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Solutions of integral equations via fixed point results in extended Branciari b-distance spaces

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Abstract

In this work, we prove the existence of the solution of integral equations via fixed point results in the framework of extended Branciari b-distance spaces. In order to do this, we introduce FG-contractive conditions in extended Branciari b-distance spaces and derive common fixed points results for triangular α-admissible mappings, followed by some suitable examples.

Keywords: Fixed point, extended Branciari b-distance spaces, triangular α-admissible mappings, integral equations.
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1 Introduction

Many authors introduce and generalize the concept of distance in the metric fixed-point theory in various ways. Czerwik[7] extends Bakhtin's [4] definition of b-metric space. Kamran et al. [10] introduced the concept of extended b-metric space by replacing the property of triangle inequality with a quadrilateral one, Branciari [5] extended the metric space and introduced the concept of the Branciari distance.

2 Preliminaries

Now we will review certain concepts and lemmas that will be useful in the following sections.

2.1 b - metric spaces

Czerwik [7] introduced the notion of b - metric space in this manner.

Definition 2.1. [7] Let X be a non empty set and s ≥ 1 be a given real number. A function $d_B : X \times X \to [0, \infty)$ is called b-metric if it satisfies the following properties for each $x, y, z \in X$

1. $d_B(x, y) = 0$ if and only if $x = y$
2. $d_B(x, y) = d_B(y, x)$ (Symmetry)

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3. $d_B(x,y) \leq s[d_B(x,z) + d_B(z,y)]$ (Triangular Inequality).

Then (X, d_B) is called a b-metric space with coefficient s. When s = 1, the concepts of b-metric space and metric space are all the same.

Example 2.2. [10] Let X = $l_p(\mathbb{R})$ with $0 < p < 1$

where $l_p(\mathbb{R}) = \{\{x_m\} \subset (\mathbb{R}) : \sum_{m=1}^{\infty} |x_m|^p < \infty\}$

Define $d_B : X \times X \to \mathbb{R}^+$ as $d_B(x,y) = \big(\sum_{m=1}^{\infty} |x_m - y_m|^p\big)^{\frac{1}{p}}$ where $x = \{x_m\}, y = \{y_m\}$.
It can be easily checked that d_B is a b-metric with coefficient $s = 2^{\frac{1}{p}}$

The class of b - metric spaces is bigger than the class of metric spaces, as seen in the example above.

2.2 Extended b - metric space

Kamran [10] termed as extended b-metric space a new form of generalized metric space.

Definition 2.3. Let X be a non-empty set and $\omega : X \times X \to [1, \infty)$. A function $d_\omega : X \times X \to [0, \infty)$ is called an extended b-metric if for all x, y, z ϵ X, it satisfies the following conditions

1. $d_\omega(x,y) = 0$ if and only if $x = y$
2. $d_\omega(x,y) = d_\omega(y,x)$ (Symmetry)
3. $d_\omega(x,z) \leq \omega(x,z)[d_\omega(x,y) + d_\omega(y,z)]$ (Triangular Inequality).

The pair (X, d_ω) is called an extended b-metric space.

Note: b - metric is a special case of the extended b-metric when $\omega(x,y) = s$, for $s \geq 1$.

Example 2.4. [2, Example 3]

Consider the set X = $\{-1, 1, 2\}$, define the function ω on $X \times X$ to be the function ω(x, y) = $|x| + |y|$. We define the function d_ω (x; y) as follows:

d_ω(2, 2) = d_ω(1, 1) = d_ω(-1, -1) = 0;

d_ω(1, 2) = $\frac{1}{2}$ = d_ω(2, 1) and

d_ω(1, -1) = d_ω(-1, 1) = d_ω(2, -1) = d_ω(-1, 2) = 1
Then it is clear that d_ω(x, y) satisfies the first two conditions of definition. We need to verify the last condition:

$$d_\omega(1,2) = \frac{1}{2} \leq 3\Big[\frac{1}{3} + \frac{1}{3}\Big] = \omega(1,2)\Big[d_\omega(1,-1) + d_\omega(-1,2)\Big]$$

$$d_\omega(1,-1) = \frac{1}{3} \leq 2\Big[\frac{1}{2} + \frac{1}{3}\Big] = \omega(1,-1)\Big[d_\omega(1,2) + d_\omega(2,-1)\Big]$$

$$d_\omega(-1,2) = \frac{1}{3} \leq 3\Big[\frac{1}{3} + \frac{1}{2}\Big] = \omega(-1,2)\Big[d_\omega(1,2) + d_\omega(2,-1)\Big]\Big[d_\omega(-1,1) + d_\omega(1,2)\Big]$$

Therefore, $d_\omega(x,y)$ satisfies the last condition of the definition and hence (X, d_ω) is an extended b-metric space.

For the mapping $T : X \to X$ and $x_0 \in X, \mathcal{O}(x_0) = \big\{x_0, T^2x_0, T^3x_0, ...\big\}$ represents the orbit of x_0.

Theorem 2.5. [10, Theorem 2] Let (X, d_ω) be a complete extended b-metric space such that d_ω is a continuous functional. Let $T : X \to X$ satisfy

$$d_\omega(T(x), T(y)) \leq k d_\omega(x, y) \text{ for each } x, y \in X$$

where $k \in [0, 1)$ be such that for $x_0 \in X$, $\lim_{n,m\to\infty} \omega(x_n, x_m) < \frac{1}{k}$, here $x_n = T^n(x_0)$,

$n = 1, 2, ...$ Then T has precisely one fixed point ξ. Moreover, for each $y \in X$ $T^n(y) \to \xi$.

2.3 Rectangular metric spaces

Branciari first introduced the concept of rectangular metric spaces in [5].

Definition 2.6. Let X be a nonempty set. A mapping $d_R : X \times X \to [0, \infty)$ is called a rectangular metric on X if for any $x, y \in X$ and such that for all distinct points $s, t \in X$ different from x and y it satisfies the following conditions:
(i) $d_R(x, y) = 0 \Longleftrightarrow x = y$
(ii) $d_R(x, y) = d_R(y, x)$
(iii) $d_R(x, y) \leq d_R(x, s) + d_R(s, t) + d_R(t, y)$ (This is known as Rectangular Inequality)

The function d_R is known as rectangular metric and the pair (X, d_R) is called a rectangular metric space. In many sources it was called " Branciari distance space".

The concept of rectangular b - metric spaces was first introduced by George et al [8]. in the following way.

Definition 2.7. A mapping $d_{RB} : X \times X \to [0, \infty)$ is called a rectangular b - metric on X if for any $x, y \in X$ if there exists a constant $\mu \geq 1$ and such that for all distinct points $s, t \in X$ different from x and y it satisfies the following conditions:
(i) $d_{RB}(x, y) = 0 \Longleftrightarrow x = y$
(ii) $d_{RB}(x, y) = d_{RB}(y, x)$
(iii) $d_{RB}(x, y) \leq \mu [d_{RB}(x, s) + d_{RB}(s, t) + d_{RB}(t, y)]$

The function d_{RB} is known as rectangular metric and the pair (X, d_{RB}) is called a rectangular b - metric space.

Abdeljawad et al. [1] introduced the notion of extended Branciari b-metric spaces as a generalization of rectangular b-metric spaces. The concepts of extended b-metric and Branciari distance were merged, to form an extended Branciari b-distance space.

Definition 2.8. A mapping $d_{R\omega} : X \times X \to [0, \infty)$ is called a extended Branciari b-distance on a non-empty set X if for any $x, y \in X$ and all distinct points $s, t\ inX$ different from x and y and a mapping $\omega : X \times X \to [1, \infty)$ if it satisfies the following conditions:
(i) $d_{R\omega}(x, y) = 0 \Longleftrightarrow x = y$
(ii) $d_{R\omega}(x, y) = d_{R\omega}(y, x)$
(iii) $d_{R\omega}(x, y) \leq \omega(x, y) [d_{R\omega}(x, s) + d_{R\omega}(s, t) + d_{R\omega}(t, y)]$

The function $d_{R\omega}$ is known as extended Branciari b-distance and the pair $(X, d_{R\omega})$ is called a extended Branciari b-distance space.

Example 2.9 (Example 2 [1]). Let $X = [0, 1]$. Define $d_{R\omega} : X \times X \longrightarrow R$ by
$\omega(x, y) = 5x + 5y + 3$, then $(X, d_{R\omega})$ is an extended Branciari b-distance space.

The quadrilateral inequality will be only proved as the other conditions are trivial.

$$\begin{aligned}
d_{R\omega}(x, y) &= |x - y|^2 \\
&= |x - z + z - w + w - y|^2 \\
&= |x - z|^2 + |z - w|^2 + |w - y|^2 + 2|x - z||z - w| + 2|z - w||w - y| + 2|w - y||x - z| \\
&\leq (5x + 5y + 3) \left[|x - z|^2 + |z - w|^2 + |w - y|^2 \right]
\end{aligned}$$

$$= \omega(x,y)\Big[d_{R\omega}(x,z) + d_{R\omega}(z,w) + d_{R\omega}(w,y)\Big]$$

Hence $d_{R\omega}(x,y) \leq \omega(x,y)\Big[d_{R\omega}(x,z) + d_{R\omega}(z,w) + d_{R\omega}(w,y)\Big]$. Therefore $(X, d_{R\omega})$ is an extended Branciari b-distance space.

Controlled rectangular b-metric spaces, which are an extension of rectangular metric spaces, were introduced by Mlaiki et al. in [14]

Definition 2.10. Let X be a nonempty set, a function $\xi : X^4 \to [1,\infty)$ and
$d_\xi : X \times X \to [0,\infty)$. We say that (X, d_ξ) a controlled rectangular b-metric space if for all distinct $x, y, s, t \in X$ it satisfies the following
(1) $d_\xi(x,y) = 0 \Longleftrightarrow x = y$
(2) $d_\xi(x,y) = d_\xi(y,x)$
(3) $d_\xi(x,y) \leq \xi(x,y,s,t)[d_\xi(x,s) + d_\xi(s,t) + d_\xi(t,y)]$

Many different forms of contractions have been used in recent years to ensure the existence and uniqueness of the fixed point of mapings in various spaces. Wardowski [19] proposed the concept of an F-contraction in 2012, and demonstrated fixed point results in metric spaces as a generalization of the Banach contraction principle.

Definition 2.11. Let (X, d) be a metric space. A mapping T : X → X is said to be an F-contraction if there exists $\tau > 0$ such that

$$\forall x, y \ \epsilon X, d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) < F(d(x,y)). \tag{2.1}$$

where F: $\mathbb{R}_+ \to \mathbb{R}$ is a mapping satisfying the following conditions
(F_1) F is strictly increasing, i.e., for all x, y ϵ $\mathbb{R}_+$ such that x < y , F(x) < F(y)
(F_2) For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers , $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$
(F_3) There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$

Wardowski's [19] key result is a generalization of the Banach Contraction Mapping Principle.

Example 2.12. Let $F : \mathbb{R}_+ \to \mathbb{R}$ be given by the formula $F(\alpha) = ln\alpha$.
It is clear that F satisfies $((F_1),(F_3))$ (F_3 satisfies for any $k \in (0,1)$).
Each mapping $T : X \to X$ satisfying (3.1) also satisfies $d(Tx,Ty) \leq e^{-\tau} d(x,y)$,
for all $x, y \in X, Tx \neq Ty$.

It is clear that for $x, y \ \epsilon \ X$ such that $Tx = Ty$ the inequality $d(Tx,Ty) \leq e^{-\tau} d(x,y)$ also holds, i.e. T is a Banach contraction.

3 Fixed Point Results for α-Admissible β- FG-Contractions

Parvaneh et al.[16] introduced the following. Let $s > 1$ be a fixed real number. We will consider the following classes of functions. Δ_F will denote the set of all functions $F : R^+ \to R$ such that

$(F\Delta_1)$ is continuous and strictly increasing.

$(F\Delta_2)$ for each sequence $\{t_n\} \subseteq R^+$, $\lim_{n\to\infty} t_n = 0 \Longleftrightarrow \lim_{n\to\infty} F(t_n) = -\infty$

Note that condition (F3) from [[19], [18]] will not be used.

$\Delta_{G,\beta}$ will denote the set of pairs (G,β), where $G : R^+ \to R$ and $\beta : [0,\infty) \to [0,1)$, such that

$(F\Delta_3)$ for each sequence $\{t_n\} \subseteq R^+$, $\limsup_{n\to\infty} G(t_n) \geq 0$ if and only if $\limsup_{n\to\infty} t_n \geq 1$

$(F\Delta_4)$ for each sequence $\{t_n\} \subseteq [0,\infty)$, $\limsup_{n\to\infty} \beta(t_n) = 1$ implies $\lim_{n\to\infty} t_n = 0$;

$(F\Delta_5)$ for each sequence $\{t_n\} \subseteq R^+$, $\sum_{n=1}^{\infty} G(\beta(t_n)) = -\infty$

Samet et al. [17] defined the α -admissible mappings class in 2012.

Definition 3.1. [17] Let $\alpha : X \times X \to [0, \infty)$ be given mapping where $X \neq 0$. A self mapping T is called α admissible if for all $x, y \in X$, we have

$\alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1$

Definition 3.2. [11] Let X be a nonempty set, $T : X \to X$ be a mapping and $\alpha : X \times X \to [0, \infty)$ be a function. Then T is called a triangular α - admissible mapping if for all $x, y \in X$,

1. $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$
2. $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ implies $\alpha(x, y) \geq 1$

Definition 3.3. For a nonempty set X, let $A, B : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be mappings. We say that (A, B) is a generalized α - admissible pair if for all $x, y \in X$, we have $\alpha(x, y) \geq 1 \Longrightarrow \alpha(Ax, By) \geq 1$

Remark 3.4. If A is α-admissible, it is obvious that (A, A) is a generalized α-admissible pair.

Definition 3.5. Let $(X, d_{R\omega})$ be an extended Branciari b-metric space. A mapping

$T : X \to X$ be a mapping on $(X, d_{R\omega})$ is said to be a generalized $FG_{R\omega}$-contraction if there exists $F \in \Delta_F$ and $(G, \beta) \in \Delta_{G,\beta}$ such that for all $x, y \in X$, $d_{R\omega}(x, y) > 0$ implies

$\mathcal{F}(\omega(x, y)^r d_{R\omega}(\mathcal{T}x, \mathcal{T}y)) \leq \mathcal{F}(\mathcal{M}_\omega(x, y)) + \mathcal{G}(\beta(\mathcal{M}_\omega(x, y)))$, where $r \geq 2$ and

$$\mathcal{M}_\omega(x, y)) = \max\left\{ d_{R\omega}(x, y), d_{R\omega}(x, \mathcal{T}x), d_{R\omega}(y, \mathcal{T}y), \frac{d_{R\omega}(y, \mathcal{T}y)[1 + d_{R\omega}(y, \mathcal{T}y)]}{\omega(x, y)[1 + d_{R\omega}(x, y)]} \right\}$$

Definition 3.6. [1] Let X be a non-empty set endowed with extended Branciari b-distance $d_{R\omega}$

1. A sequence $\{x_n\}$ in X converges to x if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_{R\omega}(x_n, x) < \epsilon$ for all $n \geq N$. For this particular case, we write $\lim_{n\to\infty} x_n = x$.
2. A sequence $\{x_n\}$ in X is called Cauchy if for every $\epsilon > 0$ $\exists$ $N = N(\epsilon) \in \mathbb{N}$ such that $d_{R\omega}(x_n, x_m) < \epsilon$ for all $m, n \geq N$.
3. A $d_{R\omega}$ - metric space $(X, d_{R\omega})$ is complete if every Cauchy sequence in X is convergent.

Lemma 3.7 ([11] Lemma 7). Let X be a nonempty set , $T : X \to X$ be a triangular α-admissible mapping and $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n$ by $x_{n+1} = Tx_n\}$ for all $n \in \mathbb{N}$. Then $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.

The existence and uniqueness of fixed points for generalized $FG_{R\omega}$ - contraction in complete extended Branciari b - distance spaces are proved by the following theorem.

Theorem 3.8. Let $(X, d_{R\omega})$ be a complete extended Branciari b-metric space,

$T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be given mappings. $F \in \Delta_F$ and $(G, \beta) \in \Delta_{G,\beta}$ such that

1. T is a triangular α - admissible mapping.
2. T is a generalized $FG_{R\omega}$ - contraction.
3. There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$
4. T is α - continuous.

Then

1. T has a fixed point $x^* \in X$ and $\lim_{n\to\infty} T^n x_0 = x^*$
2. If $\alpha(x, y) \geq 1$ for all $x, y \in Fix(T)$, T has a unique fixed point, where $Fix(T) = \{x \in X | Tx = x\}$

Proof . Define a sequence $x_n \in X$ by $x_n = T^n(x_0) = T(x_{n-1})$. As T is a triangular α - admissible mapping and there $\exists\ x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.

By using Lemma 1 we conclude that for all $m, n \in \mathbb{N}$ with $n < m$

$$\alpha(x_n, x_m) \geq 1 \tag{3.1}$$

This implies that

$$\alpha(x_n, x_{n+1}) \geq 1 \tag{3.2}$$

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_{0+1}}$, then x_{n_0} is fixed point of T and $\lim_{n\to\infty} T^n x_{n_0} = x_{n_0}$. Therefore assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$
so $d_{R\omega}(x_n, Tx_n) = d_{R\omega}(Tx_{n-1}, Tx_n) > 0$ for all $n \in \mathbb{N}$.

As T is a generalized $FG_{R\omega}$ - contraction, so we have

$$\begin{aligned} F(d_{R\omega}(x_n, x_{n+1})) &= F(d_{R\omega}(Tx_{n-1}, Tx_n)) \\ &\leq \mathcal{F}(\omega(x_{n-1}, x_n)^r d_{R\omega}(Tx_{n-1}, Tx_n)) \\ &\leq \mathcal{F}(\mathcal{M}_\omega(x_{n-1}, x_n)) + \mathcal{G}(\beta(\mathcal{M}_\omega(x_{n-1}, x_n))) \end{aligned}$$

where $\mathcal{M}_\omega(x_{n-1}, x_n) = \max\Bigg\{ d_{R\omega}(x_{n-1}, x_n), d_{R\omega}(x_{n-1}, Tx_{n-1}), d_{R\omega}(x_n, Tx_n), \dfrac{d_{R\omega}(x_n, Tx_n)[1 + d_{R\omega}(x_{n-1}, Tx_{n-1})]}{\omega(x_{n-1}, x_n)[1 + d_{R\omega}(x_{n-1}, x_n)]} \Bigg\}$

$$= \max\left\{ d_{R\omega}(x_{n-1}, x_n), d_{R\omega}(x_{n-1}, x_n), d_{R\omega}(x_n, x_{n+1}), \frac{d_{R\omega}(x_n, x_{n+1})[1 + d_{R\omega}(x_{n-1}, x_n)]}{\omega(x_{n-1}, x_n)[1 + d_{R\omega}(x_{n-1}, x_n)]} \right\}$$

$$= \max\left\{ d_{R\omega}(x_{n-1}, x_n), d_{R\omega}(x_n, x_{n+1}), \frac{d_{R\omega}(x_n, x_{n+1})}{\omega(x_{n-1}, x_n)} \right\}$$

$$= \max\left\{ d_{R\omega}(x_{n-1}, x_n), d_{R\omega}(x_n, x_{n+1}) \right\}$$

If $\mathcal{M}_\omega(x_{n-1}, x_n) = d_{R\omega}(x_n, x_{n+1})$, for some $n \geq 1$ then

$F(d_{R\omega}(x_n, x_{n+1})) \leq \mathcal{F}(d_{R\omega}(x_n, x_{n+1})) + \mathcal{G}(\beta(d_{R\omega}(x_n, x_{n+1})))$ which implies that

$\mathcal{G}(\beta(d_{R\omega}(x_n, x_{n+1}))) \geq 0$ which in turn implies $\beta(d_{R\omega}(x_n, x_{n+1})) \geq 0$.

This is a contradiction to the condition of $(F\Delta_3)$, therefore for all $n \geq 1$ we get

$$d_{R\omega}(x_n, x_{n+1}) \leq d_{R\omega}(x_{n-1}, x_n)$$

Hence we get $F(d_{R\omega}(x_n, x_{n+1})) \leq \mathcal{F}(d_{R\omega}(x_{n-1}, x_n)) + \mathcal{G}(\beta(d_{R\omega}(x_{n-1}, x_n)))$

Using the condition of $(F\Delta_1)$, we get

$$\begin{aligned} F(d_{R\omega}(x_n, x_{n+1})) &\leq \mathcal{F}(d_{R\omega}(x_{n-1}, x_n)) + \mathcal{G}(\beta(d_{R\omega}(x_{n-1}, x_n))) \\ &\leq \mathcal{F}(d_{R\omega}(x_{n-2}, x_{n-1})) + \mathcal{G}(\beta(d_{R\omega}(x_{n-2}, x_{n-1}))) + \mathcal{G}(\beta(d_{R\omega}(x_{n-1}, x_n))) \end{aligned}$$

$$\leq \dots$$

$$\leq \mathcal{F}(d_{R\omega}(x_0, x_1)) + \sum_{i=1}^{n} \mathcal{G}(\beta(d_{R\omega}(x_{i-1}, x_i)))$$

Letting the limit $n \to \infty$ in the above inequality and using the condition $(F\Delta_5)$ we get

$$\lim_{n\to\infty} F(d_{R\omega}(x_n, x_{n+1})) = -\infty.$$

Combining this with the condition $(F\Delta_2)$ we have

$$\lim_{n\to\infty} d_{R\omega}(x_n, x_{n+1}) = 0 \tag{3.3}$$

We will show that the sequence $\{x_n\}$ is a Cauchy sequence in $(X, d_{R\omega})$. On the contrary , we will assume that there exists $\epsilon > 0$ and two sub sequences $\{x_{n_t}\}$ and $\{x_{m_t}\}$ of $\{x_n\}$ such that n_t is the smallest index for which

$$n_t > m_t > t \geq 1, \quad d_{R\omega}(x_{m_t}, x_{n_t}) \geq \epsilon \tag{3.4}$$

This implies $d_{R\omega}(x_{m_t}, x_{n_t-2}) < \epsilon$

Taking the upper limit as $t \to \infty$, we get

$$\lim_{t\to\infty} \sup d_{R\omega}(x_{m_t}, x_{n_t-2}) < \epsilon \tag{3.5}$$

$$\epsilon \leq d_{R\omega}(x_{m_t}, x_{n_t}) \leq \omega(x_{m_t}, x_{n_t}) \Big[d_{R\omega}(x_{n_t}, x_{n_t-2}) + d_{R\omega}(x_{n_t-2}, x_{n_t-1}) + d_{R\omega}(x_{n_t-1}, x_{n_t}) \Big]$$

Using (3.4), we get

$$\epsilon \leq d_{R\omega}(x_{m_t}, x_{n_t}) \leq \omega(x_{m_t}, x_{n_t}) \Big[d_{R\omega}(x_{m_t}, x_{n_t-2}) \Big]$$

Taking the upper limit as $t \to \infty$ in the above equation and using(3.3) we get

$$\epsilon \leq \lim_{t\to\infty} \sup d_{R\omega}(x_{m_t}, x_{n_t}) \leq \epsilon \lim_{t\to\infty} \sup \omega(x_{m_t}, x_{n_t}) \tag{3.6}$$

From (3.4), we have

$$\epsilon \leq d_{R\omega}(x_{m_t}, x_{n_t}) \leq \omega(x_{m_t}, x_{n_t}) \Big[d_{R\omega}(x_{m_t}, x_{m_t+1}) + d_{R\omega}(x_{m_t+1}, x_{n_t+1}) + d_{R\omega}(x_{n_t+1}, x_{n_t}) \Big]$$

Taking the upper limit as $t \to \infty$ in the above equation and using(3.3) we get

$$\epsilon \leq \lim_{t\to\infty} \sup d_{R\omega}(x_{m_t}, x_{n_t}) \leq \lim_{t\to\infty} \sup \omega(x_{m_t}, x_{n_t}) \Big[\lim_{t\to\infty} \sup d_{R\omega}(x_{m_t+1}, x_{n_t+1}) \Big]$$

Therefore we get

$$\frac{\epsilon}{\lim_{t\to\infty} \sup \omega(x_{m_t}, x_{n_t})} \leq \lim_{t\to\infty} \sup d_{R\omega}(x_{m_t+1}, x_{n_t+1}) \tag{3.7}$$

Furthermore, we get

$$d_{R\omega}(x_{m_t}, x_{n_t}) \leq \omega((x_{m_t}, x_{n_t})) \Big[d_{R\omega}(x_{m_t}, x_{n_t-2}) + d_{R\omega}(x_{n_t-2}, x_{n_t-1}) + d_{R\omega}(x_{n_t-1}, x_{n_t}) \Big]$$

$$d_{R\omega}(x_{m_t+1}, x_{n_t-1}) \leq \omega((x_{m_t+1}, x_{m_t-1})) \Big[d_{R\omega}(x_{m_t+1}, x_{m_t}) + d_{R\omega}(x_{m_t}, x_{n_t-2}) + d_{R\omega}(x_{n_t-2}, x_{n_t-1}) \Big]$$

$$d_{R\omega}(x_{m_t+2}, x_{n_t-1}) \leq \omega((x_{m_t+2}, x_{n_t-1}))\Big[d_{R\omega}(x_{m_t+2}, x_{m_t-2}) + d_{R\omega}(x_{n_t-2}, x_{n_t-1}) + d_{R\omega}(x_{n_t-2}, x_{n_t})\Big]$$

$$d_{R\omega}(x_{m_t+2}, x_{n_t}) \leq \omega((x_{m_t+2}, x_{n_t}))\Big[d_{R\omega}(x_{m_t+2}, x_{n_t-2}) + d_{R\omega}(x_{n_t-2}, x_{n_t-1}) + d_{R\omega}(x_{n_t-1}, x_{n_t})\Big]$$

$$d_{R\omega}(x_{m_t+2}, x_{m_t}) \leq \omega((x_{m_t+2}, x_{m_t}))\Big[d_{R\omega}(x_{m_t+2}, x_{m_t+1}) + d_{R\omega}(x_{m_t+1}, x_{n_t+1}) + d_{R\omega}(x_{m_t+1}, x_{m_t})\Big]$$

Taking the upper limit as $t \to \infty$ in the above inequalities and using (3.3) and (3.4), we get

$$\lim_{t\to\infty} \sup\, d_{R\omega}(x_{m_t}, x_{n_t}) \leq \epsilon \lim_{t\to\infty} \sup\, \omega(x_{m_t}, x_{n_t})$$

$$\lim_{t\to\infty} \sup\, d_{R\omega}(x_{m_t+1}, x_{n_t-1}) \leq \epsilon \lim_{t\to\infty} \sup\, \omega(x_{m_t+1}, x_{n_t-1})$$

$$\lim_{t\to\infty} \sup\, d_{R\omega}(x_{m_t+2}, x_{n_t-1}) \leq \epsilon \lim_{t\to\infty} \sup\, \omega(x_{m_t+2}, x_{n_t-1})$$

$$\lim_{t\to\infty} \sup\, d_{R\omega}(x_{m_t+2}, x_{n_t}) \leq \epsilon \lim_{t\to\infty} \sup\, \omega(x_{m_t+2}, x_{n_t})$$

$$\lim_{t\to\infty} \sup\, d_{R\omega}(x_{m_t}, x_{n_t}) = 0$$

Again using

$$\epsilon \leq d_{R\omega}(x_{m_t}, x_{n_t}) \leq \omega(x_{m_t}, x_{n_t})\Big[d_{R\omega}(x_{m_t}, x_{m_t+1}) + d_{R\omega}(x_{m_t+1}, x_{n_t+1}) + d_{R\omega}(x_{n_t+1}, x_{n_t})\Big]$$

Taking the lower limit as $t \to \infty$ in the above equation and using(3.3) we get

$$\epsilon \leq \lim_{t\to\infty} \inf\, d_{R\omega}(x_{m_t}, x_{n_t}) \leq \lim_{t\to\infty} \inf\, \omega(x_{m_t}, x_{n_t})\Big[\lim_{t\to\infty} \inf\, d_{R\omega}(x_{m_t+1}, x_{n_t+1})\Big]$$

Therefore we get

$$\frac{\epsilon}{\lim_{t\to\infty} \inf\, \omega(x_{m_t}, x_{n_t})} \leq \lim_{t\to\infty} \inf\, d_{R\omega}(x_{m_t+1}, x_{n_t+1})$$

Therefore there exists $t_0 \in \mathbb{N}$ such that $d_{R\omega}(x_{m_t+1}, x_{n_t+1}) > 0m$

Consider $\mathcal{M}_\omega(x_{n_t}, x_{m_t}) = \max\Big\{d_{R\omega}(x_{n_t}, x_{m_t}), d_{R\omega}(x_{n_t}, Tx_{n_t}), d_{R\omega}(x_{m_t}, Tx_{m_t}), \frac{d_{R\omega}(x_{m_t}, Tx_{m_t})[1 + d_{R\omega}(x_{n_t}, Tx_{n_t})]}{\omega(x_{n_t}, x_{m_t})[1 + d_{R\omega}(x_{n_t}, x_{m_t})]}\Big\}$

$$\mathcal{M}_\omega(x_{n_t}, x_{m_t}) = \max\Big\{d_{R\omega}(x_{n_t}, x_{m_t}), d_{R\omega}(x_{n_t}, x_{n_t+1}), d_{R\omega}(x_{m_t}, x_{m_t+1}), \frac{d_{R\omega}(x_{m_t}, x_{m_t+1})[1 + d_{R\omega}(x_{n_t}, x_{n_t+1})]}{\omega(x_{n_t}, x_{m_t})[1 + d_{R\omega}(x_{n_t}, x_{m_t})]}\Big\}.$$

Therefore, $\mathcal{M}_\omega(x_{n_t}, x_{m_t}) = d_{R\omega}(x_{n_t}, x_{m_t})$ from [3.3].

Taking the upper limit as $t \to \infty$ in the above equation and using equation (3.6), we get

$$\lim_{t\to\infty} \sup \mathcal{M}_\omega(x_{n_t}, x_{m_t}) = \lim_{t\to\infty} \sup d_{R\omega}(x_{m_t}, x_{n_t}) < \epsilon \lim_{t\to\infty} \sup \omega(x_{m_t}, x_{n_t}). \tag{3.8}$$

Consider $\mathcal{F}\left(\omega(x_{n_t}, x_{m_t})\frac{\epsilon}{\omega(x_{n_t}, x_{m_t})}\right) \le \mathcal{F}\left(\omega(x_{n_t}, x_{m_t})^r\frac{\epsilon}{\omega(x_{n_t}, x_{m_t})}\right)$. Taking the upper limit as $t \to \infty$ in the above equation and using (3.7), we get

$$\mathcal{F}\left(\lim_{t\to\infty} \sup\omega(x_{n_t}, x_{m_t})^r \frac{\epsilon}{\lim_{t\to\infty} \sup \omega(x_{n_t}, x_{m_t})}\right)$$
$$\le \mathcal{F}\left(\lim_{t\to\infty} \sup\omega(x_{n_t}, x_{m_t})^r \lim_{t\to\infty} \sup d_{R\omega}(x_{m_t+1}, x_{n_t+1})\right)$$
$$\le \mathcal{F}(\lim_{t\to\infty} \sup\mathcal{M}_\omega(x_{n_t}, x_{m_t})) + \mathcal{G}(\beta(\lim_{t\to\infty} \sup \mathcal{M}_\omega(x_{n_t}, x_{m_t})))$$
$$\le \mathcal{F}(\epsilon \lim_{t\to\infty} \sup\omega(x_{n_t}, x_{m_t})) + \lim_{t\to\infty} \sup \mathcal{G}(\beta(\mathcal{M}_\omega(x_{n_t}, x_{m_t}))).$$

Therefore

$$\lim_{t\to\infty} \sup\mathcal{G}(\beta(\mathcal{M}_\omega(x_{n_t}, x_{m_t}))) \ge 0.$$

This implies $\lim_{t\to\infty} \sup(\beta(\mathcal{M}_\omega(x_{n_t}, x_{m_t}))) \ge 1$. As $\beta(\xi) < 1$ for all $\xi \ge 0$, we get

$$\lim_{t\to\infty} \sup(\beta(\mathcal{M}_\omega(x_{n_t}, x_{m_t}))) = 1.$$

Using the property of β, we get $\lim_{t\to\infty} \sup \mathcal{M}_\omega(x_{n_t}, x_{m_t}) = 0$, which is a contradiction to (3.8). Hence $\{x_n\}$ is a Cauchy sequence in $(X, d_{R\omega})$

As $(X, d_{R\omega})$ is a complete extended Branciari b-distance space, there exists $x^* \in X$ such that $\lim_{n\to\infty} d_{R\omega}(x_n, x^*) = 0$.

By definition of sequence $\{x_n\}$, we get $\lim_{n\to\infty} T^n(x_0) = x^*$. By using (3.2) and α continuous property of T, we get

$$\lim_{n\to\infty} T(x_n) = T(x^*) \Longrightarrow x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n) \Longrightarrow x^*$$

is a fixed point of T.

To prove uniqueness of fixed point of T, let us assume there are two fixed points x, y such that $x \neq y$ then $T(x) \neq T(y)$. As $\alpha(x, y) \ge 1$ and

$$\mathcal{F}(\omega(x, y)^r d_{R\omega}(Tx, Ty)) \le \mathcal{F}(\mathcal{M}_\omega(x, y)) + \mathcal{G}(\beta(\mathcal{M}_\omega(x, y)))$$

where $r \ge 2$ and

$$\mathcal{M}_\omega(x, y) = \max\left\{d_{R\omega}(x, y), d_{R\omega}(x, Tx), d_{R\omega}(y, Ty), \frac{d_{R\omega}(y, Ty)[1 + d_{R\omega}(x, Tx)]}{\omega(x, y)[1 + d_{R\omega}(x, y)]}\right\}.$$

Therefore,

$$\mathcal{M}_\omega(x, y) = \max\left\{d_{R\omega}(x, y), d_{R\omega}(x, x), d_{R\omega}(y, y), \frac{d_{R\omega}(y, y)[1 + d_{R\omega}(x, x)]}{\omega(x, y)[1 + d_{R\omega}(x, y)]}\right\}.$$

Hence, $\mathcal{M}_\omega(x, y) = d_{R\omega}(x, y)$. This implies $\mathcal{F}(\omega(x, y)^r d_{R\omega}(Tx, Ty)) \le \mathcal{F}(d_{R\omega}(x, y)) + \mathcal{G}(\beta(d_{R\omega}(x, y)))$. Using the increasing property of $\mathcal{F}$, we get

$$\mathcal{G}(\beta(d_{R\omega}(x, y))) \ge 0 \Longrightarrow \beta(d_{R\omega}(x, y)) \ge 1.$$

But this is a contradiction to $\beta(\xi) < 1$, for all $\xi \ge 0$. Therefore $d_{R\omega}(x, y) = 0 \Longrightarrow x = y$. This means T has a unique fixed point. □

Theorem 3.9. Let $(X, d_{R\omega})$ be a complete extended Branciari b-metric space,
$T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be two mappings. $F \in \Delta_F$ and $(G, \beta) \in \Delta_{G,\beta}$ such that

1. T is a triangular α - admissible mapping.
2. T is a generalized $FG_{R\omega}$ - contraction.
3. There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$
4. If $\{x_n\}$ is a sequence in X and $\lim\limits_{t\to\infty} x_n = x$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Then

1. T has a fixed point $x^* \in X$ and $\lim\limits_{n\to\infty} T^n x_0 = x^*$
2. If $\alpha(x, y) \geq 1$ for all $x, y \in Fix(T)$, T has a unique fixed point, where $Fix\ (T) = \{x \in X | Tx = x\}$

Proof . We conclude, as we did in the proof of Theorem 3.8, that the sequence is $\{x_n\}$ is defined by $x_n = T^n x_0 = Tx_{n-1}$ satisfying

$$\alpha(x_n, x_m) \geq 1 \tag{3.9}$$

$$\lim_{n\to\infty} d_{R\omega}(x_n, x_{n+1}) = 0 \tag{3.10}$$

for all $n, m \in \mathbb{N}$ with $n > m$ and there exists $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^* \Longrightarrow \lim_{n\to\infty} T^n x_0 = x^* \tag{3.11}$$

We will show that x^* is a fixed point of T.

Assume there exists $n_0 \in \mathbb{N}$ such that $x_{n+1} \neq Tx^*$ for all $n \geq n_0$.

This implies $d_{R\omega}(Tx_n, Tx^*) > 0$ for all $n \geq n_0$.

Using (3.9) and (3.11), we get $\alpha(x_n, x^*) \geq 1$

For all $n \geq n_0$, we get

$$\mathcal{F}(\omega(x_n, x^*)^r d_{R\omega}(Tx_n, Tx^*)) \leq \mathcal{F}(\mathcal{M}_{R\omega}(x_n, x^*)) + \mathcal{G}(\beta(\mathcal{M}_{R\omega}(x_n, x^*))) \tag{3.12}$$

where $\mathcal{M}_{R\omega}(x_n, x^*) = \max\left\{d_{R\omega}(x_n, x^*), d_{R\omega}(x_n, Tx_n), d_{R\omega}(x^*, Tx^*), \dfrac{d_{R\omega}(x^*, Tx^*)[1 + d_{R\omega}(x_n, x_n)]}{\omega(x_n, x^*)[1 + d_{R\omega}(x_n, x^*)]}\right\}$.

Therefore

$$\mathcal{M}_{R\omega}(x_n, x^*) = \max\left\{d_{R\omega}(x_n, x^*), d_{R\omega}(x_n, x_{n+1}), d_{R\omega}(x^*, Tx^*), \frac{d_{R\omega}(x^*, Tx^*)[1 + d_{R\omega}(x_n, x_{n+1})]}{\omega(x_n, x^*)[1 + d_{R\omega}(x_n, x^*)]}\right\}.$$

Taking the upper limits as $n \to \infty$ and (3.10), we get

$$\lim_{n\to\infty} \sup\ \mathcal{M}_{R\omega}(x_n, x^*) = d_{R\omega}(x^*, Tx^*). \tag{3.13}$$

Taking upper limits in (3.12) as $n \to \infty$ and using (3.13), we get

$\lim\limits_{n\to\infty} \sup\ \mathcal{F}(\omega(x_n, x^*)^r d_{R\omega}(Tx_n, Tx^*)) \leq \mathcal{F}(\lim\limits_{n\to\infty} \sup\ \mathcal{M}_{R\omega}(x_n, x^*)) + \lim\limits_{n\to\infty} \sup\ \mathcal{G}(\beta(\mathcal{M}_{R\omega}(x_n, x^*)))$

Using the increasing property of $\mathcal{F}$, we get

$\lim\limits_{n\to\infty} \sup\ \mathcal{G}(\beta(\mathcal{M}_{R\omega}(x_n, x^*))) \geq 0 \Longrightarrow \lim\limits_{n\to\infty} \sup\ \beta(\mathcal{M}_{R\omega}(x_n, x^*)) \geq 1$

As $\beta(\xi) < 1$ for all $\xi \geq 0$, we get,

$\lim\limits_{n\to\infty} \sup\ \beta(\mathcal{M}_{R\omega}(x_n, x^*)) = 1 \Longrightarrow \lim\limits_{n\to\infty} \sup\ \mathcal{M}_{R\omega}(x_n, x^*) = 0$

Using (3.13), we get $d_{R\omega}(x^*, Tx^*) = 0$. But this is a contradiction to $d_{R\omega}(x^*, Tx^*) > 0$. Therefore, x^* is a fixed point of T. For the second part of the proof proceed as in theorem 3.8. □

From the Theorem 3.8 and Theorem 3.9, we get the following corollary.

Corollary 3.10. Let $(X, d_{R\omega})$ be a complete extended Branciari b-metric space, $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be two mappings. $F \in \Delta_F$ and $(G, \beta) \in \Delta_{G,\beta}$ such that

1. T is a triangular α - admissible mapping.
2. For all $x, y \in X$ with $\omega(x, y)d_{R\omega}(Tx, Ty) \leq \gamma\mathcal{A}(x, y)$
3. There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$
4. (a) Either T is $\alpha-$ continuous or
   (b) If $\{x_n\}$ is a sequence in X and $\lim_{t\to\infty} x_n = x$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Then

1. T has a fixed point $x^* \in X$ and $\lim_{n\to\infty} T^n x_0 = x^*$
2. If $\alpha(x, y) \geq 1$ for all $x, y \in Fix(T)$, T has a unique fixed point, where $Fix(T) = \{x \in X | Tx = x\}$

4 Application to nonlinear integral equations

Let $\mathcal{C}([0, 1])$ be the set of all continuous function on $I = [0, 1]$. Let $X = \mathcal{C}(I, \mathbb{R})$ be endowed with the Extended Branciari b-metric space function defined by

$$d_{R\omega}(x, y) = \sup_{t\in I}|x(t) - y(t)|^2$$

for all $x, y \in X$ and $\omega(x, y) = |x| + |y| + 3$, where $\omega : X \times X \to [1, \infty)$

Consider the nonlinear integral equation

$$x(t) = g(t) + \lambda \int_0^1 \mathcal{L}(t, s)f(s, x(s))ds \tag{4.1}$$

where $f : I \times \mathbb{R} \to \mathbb{R}$, $\lambda \geq 0$ and $\mathcal{L} : I \times I \to [0, \infty)$ are given functions.

Suppose that the following conditions hold:

1. $g : I \to \mathbb{R}$ is a continuous function.
2. $\mathcal{L} : I \times I \to [0, \infty)$ is integrable on $[0, 1]$
3. $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that for all $x, y \in \mathcal{C}[0, 1]$.
   $\int_0^1 \left|f(s, x(s)) - f(s, y(s))\right|^2 ds \leq \frac{\rho\Theta(x(t), y(t))}{\max_{t\in[0,1]}\left(|x| + |y| + 3\right)^2}$, where
   $$\Theta(x(t), y(t)) = \max\left\{|x(t) - y(t)|^2, |x(t) - Tx(t)|^2, |y(t) - Ty(t)|^2, \frac{|y(t) - Ty(t)|^2[1 + |x(t) - Tx(t)|^2]}{\max_{t\in[0,1]}(|x| + |y| + 3)^2[1 + |x(t) - y(t)|^2]}\right\}$$
4. $Tx \in \mathcal{C}[0, 1]$ for all $x \in \mathcal{C}[0, 1]$ where $Tx(t) = g(t) + \lambda \int_0^1 \mathcal{L}(t, s)f(s, x(s))ds$
5. For all $x \in \mathcal{C}[0, 1]$ and $x(t) \geq 0$ for all $t \in [0, 1]$, we have $Tx(t) \geq 0$ for all $t \in [0, 1]$
6. Assume $\lambda^2\, \mathbf{L}^2 \leq 1$

Under the above conditions $(1) - (6)$, the nonlinear integral equation (4.1) has a unique solution in $\mathcal{C}[0, 1]$

Proof . Define a function $T : \mathcal{C}[0, 1] \to \mathcal{C}[0, 1]$ by

$$Tx(t) = g(t) + \lambda \int_0^1 \mathcal{L}(t, s)f(s, x(s))ds \text{ for all } x \in \mathcal{C}[0, 1], t \in [0, 1]$$

The existence of a solution to (4.1) is equivalent to the existence of a fixed point of T.

Define a mapping $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 2 & \text{if } x(t), y(t) \in [0,\infty) \text{ for all } t \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

We will prove that T is a triangular α-admissible mapping. Let $x, y \in \mathcal{C}[0,1]$ such that $\alpha(x,y) \geq 1$. Therefore, $x(t) \geq 0, y(t) \geq 0$ for all $t \in [0,1]$.

From condition (4) it follows that $Tx(t) \geq 0, Ty(t) \geq 0$ for all $t \in [0,1]$ this implies $\alpha(Tx, Ty) \geq 1$. Similarly, for $x, y, z \in \mathcal{C}[0,1]$ such that $\alpha(x,z) \geq 1$ and $\alpha(z,y) \geq 1$, we have $x(t) \geq 0, y(t), z(t) \geq 0$ for all $t \in [0,1]$. This implies that $\alpha(x,y) \geq 1$. Hence, T is a triangular α-admissible mapping. Now, for $x, y \in X$ we have

$$\begin{aligned}
|T(x(t) - T(y(t)|^2 &= \left| g(t) + \lambda \int_0^1 \mathcal{L}(t,s) f(s,x(s)) ds - g(t) - \lambda \int_0^1 \mathcal{L}(t,s) f(s,y(s)) ds \right|^2 \\
&\leq \lambda^2 \Big(\int_0^1 \mathcal{L}(t,s) \big| f(s,x(s)) - f(s,y(s)) \big| ds \Big)^2 \\
&\leq \lambda^2 \Big(\int_0^1 \mathcal{L}(t,s) ds \Big)^2 \Big(\int_0^1 \big| f(s,x(s)) - f(s,y(s)) \big|^2 ds \Big) \\
&\leq \lambda^2 \Big(\sup_{t \in I} \int_0^1 \mathcal{L}(t,s) ds \Big)^2 \frac{\rho \Theta(x(t), y(t))}{\max_{t \in [0,1]} \big(|x| + |y| + 3 \big)^2} \\
&\leq \lambda^2 \mathbf{L}^2 \frac{\rho \Theta(x(t), y(t))}{\max_{t \in [0,1]} \big(|x| + |y| + 3 \big)^2} \\
&\leq \frac{\rho\ \Theta(x(t), y(t))}{(|x| + |y| + 3)^2}.
\end{aligned}$$

Therefore

$$\sup_{t \in [0,1]} |T(x(t) - T(y(t)|^2 \leq \sup_{t \in [0,1]} \Big(\frac{\rho Theta(x(t), y(t))}{(|x| + |y| + 3)^2} \Big) = \frac{\rho\ Theta(x,y)}{(|x| + |y| + 3)^2}.$$

By the above, we conclude that all the assumptions in corollary (3.10) are satisfied. Thus, T has a fixed point $x \in C[0,1]$ and hence equation (4.1) has a solution $x \in C[0,1]$. □

5 Conclusion

We proved the existence and uniqueness of a fixed point in extended Branciari b-metric space in this manuscript, which generalises many previous results. We also presented an application of our integral equations results.

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