[\left(1 - \alpha^{-\theta^{x^\beta}}\right) - \left(1 - \alpha^{-\theta^{(x+1)^\beta}}\right)\right] \end{aligned} \tag{3.13}$$

Where $\Theta_m^{(n,k-1)} = (-1)^m \binom{k-1}{m} \frac{n!}{(k-1)!(n-k)!}$.
So, the q^{th} moments of $X_{i:n}$ can be given in terms

$$E\left[X_{i:n}^q\right] = \sum_{x=0}^{\infty}\sum_{m=0}^{k-1}\Theta_m^{(n,k-1)}x^q\frac{\alpha^{n+m-k+1}}{(\alpha-1)^{n+m-k+1}}\left(1-\alpha^{-\theta^{(x+1)^\beta}}\right)^{n+m-k} \left[\left(1-\alpha^{-\theta^{x^\beta}}\right)-\left(1-\alpha^{-\theta^{(x+1)^\beta}}\right)\right] \tag{3.14}$$

4 Estimation

for unknown parameters of DAPW $(x, \alpha, \Theta, \beta)$, some methods of estimation were used to find their values .

4.1 MLE Method.

let $X_1, X_2, \ldots.., X_n$ represent (n) units based on the DAPW $(x, \alpha,\ \theta, \beta)$ distribution. Then the following is the appropriate log-likelihood function:

$$p(x) = \frac{\alpha}{\alpha-1}\left[\left(1-\alpha^{-\theta^{x^\beta}}\right)-\left(1-\alpha^{-\theta^{(x+1)^\beta}}\right)\right]$$
$$L\left[P\left(X=x\right)\right] = \prod_{i=1}^{n}p(x_i) = \frac{\alpha^n}{(\alpha-1)^n}\prod_{i=1}^{n}\left(1-\alpha^{-\theta^{x_i{}^\beta}}\right)-\left(1-\alpha^{-\theta^{(x_i+1)^\beta}}\right) \tag{4.1}$$

$$l(x,\alpha,\ \theta,\beta) = n\ln\alpha - n\ln(\alpha-1) + \sum_{i=1}^{n}\ln\left(\left(1-\alpha^{-\theta^{x_i{}^\beta}}\right)-\left(1-\alpha^{-\theta^{(x_i+1)^\beta}}\right)\right) \tag{4.2}$$

Likelihood equations are then obtained as follows:

$$\frac{\delta 1}{\delta\alpha} = \frac{n}{\alpha} - \frac{n}{\alpha-1} + \sum_{i=1}^{n}\frac{\left(\theta^{x_i{}^\beta}\alpha^{-\theta^{x_i{}^\beta}-1}\right)-\left(\theta^{(x_i+1)^\beta}\alpha^{-\theta^{(x_i+1)^\beta}-1}\right)}{\left(1-\alpha^{-\theta^{x_i{}^\beta}}\right)-\left(1-\alpha^{-\theta^{(x_i+1)^\beta}}\right)} \tag{4.3}$$

$$\frac{\delta 1}{\delta\beta} = \sum_{i=1}^{n}\frac{\theta^{x_i{}^\beta}\alpha^{-\theta^{x_i{}^\beta}}(\ln\alpha)(\ln\theta)(lnx_i)x_i{}^\beta - \theta^{(x_i+1)^\beta}\alpha^{-\theta^{(x_i+1)^\beta}}(\ln\alpha)(\ln\theta)(\ln x_i+1)(x_i+1)^\beta}{\left(1-\alpha^{-\theta^{x_i{}^\beta}}\right)-\left(1-\alpha^{-\theta^{(x_i+1)^\beta}}\right)}$$

It's possible to write it as

$$\frac{\delta 1}{\delta\beta} = \sum_{i=0}^{n}\frac{(\ln\alpha)(\ln\theta)\left[\theta^{x_i{}^\beta}\alpha^{-\theta^{x_i{}^\beta}}(lnx_i)x_i{}^\beta - \theta^{(x_i+1)^\beta}\alpha^{-\theta^{(x_i+1)^\beta}}(lnx_i+1)(x_i+1)^\beta\right]}{\left(1-\alpha^{-\theta^{x_i{}^\beta}}\right)-\left(1-\alpha^{-\theta^{(x_i+1)^\beta}}\right)} = 0; \tag{4.4}$$

$$\frac{\delta 1}{\delta\theta} = \sum_{i=0}^{n}\frac{\alpha^{-\theta^{x_i{}^\beta}}(\ln\alpha)\,x_i{}^\beta\theta^{x_i{}^\beta-1} - \alpha^{-\theta^{(x_i+1)^\beta}}(x_i+1)^\beta\theta^{(x_i+1)^\beta-1}}{\left(1-\alpha^{-\theta^{x_i{}^\beta}}\right)-\left(1-\alpha^{-\theta^{(x_i+1)^\beta}}\right)} = 0; \tag{4.5}$$

can solve the above equations numerically because the equations do not have a final form.

4.2 Method of Moments Estimation

The following equations are used to calculate the moments estimates (MME_s) of $(\alpha,\ \theta,\beta)$

$$\frac{\alpha}{\alpha-1}\sum_{i=1}^{\infty}x_i\left[\left(1-\alpha^{-\theta^{x_i{}^\beta}}\right)-\left(1-\alpha^{-\theta^{(x_i+1)^\beta}}\right)\right] \quad = \mu_1{}^{[1]},$$

And

$$\frac{\alpha}{\alpha-1}\sum_{i=1}^{\infty} x_i^2 \left[\left(1-\alpha^{-\theta^{x_i\beta}}\right)-\left(1-\alpha^{-\theta^{(x_i+1)^\beta}}\right)\right] \quad = {\mu_2}^{[2]},$$

$$\frac{\alpha}{\alpha-1}\sum_{i=1}^{\infty} x_i^3 \left[\left(1-\alpha^{-\theta^{x_i\beta}}\right)-\left(1-\alpha^{-\theta^{(x_i+1)^\beta}}\right)\right] \quad = {\mu_3}^{[3]},$$

Where ${\mu_1}^{[1]}$, ${\mu_2}^{[2]}$ and${\mu_3}^{[3]}$ imply the first, second, and third sample moments respectively.

5 Simulation Study

To evaluate the performance of the maximum-likelihood estimate, use simulation study:

1. Using Equation (2.3), generate 10000 samples of size n. To create samples, the inversion approach is employed; that is, varites of the discrete exponentiated exponential distribution are obtained using this method.$X = \left\{\frac{\ln\left(1-u^{\frac{1}{\alpha}}\right)}{\ln\theta}-1\right\}$; $\quad 0 < u < 1$

   Where $U \sim U(0,1)$ is a variable with a uniform distribution on the unit interval;

2. Calculate the maximum-likelihood estimates for 10000 samples, say $\widehat{\theta}_i$ for $i = 1, 2, \ldots, 10000$,
3. Determine the biases and mean-squared errors given by

$$\text{bias}\,(n) = \frac{1}{10000}\sum_{i=1}^{10000}(\widehat{\theta}_i - \theta_i), \quad \text{MSE}\,(n) = \frac{1}{10000}\sum_{i=1}^{10000}(\widehat{\theta}_i - \theta_i)^2.$$

Table 2: The averages bias and averages MSE in parenthesis for simulated results of ML estimates.

(α,θ,β)	$\widehat{\alpha}$	$\widehat{\theta}$	$\widehat{\beta}$
		n=10	
(0.5, 0.2, 1)	*-0.477(0.694)*	*-0.266 (0.127)*	*0.354 (1.839)*
(0.5, 0.1, 1.5)	*-0.387 (0.173)*	*-0.095 (0.012)*	*0.134 (0.783)*
(0.5, 0.2, 2)	*-0.437 (0.207)*	*-0.25 (0.073)*	*-1.273 (2.938)*
		n=15	
(0.5, 0.2, 1)	*-0.494 (0.263)*	*-0.25 (0.071)*	*0.13 (0.914)*
(0.5, 0.1, 1.5)	*-0.401 (0.18)*	*-0.098 (0.012)*	*0.076 (0.616)*
(0.5, 0.2, 2)	*-0.447 (0.215)*	*-0.257 (0.076)*	*-1.248 (2.837)*
		n=30	
(0.5, 0.2, 1)	*-0.501 (0.256)*	*-0.227 (0.055)*	*-0.118 (0.228)*
(0.5, 0.1, 1.5)	*-0.394 (0.166)*	*-0.092 (0.009246)*	*-0.162 (0.2)*
(0.5, 0.2, 2)	*-0.446 (0.217)*	*-0.274 (0.089)*	*-1.192 (2.74)*
		n=50	
(0.5, 0.2, 1)	*-0.497 (0.249)*	*-0.224 (0.052)*	*-0.17 (0.103)*
(0.5, 0.1, 1.5)	*-0.397 (0.162)*	*-0.091 (0.00847)*	*-0.264 (0.14)*
(0.5, 0.2, 2)	*-0.439 (0.24)*	*-0.286 (0.102)*	*-1.257 (2.667)*

The following observations can be made based on Table 2:

- As $n \longrightarrow \infty$. increases, the magnitude of the bias decreases to zero.
- As $n \longrightarrow \infty$., the MSEs always decrease to zero. This demonstrates the estimators' consistency.

Table 3: Data set 1.

29	25	50	15	13	27
15	18	7	7	8	19
12	18	5	21	15	86
21	15	14	39	15	14
70	44	6	23	58	19
50	23	11	6	34	18
28	34	12	37	4	60
20	23	40	65	19	31

6 Real Data Example.

This section shows how DAPW Distribution outperforms traditional distributions such as Poisson and Geometric besides the models we have recently obtained in different researches such as (Discrete Gamma, Discrete Weibull, Discrete Logistic, and Discrete Lindely).
Table 3 contains the degrees of 48 students in mathematics at the Indian institute of technology in kampur. gupta and kundu [11] provided the data set.

The MLE, of α, μ, β, and λ All of these values have been computed. In each case, the distance of (K-S) between the CDF and the fitted distribution function is computed, as well as the associated P-value. Table 4 summarizes the outcome.

Table 4: Shows the data set 1 fitted estimates.

Distribution	$p(x)$	*Parameter Estimates*	*p value*	*K-S statistics*
Discrete Alpha-power Weibull Distribution	(2.2)	$\alpha = 4.677,\ \beta = 1.248,\ \theta = .978$	.339044	.133417096
Poisson	$\lambda^x e^{-\lambda}/x!$	$\lambda = 25.8958$	2.4013×10^{-7}	.3998
Geometric	$p\,(1-p)^x$	$p = .0372$	.0145	.2223
Discrete Weibull	$q^{x^\beta} - q^{(x+1)^\beta}$	$q = .6488,\ \beta = .6758$	2.9221×10^{-24}	.7419
Discrete Gamma	$\frac{\gamma(\alpha,\beta(x+1))}{\Gamma(\alpha)} - \frac{\gamma(\alpha,\beta(x))}{\Gamma(\alpha)}$	$\alpha = .8098,\ \beta = .0350$	2.6082×10^{-4}	.2993

Some summary statistics of this data: minimum is 4, maximum is 86, mean is 25.90, Std Deviation is 18.605, Variance is 346.138, Skewness is 1.375 and Kurtosis is 1.608.
Dataset 2 Table 5 refers to the uncensored data set released by Maguire et al. [19] that corresponds to intervals in days between 109 consecutive coal-mining tragedies in Great Britain from 1875 to 1951. Following is a list of the sorted data:

Table 5: Data set 2.

1	4	4	7	11	13	15	15	17	18	19	19	20	20	22
23	28	29	31	32	36	37	47	48	49	50	54	54	55	59
59	61	61	66	72	72	75	78	78	81	93	96	99	108	113
114	120	120	120	123	124	129	131	137	145	151	156	171	176	182
188	189	195	203	208	215	217	217	224	228	233	255	271	275	275
275	286	291	312	312	312	315	326	326	329	330	336	338	345	348
354	361	364	369	378	390	457	467	498	517	566	644	745	871	1312
1357	1613	1630												

In each of these examples, The MLE, of $\alpha, \mu, \beta,$ *and* λ values has been computed. In each scenario, the Kolmogorov-Smirnov (K-S) distance between the empirical cumulative distribution function and the fitted distribution function is obtained, as well as the P-value. Table 6 summarizes the outcome

Table 6: Shows the data set 2 fitted estimates.

Distribution	$p(x)$	*Estimated value of parameter*	p *value*	*K-S statistics*
Discrete Alpha-power Weibull Distribution	*(2.2)*	$\alpha = 4.792,\ \beta = .701,\ \theta = .967$	.5879822290	.07414
Geometric	$p\,(1-p)^{x}$	$p = 3.992 * 10^{-3}$	.2508886128	.09651
Discrete Lindely	$\frac{p^{x}}{1+\theta}\,[\theta\,(1-2p)+(1-p)(1\ +\theta x)]$	$\theta = 7.937 * 10^{-3},\ p = .992$	.000007113	0.237266
Discrete Logistic	$\frac{(1-p)p^{y-\mu}}{(1+p^{y-\mu})(1+p^{y-\mu+1})}$	$p = 2.468 * 10^{-9},\ \mu = -6.67 * 10^{-8}$	$3.22657 * 10^{-97}$	1

Some summary statistics of this data: minimum is 1, maximum is 1630, mean is 233.32, Std Deviation is 296.434, Variance is 87873.331, Skewness is 2.999 and Kurtosis is 10.526.
Figures 3- 4 show a comparison between CDF graphs and the fitted and observed distribution functions.

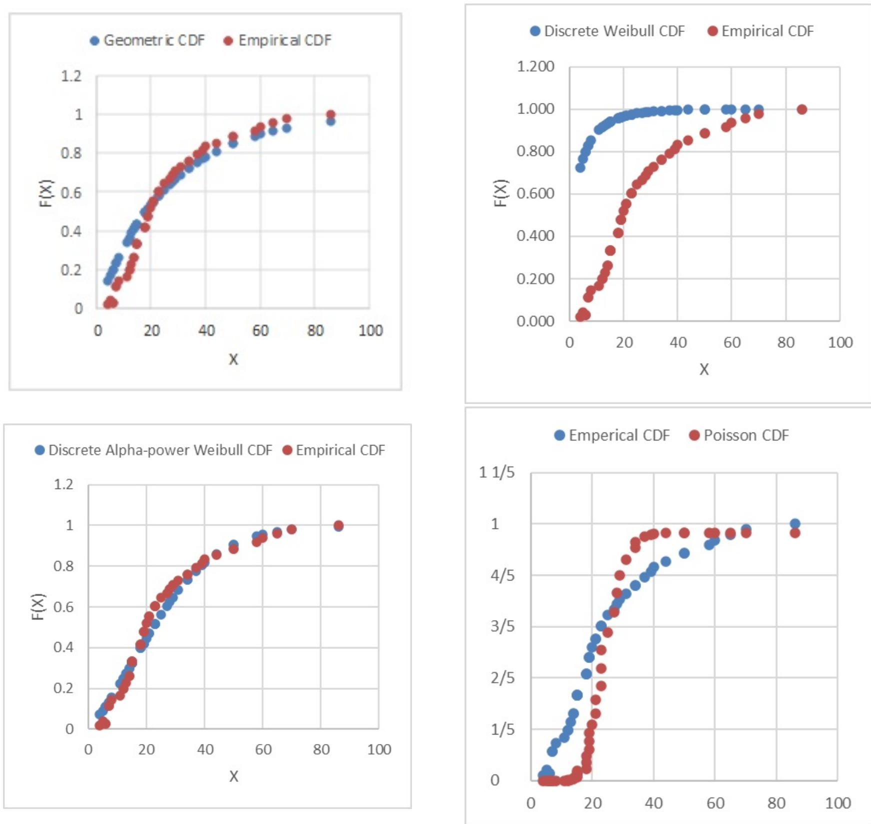

Figure 3: Distribution plots for data set 1.

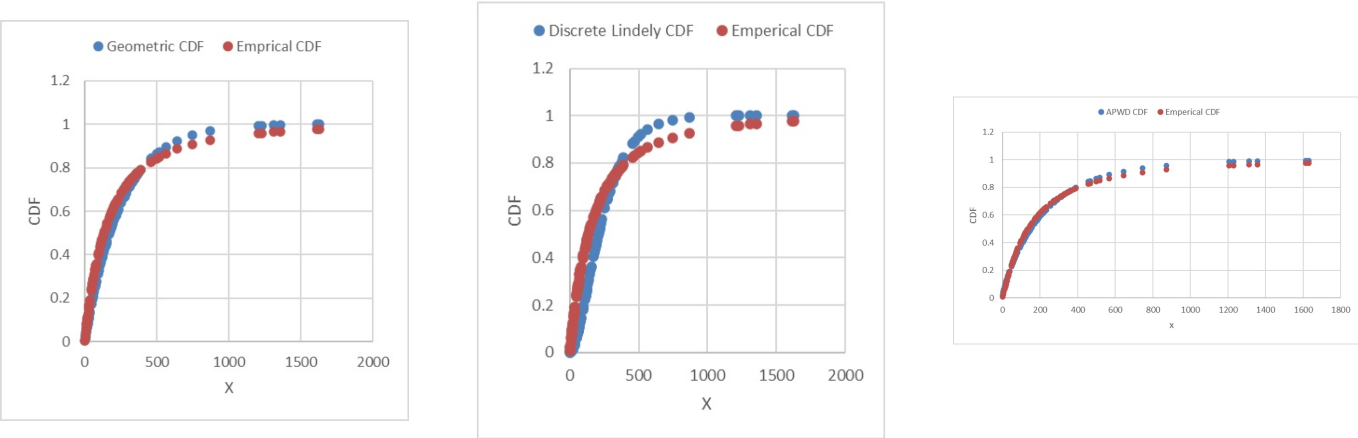

Figure 4: Data set 2 distribution plots

For the 1st and 2nd data sets, the discrete alpha-power Weibull distribution yields acceptable p-values and is the best fit between the competing distributions.

Relying on the results obtained, we deduce that the discrete Alpha-power Weibull distribution, when compared to its sub models, provides the best fit.

7 Conclusion

In this paper, a new discrete distribution, the Alpha-power Weibull distribution, is presented. After obtaining it, the probabilistic properties of the parameter are studied and its parameters are estimated. The reliability function was also studied using the stress-strength model based on the new distribution, and during the review of the steps, we studied the statistics arranged for the APW distribution.

It was also applied to the stress-strength model using data produced from a simulation model by the Monte-Carlo method. Then we applied the proposed distribution to two sets of real data and found that the distribution would be a strong competitor to known discrete distributions. It can be used in applications for materials that affect the environment such as coal or in general, it can be used to study natural disasters, and the univariate case has been presented. It is more important to show how it can be generalized to the multivariate case. More work is required in this area in the future.

References

[1] B. Abebe, *A discrete Lindely distribution with applications in biological sciences biometrics and biostatistics*, Biomet. Biostatist. Int. J. **7** (2018), no. 1, 48–52.

[2] Z. Ahmed and B. Iqpal, *Generalized flexible Weibull extension distribution*, Circ. Comput. Sci. **2** (2017), no. 4, 68–75.

[3] G.R. Aryal and C.P. Tsokos, *Transmuted Weibull distribution: A generalization of the Weibull probability distribution*, Eur. J. Pure Appl. Math. **4** (2011), no. 2, 89–102.

[4] M. Bebbington, C.D. Lai and R. Zitikis, *A flexible Weibull extension*, Reliab. Engin. Syst. Safety **92** (2007), 719–726.

[5] S. Chakraborty and D. Chakraborty, *A new discrete probability distribution with integer support on* $(-\infty, \infty)$, Commun. Statist. Theory Method. **45** (2016), no. 2, 492–505.

[6] M. El-Morshedy, M.S. Eliwa and H. Nagy, *A new two-parameter exponentiated discrete Lindley distribution: properties, estimation and applications*, J. Appl. Statist. **47** (2020), no. 2, 354–375.

[7] A. Flaih, H. Elsalloukh, E. Mendi and M. Milanova, *The exponetiated inverted Weibull distribution*, Appl. Math. Inf. Sci. **6** (2012), no. 2, 167–171.

[8] B. Gnedenko and I. Ushakov, *Probabilistic reliability engineering*, New York: Wiley, 1995.

[9] E. Gómez-Déniz & E. Calderín-Ojeda,*The discrete lindley distribution: Properties and applications*, J. Statist. Comput. Simul. **81** (2011), no. 11, 1405–1416.

[10] E. Gómez-Déniz and E. Calderín Ojeda, *The compound DGL/Erlang distribution in the collective risk model*, Revist. Methods Cuantt. Para´ la econom ´ la y la empresa, **16** (2013), 121–142.

[11] R.D. Gupta and D. Kundu, *A new class of weighted exponential distribution*, Statist. **43** (2009), 621–634.

[12] S. Hassan, M. Bakouch, A. Jazi and S. Nadarajah, *A new discrete Distribution*, Statist. **48** (2014), no. 1, 200–240.

[13] K.C. Kapur and L.R. Lamberson, *Reliability in engineering design*, John Wiley and Sons, Inc., New York, 1977.

[14] H. Krishna and P. Singh Pundir, *Discrete Burr and discrete Pareto distributions*, Statist. Meth. **6** (2009), no. 2, 177–188.

[15] S. Kotz and M. Pensky, *The stress-strength model and its generalization: Theory and application*, World Scientific, 2003.

[16] A. Lai and M.H. Alamatsaz, *A discrete inverse Weibull distribution and estimation of its parameters*, Statist. Meth. **7** (2010), no. 2, 121–132.

[17] J.F. Lawless,*Statistical models and methods for lifetime data*, John Wiley and Sons, Inc., New York, 1982.

[18] A. Mahdavi, D. Kundu, *A new method for generating distributions with an application to exponential distribution*, Commun. Statist. Theory Method. **46** (2015), no. 13, 6543–6557.

[19] B.A. Maguire, E. Pearson and A. Wynn, *The time intervals between industrial accidents*, Biometrica **39** (1952), no. 1/2, 168–180.

[20] M.O. Mohamed, *Inference for reliability and stress-strength for geometric distribution*, Sylwan **159** (2015), no. 2, 281–289.

[21] M.O. Mohamed, *Estimation of R for geometric distribution under lower record values*, J. Appl. Res. Technol. 18 (2020), no. 6, 368–375.

[22] B. Munindra, R.S. Kirshna and J. Junali, *A study on two parameters discrete quasi Lindely distribution and its derived distribution*, Int. J. Math. Arch. **6** (2015), no. 12, 149–156.

[23] V. Nekoukhou and H. Bidram, *Exponential-discrete generalized exponential distribution: A new compound model*, J. Statist. Theory Appl. **15** (2016), no. 2, 169.

[24] M. Nassar, A. Alzaatreh, M. Mead and O. Abo-Kasem, *Alpha power Weibull Distribution: Properties and Applications*, Commun. Statist. Theory Method. **46** (2017), no. 20, 10236–10252.

[25] B.A. Para and T.R. Jan, *Discretization of Burr-Type III distribution*, J. Reliab. Statist. Stud. **7** (2014), no. 2, 87–94.

[26] B.A. Para and T.R. Jan, *On discrete three parameter Burr type XII and discrete Lomax distributions and their applications to model count data from medical science*, Biomet. Biostatist. Int. J. **4** (2016), no. 2.

[27] B. A. Para and T. R. Jan, *Discrete Inverse Weibull Minimax Distribution: Properties and Applications*, J. Statist. Appl. Prob. **6** (2017), no. 1, 205–218.

[28] M. Pal, M.M. Ali and J. Woo, *Exponentiated Weibull distribution*, Statistica **66** (2006), no. 2, 139–147.

[29] D. Roy and R.P. Gupta, *Classifications of discrete lives*, Micro Electron. Reliab. **32** (1992), 1459–1473.

[30] D. Roy, *Reliability measures in the discrete bivariate set up and related characterization results for a bivariate geometric distribution*, Multivar. Anal. **46** (1993), 362–373.

[31] D. Roy, *The discrete normal distribution*, Commun. Statist. Theory Method. **32** (2003), no. 10, 1871–1883.

[32] D. Roy, *Discrete Rayleigh distribution*, IEEE Trans. Reliab. **53** (2004), no. 2, 255–260.

[33] M. Shafaei Nooghabi, G.R. Mohtashami Borzadaran and A. Hamid Rezaei Roknabadi, *Discrete modified Weibull*, METRON, **69** (2011), 207.

[34] S.K. Sinha, *Reliability and life testing*, Wiley Eastern Limited, New Delhi, 1986.

[35] W. Barreto-Souza, A.L. de Morais and M. Gauss Cordeiro, *The Weibull-geometric distribution*, J. Statist. Comput. Simul. **81** (2011), no. 5, 645–657.