Solutions of integral equations via fixed point results in extended Branciari $b$-distance spaces

Usha Bag*, Reena Jain$^{*, *}$

$^*$VIT Bhopal University, Mathematics Division, SASL, India

(Communicated by Hemant Kumar Nashine)

Abstract

In this work, we prove the existence of the solution of integral equations via fixed point results in the framework of extended Branciari $b$-distance spaces. In order to do this, we introduce $FG$-contractive conditions in extended Branciari $b$-distance spaces and derive common fixed points results for triangular $\alpha$-admissible mappings, followed by some suitable examples.

Keywords: Fixed point, extended Branciari $b$-distance spaces, triangular $\alpha$-admissible mappings, integral equations.

2020 MSC: Primary 47H10, Secondary 54H25

1 Introduction


2 Preliminaries

Now we will review certain concepts and lemmas that will be useful in the following sections.

2.1 $b$ - metric spaces

Czerwik [4] introduced the notion of $b$ - metric space in this manner.

Definition 2.1. [7] Let $X$ be a non empty set and $s \geq 1$ be a given real number. A function $d_B : X \times X \to [0, \infty)$ is called $b$-metric if it satisfies the following properties for each $x, y, z \in X$

1. $d_B(x, y) = 0$ if and only if $x = y$
2. $d_B(x, y) = d_B(y, x)$ (Symmetry)

*Corresponding author
Email addresses: usha.bag2019@vitbhopal.ac.in (Usha Bag), reena.jain@vitbhopal.ac.in (Reena Jain)

Received: August 2021    Accepted: January 2022
3. \( d_B(x, y) \leq s[d_B(x, z) + d_B(z, y)] \) (Triangular Inequality).

Then \((X, d_B)\) is called a b-metric space with coefficient \(s\). When \(s = 1\), the concepts of b-metric space and metric space are all the same.

**Example 2.2.** Let \(X = l_p(\mathbb{R})\) with \(0 < p < 1\)

where \(l_p(\mathbb{R}) = \{\{x_m\} \subset (\mathbb{R}) : \sum_{m=1}^{\infty} |x_m|^p < \infty\}\)

Define \(d_B : X \times X \rightarrow \mathbb{R}^+\) as \(d_B(x, y) = \left( \sum_{m=1}^{\infty} |x_m - y_m|^p \right)^{\frac{1}{p}}\) where \(x = \{x_m\}, y = \{y_m\}\).

It can be easily checked that \(d_B\) is a b-metric with coefficient \(s = 2^\frac{1}{p}\)

The class of b-metric spaces is bigger than the class of metric spaces, as seen in the example above.

### 2.2 Extended b-metric space

Kamran termed as extended b-metric space a new form of generalized metric space.

**Definition 2.3.** Let \(X\) be a non-empty set and \(\omega : X \times X \rightarrow [1, \infty)\). A function \(d_\omega : X \times X \rightarrow [0, \infty)\) is called an extended b-metric if for all \(x, y, z \in X\), it satisfies the following conditions:

1. \(d_\omega(x, y) = 0\) if and only if \(x = y\)
2. \(d_\omega(x, y) = d_\omega(y, x)\) (Symmetry)
3. \(d_\omega(x, z) \leq \omega(x, z)[d_\omega(x, y) + d_\omega(y, z)]\) (Triangular Inequality).

The pair \((X, d_\omega)\) is called an extended b-metric space.

**Note:** b-metric is a special case of the extended b-metric when \(\omega(x, y) = s\), for \(s \geq 1\).

**Example 2.4.** Consider the set \(X = \{-1, 1, 2\}\), define the function \(\omega\) on \(X \times X\) to be the function \(\omega(x, y) = |x| + |y|\). We define the function \(d_\omega\) \((x; y)\) as follows:

\[
\begin{align*}
d_\omega(2, 2) &= d_\omega(1, 1) = d_\omega(-1, -1) = 0; \\
d_\omega(1, 2) &= \frac{1}{2} = d_\omega(2, 1) \\
d_\omega(1, -1) &= d_\omega(-1, 1) = d_\omega(2, -1) = d_\omega(-1, 2) = 1
\end{align*}
\]

Then it is clear that \(d_\omega(x, y)\) satisfies the first two conditions of definition. We need to verify the last condition:

\[
\begin{align*}
d_\omega(1, 2) &= \frac{1}{2} \leq 3 \left[ \frac{1}{3} + \frac{1}{3} \right] = \omega(1, 2) \left[ d_\omega(1, -1) + d_\omega(-1, 2) \right] \\
d_\omega(1, -1) &= \frac{1}{3} \leq 2 \left[ \frac{1}{2} + \frac{1}{3} \right] = \omega(1, -1) \left[ d_\omega(1, 2) + d_\omega(2, -1) \right] \\
d_\omega(-1, 2) &= \frac{1}{3} \leq 3 \left[ \frac{1}{3} + \frac{1}{2} \right] = \omega(-1, 2) \left[ d_\omega(1, 2) + d_\omega(2, -1) \right] \left[ d_\omega(-1, 1) + d_\omega(1, 2) \right]
\end{align*}
\]

Therefore, \(d_\omega(x, y)\) satisfies the last condition of the definition and hence \((X, d_\omega)\) is an extended b-metric space.

For the mapping \(T : X \rightarrow X\) and \(x_0 \in X\), \(O(x_0) = \{x_0, T^2x_0, T^3x_0, \ldots\}\) represents the orbit of \(x_0\).

**Theorem 2.5.** Let \((X, d_\omega)\) be a complete extended b-metric such that \(d_\omega\) is a continuous functional. Let \(T : X \rightarrow X\) satisfy
where \( k \in [0, 1) \) be such that for \( x_0 \in X \), \( \lim_{n,m \to \infty} \omega(x_n, x_m) < \frac{1}{k} \), here \( x_n = T^n(x_0) \), \( n = 1, 2, \ldots \) Then \( T \) has precisely one fixed point \( \xi \). Moreover, for each \( y \in X \) \( T^n(y) \to \xi \).

### 2.3 Rectangular metric spaces

Branciari first introduced the concept of rectangular metric spaces in [5].

**Definition 2.6.** Let \( X \) be a nonempty set. A mapping \( d_R : X \times X \to [0, \infty) \) is called a rectangular metric on \( X \) if for any \( x, y \in X \) and such that for all distinct points \( s, t \in X \) different from \( x \) and \( y \) it satisfies the following conditions:

(i) \( d_R(x, y) = 0 \iff x = y \)

(ii) \( d_R(x, y) = d_R(y, x) \)

(iii) \( d_R(x, y) \leq d_R(x, s) + d_R(s, t) + d_R(t, y) \) (This is known as Rectangular Inequality)

The function \( d_R \) is known as rectangular metric and the pair \( (X, d_R) \) is called a rectangular metric space. In many sources it was called ” Branciari distance space”.

The concept of rectangular b - metric spaces was first introduced by George et al [8], in the following way.

**Definition 2.7.** A mapping \( d_{RB} : X \times X \to [0, \infty) \) is called a rectangular b - metric on \( X \) if for any \( x, y \in X \) if there exists a constant \( \mu \geq 1 \) and such that for all distinct points \( s, t \in X \) different from \( x \) and \( y \) it satisfies the following conditions:

(i) \( d_{RB}(x, y) = 0 \iff x = y \)

(ii) \( d_{RB}(x, y) = d_{RB}(y, x) \)

(iii) \( d_{RB}(x, y) \leq \mu [d_{RB}(x, s) + d_{RB}(s, t) + d_{RB}(t, y)] \)

The function \( d_{RB} \) is known as rectangular metric and the pair \( (X, d_{RB}) \) is called a rectangular b - metric space.

Abdeljawad et al. [1] introduced the notion of extended Branciari b-metric spaces as a generalization of rectangular b-metric spaces. The concepts of extended b-metric and Branciari distance were merged, to form an extended Branciari b-distance space.

**Definition 2.8.** A mapping \( d_{R\omega} : X \times X \to [0, \infty) \) is called a extended Branciari b-distance on a non-empty set \( X \) if for any \( x, y \in X \) and all distinct points \( s, t \in X \) different from \( x \) and \( y \) and a mapping \( \omega : X \times X \to [1, \infty) \) if it satisfies the following conditions:

(i) \( d_{R\omega}(x, y) = 0 \iff x = y \)

(ii) \( d_{R\omega}(x, y) = d_{R\omega}(y, x) \)

(iii) \( d_{R\omega}(x, y) \leq \omega(x, y) [d_{R\omega}(x, s) + d_{R\omega}(s, t) + d_{R\omega}(t, y)] \)

The function \( d_{R\omega} \) is known as extended Branciari b-distance and the pair \( (X, d_{R\omega}) \) is called a extended Branciari b-distance space.

**Example 2.9 (Example 2 [1]).** Let \( X = [0, 1] \). Define \( d_{R\omega} : X \times X \to R \) by

\[
\omega(x, y) = 5x + 5y + 3
\]

then \( (X, d_{R\omega}) \) is an extended Branciari b-distance space. The quadrilateral inequality will be only proved as the other conditions are trivial.

\[
d_{R\omega}(x, y) = |x - y|^2
\]

\[
= |x - z + z - w + w - y|^2
\]

\[
= |x - z|^2 + |z - w|^2 + |w - y|^2 + 2|x - z||z - w| + 2|z - w||w - y| + 2|w - y||x - z|
\]

\[
\leq (5x + 5y + 3) \left[ |x - z|^2 + |z - w|^2 + |w - y|^2 \right]
\]
Hence $d_{Rω}(x, y) \leq \omega(x, y)\left[d_{Rω}(x, z) + d_{Rω}(z, w) + d_{Rω}(w, y)\right]$. Therefore $(X, d_{Rω})$ is an extended Branciari b-distance space.

Controlled rectangular b-metric spaces, which are an extension of rectangular metric spaces, were introduced by Mlaiki et al. in [14]

**Definition 2.10.** Let $X$ be a nonempty set, a function $d : X \times X \to [0, \infty)$.

Let $(X, d)$ be a metric space. A mapping $T : X \to X$ is said to be an $F$-contraction if there exists $\tau > 0$ such that

$$\forall x, y \in X, d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) < F(d(x, y)).$$

where $F : \mathbb{R}_+ \to \mathbb{R}$ is a mapping satisfying the following conditions

(F1) $F$ is strictly increasing, i.e., for all $x, y \in \mathbb{R}_+$ such that $x < y$, $F(x) < F(y)$

(F2) For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n \to \infty} \alpha_n = 0$ if and only if $\lim_{n \to \infty} F(\alpha_n) = -\infty$

(F3) There exists $k \in (0, 1)$ such that $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$

Wardowski’s [19] key result is a generalization of the Banach Contraction Mapping Principle.

**Example 2.12.** Let $F : \mathbb{R}_+ \to \mathbb{R}$ be given by the formula $F(\alpha) = \ln\alpha$.

It is clear that $F$ satisfies (F1), (F3) (F3 satisfies for any $k \in (0, 1)$).

Each mapping $T : X \to X$ satisfying (3.1) also satisfies $d(Tx, Ty) \leq e^{-\tau}d(x, y)$, for all $x, y \in X, Tx \neq Ty$.

It is clear that for $x, y \in X$ such that $Tx = Ty$ the inequality $d(Tx, Ty) \leq e^{-\tau}d(x, y)$ also holds, i.e. $T$ is a Banach contraction.

### 3 Fixed Point Results for $\alpha$-Admissible $\beta$- FG-Contractions

Parvaneh et al. [16] introduced the following. Let $s > 1$ be a fixed real number. We will consider the following classes of functions: $\Delta_{F}$ will denote the set of all functions $F : R^+ \to R$ such that

$(F_{\Delta_1})$ is continuous and strictly increasing.

$(F_{\Delta_2})$ for each sequence $\{t_n\} \subseteq R^+$, $\lim_{n \to \infty} t_n = 0 \iff \lim_{n \to \infty} F(t_n) = -\infty$

Note that condition (F3) from [19], [18] will not be used.

$\Delta_{G, \beta}$ will denote the set of pairs $(G, \beta)$, where $G : R^+ \to R$ and $\beta : [0, \infty) \to [0, 1)$, such that

$(F_{\Delta_4})$ for each sequence $\{t_n\} \subseteq R^+$, $\lim sup_{n \to \infty} G(t_n) \geq 0$ if and only if $\lim sup_{n \to \infty} t_n \geq 1$

$(F_{\Delta_4})$ for each sequence $\{t_n\} \subseteq [0, \infty)$, $\lim sup_{n \to \infty} \beta(t_n) = 1$ implies $\lim_{n \to \infty} t_n = 0$

$(F_{\Delta_5})$ for each sequence $\{t_n\} \subseteq R^+$ and $\sum_{n=1}^{\infty} G(\beta(t_n)) = -\infty$
Definition 3.1. [17] Let \( \alpha : X \times X \to [0, \infty) \) be given mapping where \( X \neq 0 \). A self mapping \( T \) is called \( \alpha \) admissible if for all \( x, y \in X \), we have
\[
\alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1
\]

Definition 3.2. [11] Let \( X \) be a nonempty set, \( T : X \to X \) be a mapping and \( \alpha : X \times X \to [0, \infty) \) be a function. Then \( T \) is called a triangular \( \alpha \) - admissible mapping if for all \( x, y \in X \),
\[
\begin{align*}
1. & \quad \alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1 \\
2. & \quad \alpha(x, z) \geq 1 \quad \text{and} \quad \alpha(z, y) \geq 1 \implies \alpha(x, y) \geq 1
\end{align*}
\]

Definition 3.3. For a nonempty set \( X \), let \( A, B : X \to X \) and \( \alpha : X \times X \to [0, \infty) \) be mappings. We say that \( (A, B) \) is a generalized \( \alpha \) - admissible pair if for all \( x, y \in X \), we have \( \alpha(x, y) \geq 1 \implies \alpha(Ax, By) \geq 1 \)

Remark 3.4. If \( A \) is \( \alpha \)-admissible, it is obvious that \( (A, A) \) is a generalized \( \alpha \)-admissible pair.

Definition 3.5. Let \( (X, d_{R_0}) \) be an extended Branciari b-metric space. A mapping
\( T : X \to X \) be a mapping on \( (X, d_{R_0}) \) is said to be a generalized \( FG_{R_0}\)-contraction if there exists \( F \in \Delta_F \) and \( (G, \beta) \in \Delta_{G, \beta} \) such that for all \( x, y \in X \), \( d_{R_0}(x, y) > 0 \) implies
\[
F(\omega(x, y) d_{R_0}(Tx, Ty)) \leq F(M_\omega(x, y)) + G(\beta(M_\omega(x, y))), \quad \text{where } r \geq 2
\]
and
\[
M_\omega(x, y) = \max \left\{ d_{R_0}(x, y), d_{R_0}(x, Tx), d_{R_0}(y, Ty), \frac{d_{R_0}(y, Ty) + d_{R_0}(x, y)}{\omega(x, y) + d_{R_0}(x, y)} \right\}
\]

Definition 3.6. [11] Let \( X \) be a non-empty set endowed with extended Branciari b-distance \( d_{R_0} \)
\[
1. \quad \text{A sequence } \{x_n\} \text{ in } X \text{ converges to } x \text{ if for every } \epsilon > 0 \text{ there exists } N = N(\epsilon) \in \mathbb{N} \text{ such that } d_{R_0}(x_n, x) < \epsilon \text{ for all } n \geq N. \text{ For this particular case, we write } \lim_{n \to \infty} x_n = x. \\
2. \quad \text{A sequence } \{x_n\} \text{ in } X \text{ is called Cauchy if for every } \epsilon > 0 \exists N = N(\epsilon) \in \mathbb{N} \text{ such that } d_{R_0}(x_n, x_m) < \epsilon \text{ for all } m, n \geq N. \\
3. \quad \text{A } d_{R_0} \text{- metric space } (X, d_{R_0}) \text{ is complete if every Cauchy sequence in } X \text{ is convergent.}
\]

Lemma 3.7 ([11] Lemma 7). Let \( X \) be a nonempty set , \( T : X \to X \) be a triangular \( \alpha \)-admissible mapping and \( x_0 \in X \) such that \( \alpha(x_0, Tx_0) \geq 1 \). Define a sequence \( \{x_n \text{ by } x_{n+1} = Tx_n \} \) for all \( n \in \mathbb{N} \). Then \( \alpha(x_n, x_m) \geq 1 \) for all \( m, n \in \mathbb{N} \) with \( n < m \).

The existence and uniqueness of fixed points for generalized \( FG_{R_0} \)-contraction in complete extended Branciari b-distance spaces are proved by the following theorem.

Theorem 3.8. Let \( (X, d_{R_0}) \) be a complete extended Branciari b-metric space, \( T : X \to X \) and \( \alpha : X \times X \to [0, \infty) \) be given mappings. \( F \in \Delta_F \) and \( (G, \beta) \in \Delta_{G, \beta} \) such that
\[
1. \quad T \text{ is a triangular } \alpha \text{- admissible mapping.} \\
2. \quad T \text{ is a generalized } FG_{R_0} \text{- contraction.} \\
3. \quad \text{There exists } x_0 \in X \text{ such that } \alpha(x_0, Tx_0) \geq 1 \\
4. \quad T \text{ is } \alpha \text{- continuous.}
\]

Then
\[
1. \quad T \text{ has a fixed point } x^* \in X \text{ and } \lim_{n \to \infty} x_n = x^* \\
2. \quad \text{If } \alpha(x, y) \geq 1 \text{ for all } x, y \in \text{Fix}(T), \text{ } T \text{ has a unique fixed point, where } \text{Fix}(T) = \{x \in X | Tx = x\}
\]
\textbf{Proof.} Define a sequence \( x_n \in X \) by \( x_n = T^n(x_0) = T(x_{n-1}) \). As \( T \) is a triangular \( \alpha \)-admissible mapping and there \( \exists \ x_0 \in X \) such that \( \alpha(x_0, T x_0) \geq 1 \).

By using Lemma 1 we conclude that for all \( m, n \in \mathbb{N} \) with \( n < m \)
\[ \alpha(x_n, x_m) \geq 1 \] (3.1)
This implies that
\[ \alpha(x_n, x_{n+1}) \geq 1 \] (3.2)

If there exists \( n_0 \in \mathbb{N} \) such that \( x_{n_0} = x_{n_0+1} \), then \( x_{n_0} \) is fixed point of \( T \) and \( \lim_{n \to \infty} T^n x_{n_0} = x_{n_0} \).

Therefore assume that \( x_n \neq x_{n+1} \) for all \( n \in \mathbb{N} \)
so \( d_{R\omega}(x_n, T x_n) = d_{R\omega}(T x_{n-1}, T x_n) > 0 \) for all \( n \in \mathbb{N} \).

As \( T \) is a generalized \( F \mathcal{G}_{R\omega} \) - contraction, so we have
\[ F(d_{R\omega}(x_n, x_{n+1})) = F(d_{R\omega}(T x_{n-1}, T x_n)) \leq F(\omega(x_{n-1}, x_n)) d_{R\omega}(T x_{n-1}, T x_n) \leq F(\mathcal{M}_\omega(x_{n-1}, x_n)) + \mathcal{G}(\beta(\mathcal{M}_\omega(x_{n-1}, x_n))) \]
where \( \mathcal{M}_\omega(x_{n-1}, x_n) = \max \left\{ d_{R\omega}(x_{n-1}, x_n), d_{R\omega}(x_{n-1}, T x_n), d_{R\omega}(x_n, T x_n), \right. \]
\[ \frac{d_{R\omega}(x_n, T x_n)}{\omega(x_{n-1}, x_n)} \left[ 1 + d_{R\omega}(x_{n-1}, x_n) \right] \}
\[ = \max \left\{ d_{R\omega}(x_{n-1}, x_n), d_{R\omega}(x_{n-1}, x_n), d_{R\omega}(x_n, x_{n+1}), \right. \]
\[ \frac{d_{R\omega}(x_n, x_{n+1})}{\omega(x_{n-1}, x_n)} \left[ 1 + d_{R\omega}(x_{n-1}, x_n) \right] \}
\[ = \max \left\{ d_{R\omega}(x_{n-1}, x_n), d_{R\omega}(x_{n-1}, x_n) \right\} \]
\[ = \max \left\{ d_{R\omega}(x_{n-1}, x_n), d_{R\omega}(x_{n-1}, x_n) \right\} \]

If \( \mathcal{M}_\omega(x_{n-1}, x_n) = d_{R\omega}(x_n, x_{n+1}) \), for some \( n \geq 1 \) then
\[ F(d_{R\omega}(x_n, x_{n+1})) \leq F(d_{R\omega}(x_n, x_{n+1})) + \mathcal{G}(\beta(d_{R\omega}(x_n, x_{n+1}))) \]
which implies that
\[ \mathcal{G}(\beta(d_{R\omega}(x_n, x_{n+1}))) \geq 0 \]
which in turn implies \( \beta(d_{R\omega}(x_n, x_{n+1})) \geq 0 \).

This is a contradiction to the condition of \( (F\Delta_3) \), therefore for all \( n \geq 1 \) we get
\[ d_{R\omega}(x_n, x_{n+1}) \leq d_{R\omega}(x_{n-1}, x_n) \]
Hence we get
\[ F(d_{R\omega}(x_n, x_{n+1})) \leq F(d_{R\omega}(x_{n-1}, x_n)) + \mathcal{G}(\beta(d_{R\omega}(x_{n-1}, x_n))) \]

Using the condition of \( (F\Delta_1) \), we get
\[ F(d_{R\omega}(x_n, x_{n+1})) \leq F(d_{R\omega}(x_{n-1}, x_n)) + \mathcal{G}(\beta(d_{R\omega}(x_{n-1}, x_n))) \]
\[ \leq F(d_{R\omega}(x_{n-2}, x_{n-1})) + \mathcal{G}(\beta(d_{R\omega}(x_{n-2}, x_{n-1}))) + \mathcal{G}(\beta(d_{R\omega}(x_{n-1}, x_n))) \]
there exists }\epsilon > 0\text{.}

Letting the limit } n \to \infty \text{ in the above inequality and using the condition } (F\Delta_5) \text{ we get}

\lim_{n \to \infty} F(d_{R\omega}(x_n, x_{n+1})) = -\infty.

Combining this with the condition } (F\Delta_2) \text{ we have}

\lim_{n \to \infty} d_{R\omega}(x_n, x_{n+1}) = 0 \quad (3.3)

We will show that the sequence } \{x_n\} \text{ is a Cauchy sequence in } (X, d_{R\omega}). \text{ On the contrary, we will assume that there exists } \epsilon > 0 \text{ and two sub sequences } \{x_{n_t}\} \text{ and } \{x_{m_t}\} \text{ of } \{x_n\} \text{ such that } n_t \text{ is the smallest index for which}

n_t > m_t > t \geq 1, \quad d_{R\omega}(x_{m_t}, x_{n_t}) \geq \epsilon \quad (3.4)

This implies } d_{R\omega}(x_{m_t}, x_{n_t-2}) < \epsilon\text{.}

Taking the upper limit as } t \to \infty \text{, we get}

\lim_{t \to \infty} \sup d_{R\omega}(x_{m_t}, x_{n_t-2}) < \epsilon \quad (3.5)

\epsilon \leq d_{R\omega}(x_{m_t}, x_{n_t}) \leq \omega(x_{m_t}, x_{n_t}) \left[d_{R\omega}(x_{n_t}, x_{n_t-2}) + d_{R\omega}(x_{n_t-2}, x_{n_t-1}) + d_{R\omega}(x_{n_t-1}, x_{n_t})\right]

Using (3.4), we get

\epsilon \leq d_{R\omega}(x_{m_t}, x_{n_t}) \leq \omega(x_{m_t}, x_{n_t}) \left[d_{R\omega}(x_{m_t}, x_{n_t-2})\right]

Taking the upper limit as } t \to \infty \text{ in the above equation and using (3.3) we get}

\epsilon \leq \lim_{t \to \infty} \sup d_{R\omega}(x_{m_t}, x_{n_t}) \leq \epsilon \lim_{t \to \infty} \sup \omega(x_{m_t}, x_{n_t}) \quad (3.6)

From (3.4), we have

\epsilon \leq d_{R\omega}(x_{m_t}, x_{m_t}) \leq \omega(x_{m_t}, x_{m_t}) \left[d_{R\omega}(x_{m_t}, x_{m_t+1}) + d_{R\omega}(x_{m_t+1}, x_{m_t+1}) + d_{R\omega}(x_{m_t+1}, x_{m_t})\right]

Taking the upper limit as } t \to \infty \text{ in the above equation and using (3.3) we get}

\epsilon \leq \lim_{t \to \infty} \sup d_{R\omega}(x_{m_t}, x_{m_t}) \leq \lim_{t \to \infty} \sup \omega(x_{m_t}, x_{n_t}) \left[\lim_{t \to \infty} \sup d_{R\omega}(x_{m_t+1}, x_{n_t+1})\right]

Therefore we get

\frac{\epsilon}{\lim_{t \to \infty} \sup \omega(x_{m_t}, x_{n_t})} \leq \lim_{t \to \infty} \sup d_{R\omega}(x_{m_t+1}, x_{n_t+1}) \quad (3.7)

Furthermore, we get

\begin{align*}
d_{R\omega}(x_{m_t}, x_{n_t}) & \leq \omega((x_{m_t}, x_{m_t})) \left[d_{R\omega}(x_{m_t}, x_{n_t-2}) + d_{R\omega}(x_{n_t-2}, x_{n_t-1}) + d_{R\omega}(x_{n_t-1}, x_{n_t})\right] \\
d_{R\omega}(x_{m_t+1}, x_{n_t-1}) & \leq \omega((x_{m_t+1}, x_{m_t-1})) \left[d_{R\omega}(x_{m_t+1}, x_{m_t}) + d_{R\omega}(x_{m_t}, x_{n_t-2}) + d_{R\omega}(x_{n_t-2}, x_{n_t-1})\right]
\end{align*}
\[d_{R\omega}(x_{m+1,2}, x_{n-1}) \leq \omega((x_{m+1,2}, x_{n-1}))[d_{R\omega}(x_{m+1,2}, x_{m-2}) + d_{R\omega}(x_{n-2, x_{n-1}}) + d_{R\omega}(x_{n-2, x_{n-1}})]\]
\[d_{R\omega}(x_{m+1,2}, x_{n}) \leq \omega((x_{m+2, x_{n}}))[d_{R\omega}(x_{m+1,2}, x_{m-2}) + d_{R\omega}(x_{n-2, x_{n-1}}) + d_{R\omega}(x_{n-1, x_{n}})]\]
\[d_{R\omega}(x_{m+1,2}, x_{m}) \leq \omega((x_{m+2, x_{m}}))[d_{R\omega}(x_{m+1,2}, x_{m+1}) + d_{R\omega}(x_{m+1, x_{n+1}}) + d_{R\omega}(x_{m+1, x_{n}})]\]

Taking the upper limit as \(t \to \infty\) in the above inequalities and using (3.3) and (3.4), we get
\[
\lim_{t \to \infty} \sup_{m} d_{R\omega}(x_{m}, x_{n}) \leq \epsilon \lim_{t \to \infty} \sup_{m} \omega(x_{m}, x_{n})
\]
\[
\lim_{t \to \infty} \sup_{m} d_{R\omega}(x_{m+1,1}, x_{n-1}) \leq \epsilon \lim_{t \to \infty} \sup_{m} \omega(x_{m+1,1}, x_{n-1})
\]
\[
\lim_{t \to \infty} \sup_{m} d_{R\omega}(x_{m+1,2}, x_{n-1}) \leq \epsilon \lim_{t \to \infty} \sup_{m} \omega(x_{m+1,2}, x_{n-1})
\]
\[
\lim_{t \to \infty} \sup_{m} d_{R\omega}(x_{m+1,2}, x_{n}) \leq \epsilon \lim_{t \to \infty} \sup_{m} \omega(x_{m+1,2}, x_{n})
\]
\[
\lim_{t \to \infty} \sup_{m} d_{R\omega}(x_{m+1,2}, x_{n}) = 0
\]

Again using
\[
\epsilon \leq d_{R\omega}(x_{m}, x_{n}) \leq \omega(x_{m}, x_{n})\left[d_{R\omega}(x_{m}, x_{n+1}) + d_{R\omega}(x_{m+1,1}, x_{n+1}) + d_{R\omega}(x_{m+1,1}, x_{n})\right]
\]

Taking the lower limit as \(t \to \infty\) in the above equation and using (3.3) we get
\[
\epsilon \leq \lim_{t \to \infty} \inf_{m} d_{R\omega}(x_{m}, x_{n}) \leq \lim_{t \to \infty} \inf_{m} \omega(x_{m}, x_{n})\left[\lim_{t \to \infty} \inf_{m} d_{R\omega}(x_{m+1,1}, x_{n+1})\right]
\]
Therefore we get
\[
\frac{\epsilon}{\lim_{t \to \infty} \inf_{m} \omega(x_{m}, x_{n})} \leq \lim_{t \to \infty} \inf_{m} d_{R\omega}(x_{m+1,1}, x_{n+1})
\]

Therefore there exists \(t_0 \in \mathbb{N}\) such that \(d_{R\omega}(x_{m+1,1}, x_{n+1}) > 0\).

Consider \(M_{\omega}(x_{n}, x_{m}) = \max\left\{d_{R\omega}(x_{n}, x_{m}), d_{R\omega}(x_{n}, T x_{m}), d_{R\omega}(x_{m}, T x_{n}), \right\}\)
\[
\frac{d_{R\omega}(x_{m}, T x_{n})[1 + d_{R\omega}(x_{n}, T x_{n})]}{\omega(x_{n}, x_{m})[1 + d_{R\omega}(x_{n}, x_{m})]}
\]
\[
M_{\omega}(x_{m}, x_{m}) = \max\left\{d_{R\omega}(x_{m}, x_{m}), d_{R\omega}(x_{m}, x_{m+1}), d_{R\omega}(x_{m}, x_{m+1}), \right\}\]
\[
\frac{d_{R\omega}(x_{m}, x_{m+1})[1 + d_{R\omega}(x_{m}, x_{m+1})]}{\omega(x_{m}, x_{m})[1 + d_{R\omega}(x_{m}, x_{m})]}
\]
Therefore, \(M_{\omega}(x_{m}, x_{m}) = d_{R\omega}(x_{m}, x_{m})\) from (3.3).
Taking the upper limit as \( t \to \infty \) in the above equation and using equation (3.6), we get
\[
\lim_{t \to \infty} \sup_{x_n, x_{m_1}} \mathcal{M}_{\omega}(x_{n}, x_{m_1}) = \lim_{t \to \infty} \sup_{x_n, x_{m_1}} d_{R_{\omega}}(x_{m_1}, x_{n}) < \epsilon \lim_{t \to \infty} \sup_{x, x'} \omega(x, x'). \tag{3.8}
\]
Consider \( F(\omega(x_{n}, x_{m_1}) \leq \epsilon \omega(x_{n}, x_{m_1})) \leq F(\omega(x_{n}, x_{m_1}) \leq \epsilon \omega(x_{n}, x_{m_1})). \) Taking the upper limit as \( t \to \infty \) in the above equation and using (3.7), we get
\[
F\left(\lim_{t \to \infty} \sup_{x_n, x_{m_1}} \omega(x_{n}, x_{m_1}) \right) \leq F\left(\lim_{t \to \infty} \sup_{x_n, x_{m_1}} d_{R_{\omega}}(x_{m_1+1}, x_{n+1})\right) \leq F\left(\lim_{t \to \infty} \sup_{x_n, x_{m_1}} \mathcal{M}_{\omega}(x_{n}, x_{m_1}) + G(\beta(\lim_{t \to \infty} \sup_{x_n, x_{m_1}} \mathcal{M}_{\omega}(x_{n}, x_{m_1})))\right)
\]
\[
\leq F(\epsilon \lim_{t \to \infty} \sup_{x_n, x_{m_1}} \mathcal{M}_{\omega}(x_{n}, x_{m_1})) + \sup_{x_n, x_{m_1}} G(\beta(\mathcal{M}_{\omega}(x_{n}, x_{m_1}))).
\]
Therefore
\[
\lim_{t \to \infty} \sup_{x_n, x_{m_1}} G(\beta(\mathcal{M}_{\omega}(x_{n}, x_{m_1}))) \geq 0.
\]
This implies \( \lim_{t \to \infty} \sup_{x_n, x_{m_1}} (\beta(\mathcal{M}_{\omega}(x_{n}, x_{m_1}))) \geq 1. \) As \( \beta(\xi) < 1 \) for all \( \xi \geq 0, \) we get
\[
\lim_{t \to \infty} \sup_{x_n, x_{m_1}} (\beta(\mathcal{M}_{\omega}(x_{n}, x_{m_1}))) = 1.
\]
Using the property of \( \beta, \) we get \( \lim_{t \to \infty} \sup_{x_n, x_{m_1}} \mathcal{M}_{\omega}(x_{n}, x_{m_1}) = 0, \) which is a contradiction to (3.8). Hence \( \{x_n\} \) is a Cauchy sequence in \( (X, d_{R_{\omega}}) \)
As \( (X, d_{R_{\omega}}) \) is a complete extended Branciari b-distance space, there exists \( x^* \in X \) such that \( \lim_{n \to \infty} d_{R_{\omega}}(x_n, x^*) = 0. \)
By definition of sequence \( \{x_n\}, \) we get \( \lim_{n \to \infty} T^{n} x_0 = x^*. \) By using (3.2) and \( \alpha \) continuous property of \( T, \) we get
\[
\lim_{n \to \infty} T(x_n) = T(x^*) \Rightarrow x^* = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} T(x_n) \Rightarrow x^*
\]
is a fixed point of \( T. \)
To prove uniqueness of fixed point of \( T, \) let us assume there are two fixed points \( x, y \) such that \( x \neq y \) then \( T(x) \neq T(y). \) As \( \alpha(x, y) \geq 1 \) and
\[
F(\omega(x, y)^{r} d_{R_{\omega}}(T x, T y)) \leq F(\mathcal{M}_{\omega}(x, y)) + G(\beta(\mathcal{M}_{\omega}(x, y)))
\]
where \( r \geq 2 \) and
\[
\mathcal{M}_{\omega}(x, y) = \max \left\{ d_{R_{\omega}}(x, y), d_{R_{\omega}}(x, T x), d_{R_{\omega}}(y, T y), \frac{d_{R_{\omega}}(y, T y)[1 + d_{R_{\omega}}(x, T x)]}{\omega(x, y)[1 + d_{R_{\omega}}(x, y)]} \right\}.
\]
Therefore,
\[
\mathcal{M}_{\omega}(x, y) = \max \left\{ d_{R_{\omega}}(x, y), d_{R_{\omega}}(x, x), d_{R_{\omega}}(y, y), \frac{d_{R_{\omega}}(y, y)[1 + d_{R_{\omega}}(x, x)]}{\omega(x, y)[1 + d_{R_{\omega}}(x, y)]} \right\}.
\]
Hence, \( \mathcal{M}_{\omega}(x, y) = d_{R_{\omega}}(x, y). \) This implies \( F(\omega(x, y)^{r} d_{R_{\omega}}(T x, T y)) \leq F(d_{R_{\omega}}(x, y)) + G(\beta(d_{R_{\omega}}(x, y))). \) Using the increasing property of \( F, \) we get
\[
G(\beta(d_{R_{\omega}}(x, y))) \geq 0 \Rightarrow \beta(d_{R_{\omega}}(x, y)) \geq 1.
\]
But this is a contradiction to \( \beta(\xi) < 1, \) for all \( \xi \geq 0. \) Therefore \( d_{R_{\omega}}(x, y) = 0 \Rightarrow x = y. \) This means \( T \) has a unique fixed point. \( \square \)

**Theorem 3.9.** Let \( (X, d_{R_{\omega}}) \) be a complete extended Branciari b-metric space,
\( T : X \to X \) and \( \alpha : X \times X \to [0, \infty) \) be two mappings. \( F \in \Delta_F \) and \( (G, \beta) \in \Delta_{G, \beta} \) such that
1. $T$ is a triangular $\alpha$ - admissible mapping.
2. $T$ is a generalized $FG_{R\omega}$ - contraction.
3. There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$
4. If $\{x_n\}$ is a sequence in $X$ and $\lim_{t\to\infty} x_n = x$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Then
1. $T$ has a fixed point $x^* \in X$ and $\lim_{n\to\infty} T^n x_0 = x^*$
2. If $\alpha(x, y) \geq 1$ for all $x, y \in Fix(T)$, $T$ has a unique fixed point, where $Fix(T) = \{x \in X | Tx = x\}$

**Proof.** We conclude, as we did in the proof of Theorem 3.8, that the sequence is $\{x_n\}$ is defined by $x_n = T^n x_0 = T x_{n-1}$ satisfying

$$\alpha(x_n, x_{n+1}) \geq 1$$

(3.9)

$$\lim_{n\to\infty} d_{R\omega}(x_n, x_{n+1}) = 0$$

(3.10)

for all $n, m \in \mathbb{N}$ with $n > m$ and there exists $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^* \implies \lim_{n\to\infty} T^n x_0 = x^*$$

(3.11)

We will show that $x^*$ is a fixed point of $T$.

Assume there exists $n_0 \in \mathbb{N}$ such that $x_{n+1} \neq Tx^*$ for all $n \geq n_0$.

This implies $d_{R\omega}(Tx_n, Tx^*) > 0$ for all $n \geq n_0$.

Using (3.9) and (3.11), we get $\alpha(x_n, x^*) \geq 1$.

For all $n \geq n_0$, we get

$$F(\omega(x_n, x^*)^r d_{R\omega}(Tx_n, Tx^*)) \leq F(M_{R\omega}(x_n, x^*)) + G(\beta(M_{R\omega}(x_n, x^*)))$$

(3.12)

where $M_{R\omega}(x_n, x^*) = \max\left\{d_{R\omega}(x_n, x^*), d_{R\omega}(x_n, Tx_n), d_{R\omega}(x^*, Tx^*), \frac{d_{R\omega}(x^*, Tx^*)(1 + d_{R\omega}(x_n, x^*))}{\omega(x_n, x^*)[1 + d_{R\omega}(x_n, x^*)]} \right\}$.

Therefore

$$M_{R\omega}(x_n, x^*) = \max\left\{d_{R\omega}(x_n, x^*), d_{R\omega}(x_n, x_{n+1}), d_{R\omega}(x^*, Tx^*), \frac{d_{R\omega}(x^*, Tx^*)(1 + d_{R\omega}(x_n, x_{n+1}))}{\omega(x_n, x^*)[1 + d_{R\omega}(x_n, x^*)]} \right\}.$$ 

(3.13)

Taking the upper limits as $n \to \infty$ and (3.10), we get

$$\lim_{n\to\infty} \sup M_{R\omega}(x_n, x^*) = d_{R\omega}(x^*, Tx^*).$$

Using (3.12) as $n \to \infty$ and using (3.1), we get

$$\lim_{n\to\infty} F(\omega(x_n, x^*)^r d_{R\omega}(Tx_n, Tx^*)) \leq F(\lim_{n\to\infty} \sup M_{R\omega}(x_n, x^*)) + \lim_{n\to\infty} \sup G(\beta(M_{R\omega}(x_n, x^*)))$$

Using the increasing property of $F$, we get

$$\lim_{n\to\infty} \sup G(\beta(M_{R\omega}(x_n, x^*))) \geq 0 \implies \lim_{n\to\infty} \sup \beta(M_{R\omega}(x_n, x^*)) \geq 1$$

As $\beta(\xi) < 1$ for all $\xi \geq 0$, we get,

$$\lim_{n\to\infty} \beta(M_{R\omega}(x_n, x^*)) = 1 \implies \lim_{n\to\infty} \sup M_{R\omega}(x_n, x^*) = 0$$

Using (3.13), we get $d_{R\omega}(x^*, Tx^*) = 0$. But this is a contradiction to $d_{R\omega}(x^*, Tx^*) > 0$. Therefore, $x^*$ is a fixed point of $T$. For the second part of the proof proceed as in theorem 3.8.

From the Theorem 3.8 and Theorem 3.9, we get the following corollary.
Corollary 3.10. Let \((X, d_{R_w})\) be a complete extended Branciari b-metric space, \(T : X \to X\) and \(\alpha : X \times X \to [0, \infty)\) be two mappings. \(F \in \Delta_F\) and \((G, \beta) \in \Delta_{G, \beta}\) such that

1. \(T\) is a triangular \(\alpha\) - admissible mapping.
2. For all \(x, y \in X\) with \(\omega(x, y)d_{R_w}(Tx, Ty) \leq \gamma \mathcal{A}(x, y)\)
3. There exists \(x_0 \in X\) such that \(\alpha(x_0, Tx_0) \geq 1\)
4. (a) Either \(T\) is \(\alpha\) - continuous or
   (b) If \(\{x_n\}\) is a sequence in \(X\) and \(\lim_{n \to \infty} x_n = x\) such that \(\alpha(x_n, x_{n+1}) \geq 1\) for all \(n \in \mathbb{N}\), then \(\alpha(x_n, x) \geq 1\) for all \(n \in \mathbb{N}\).

Then
1. \(T\) has a fixed point \(x^* \in X\) and \(\lim_{n \to \infty} T^n x_0 = x^*\)
2. If \(\alpha(x, y) \geq 1\) for all \(x, y \in Fix(T)\), \(T\) has a unique fixed point, where
   \[Fix(T) = \{x \in X | Tx = x\}\]

4 Application to nonlinear integral equations

Let \(\mathcal{C}([0, 1])\) be the set of all continuous function on \(I = [0, 1]\). Let \(X = \mathcal{C}(I, \mathbb{R})\) be endowed with the Extended Branciari b-metric space function defined by

\[d_{R_w}(x, y) = \sup_{t \in I} |x(t) - y(t)|^2\]

for all \(x, y \in X\) and \(\omega(x, y) = |x| + |y| + 3\), where \(\omega : X \times X \to [1, \infty)\)

Consider the nonlinear integral equation

\[x(t) = g(t) + \lambda \int_0^1 \mathcal{L}(t, s)f(s, x(s))ds\]  \hspace{1cm} (4.1)

where \(f : I \times \mathbb{R} \to \mathbb{R}, \lambda \geq 0\) and \(\mathcal{L} : I \times I \to [0, \infty)\) are given functions.

Suppose that the following conditions hold:
1. \(g : I \to \mathbb{R}\) is a continuous function.
2. \(\mathcal{L} : I \times I \to [0, \infty)\) is integrable on \([0, 1]\)
3. \(f : I \times \mathbb{R} \to \mathbb{R}\) is a continuous function such that for all \(x, y \in \mathcal{C}[0, 1]\).
   \[\int_0^1 |f(s, x(s)) - f(s, y(s))|^2ds \leq \frac{\rho \Theta(x(t), y(t))}{\max_{t \in [0, 1]}(|x| + |y| + 3)^2},\]
   where
   \[\Theta(x(t), y(t)) = \max \left\{|x(t) - y(t)|^2, |x(t) - Tx(t)|^2, |y(t) - Ty(t)|^2,\right.\]
   \[\left.\frac{|y(t) - Ty(t)|^2(1 + |x(t) - Tx(t)|^2)}{\max_{t \in [0, 1]}(|x| + |y| + 3)^2[1 + |x(t) - y(t)|^2]}\right\}\]
4. \(Tx \in \mathcal{C}[0, 1]\) for all \(x \in \mathcal{C}[0, 1]\) where \(Tx(t) = g(t) + \lambda \int_0^1 \mathcal{L}(t, s)f(s, x(s))ds\)
5. For all \(x \in \mathcal{C}[0, 1]\) and \(x(t) \geq 0\) for all \(t \in [0, 1]\), we have \(Tx(t) \geq 0\) for all \(t \in [0, 1]\)
6. Assume \(\lambda^2 \mathcal{L}^2 \leq 1\)

Under the above conditions \((1) - (6)\), the nonlinear integral equation \((4.1)\) has a unique solution in \(\mathcal{C}[0, 1]\)

**Proof.** Define a function \(T : \mathcal{C}[0, 1] \to \mathcal{C}[0, 1]\) by

\[Tx(t) = g(t) + \lambda \int_0^1 \mathcal{L}(t, s)f(s, x(s))ds\]  \hspace{1cm} for all \(x \in \mathcal{C}[0, 1], t \in [0, 1]\)

The existence of a solution to \((4.1)\) is equivalent to the existence of a fixed point of \(T\).
Define a mapping \( \alpha : X \times X \to [0, \infty) \) by
\[
\alpha(x, y) = \begin{cases} 
2 & \text{if } x(t), y(t) \in [0, \infty) \text{ for all } t \in [0, 1] \\
0 & \text{otherwise}
\end{cases}
\]

We will prove that \( T \) is a triangular \( \alpha \)-admissible mapping. Let \( x, y \in C[0, 1] \) such that \( \alpha(x, y) \geq 1 \). Therefore, \( x(t) \geq 0, y(t) \geq 0 \) for all \( t \in [0, 1] \).

From condition (4) it follows that \( Tx(t) \geq 0, Ty(t) \geq 0 \) for all \( t \in [0, 1] \) this implies \( \alpha(Tx, Ty) \geq 1 \). Similarly, for \( x, y, z \in C[0, 1] \) such that \( \alpha(x, z) \geq 1 \) and \( \alpha(z, y) \geq 1 \), we have \( x(t) \geq 0, y(t), z(t) \geq 0 \) for all \( t \in [0, 1] \). This implies that \( \alpha(x, y) \geq 1 \). Hence, \( T \) is a triangular \( \alpha \)-admissible mapping. Now, for \( x, y \in X \) we have
\[
|T(x(t)) - T(y(t))|^2 = \left| g(t) + \lambda \int_0^1 \mathcal{L}(t, s)f(s, x(s))ds - g(t) - \lambda \int_0^1 \mathcal{L}(t, s)f(s, y(s))ds \right|^2 \\
\leq \lambda^2 \left( \int_0^1 \mathcal{L}(t, s)|f(s, x(s)) - f(s, y(s))|ds \right)^2 \\
\leq \lambda^2 \left( \int_0^1 \mathcal{L}(t, s)ds \right)^2 \left( \int_0^1 |f(s, x(s)) - f(s, y(s))|^2 ds \right) \\
\leq \lambda^2 \left( \sup_{t \in I} \int_0^1 \mathcal{L}(t, s)ds \right)^2 \left( \int_0^1 |f(s, x(s)) - f(s, y(s))|^2 ds \right) \\
\leq \lambda^2 L^2 \frac{\rho \Theta(x(t), y(t))}{\max_{t \in [0, 1]} (|x| + |y| + 3)^2} \\
\leq \rho \frac{\Theta(x(t), y(t))}{(|x| + |y| + 3)^2}.
\]

Therefore
\[
\sup_{t \in [0, 1]} |T(x(t)) - T(y(t))|^2 \leq \sup_{t \in [0, 1]} \left( \frac{\rho \Theta(x(t), y(t))}{(|x| + |y| + 3)^2} \right) = \frac{\rho \Theta(x, y)}{(|x| + |y| + 3)^2}.
\]

By the above, we conclude that all the assumptions in corollary (3.10) are satisfied. Thus, \( T \) has a fixed point \( x \in C[0, 1] \) and hence equation (4.1) has a solution \( x \in C[0, 1] \). \( \square \)

5 Conclusion

We proved the existence and uniqueness of a fixed point in extended Branciari \( b \)-metric space in this manuscript, which generalises many previous results. We also presented an application of our integral equations results.

Acknowledgment The authors are grateful to the referees for their assistance in improving the paper in several places. The authors are grateful to the manuscript reviewers and the Journal’s editorial board for their thoughtful remarks and consideration.

References


[2] A.A. Abdou and M.F.S. Alasmari, Fixed point theorems for generalized \( \alpha-\psi \)-contractive mappings in extended \( b \)-metric spaces with applications, AIMS Math. 6 (2021), no. 6, 5465–5478.


Solutions of integral equations via fixed point results in extended Branciari b-distance spaces


