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A novel numerical technique and stability criterion of VF type integro-differential equations of non-integer order

Dipankar Saha[a], Nimai Sarkar[a,*], Mausumi Sen[a], Subhankar Saha[b]

[a]Department of Mathematics, National Institute of Technology Silchar, Assam, India

[b]Department of Mechanical Engineering, National Institute of Technology Silchar, Assam, India

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Abstract

In this article, Ulam Hyers stability of Volterra Fredholm (VF) type fractional integro-differential equation is studied by the fixed point notion in the generalized metric space. In addition, the efficiency of the Laplace decomposition method in the context of solving some integral equations of the Volterra Fredholm type is shown. Further convergence analysis of the numerical scheme is shown.

Keywords: Fixed point, Ulam Hyers stability, Metric space
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1 Introduction

Integral and integro-differential equations have been used through decades to model variety of physical problems. Specifically, fractional integral and integro-differential equations often arise to model problems related to mathematical physics, such as heat conduction with some hereditary properties, viscoelasticity etc. As a sequel the study of existence of solution of such equation is of utmost significance. Some recent works related to the existence study can be found in [6, 9, 10, 13, 11, 15]. Also, the essence of developing various numerical schemes [2, 3, 7, 8, 14, 12] for solving such equations with accuracy is of high interest till date. Mittal and Nigam [8] implemented the ADM to deal with integro-differential equations of fractional order. Ma and Huang [7] adopted hybridization method to solve concerned class of equations. Hamoud and Ghadle [3] implemented homotopy analysis method to approximate solution of the fractional VF integro-differential equation. Hamoud and Ghadle [2] proposed analytic technique to approximate solution of fractional integro-differential equation. They further [5] proposed modified adomian decomposition method to solve such equations. A part from this many authors deal with the stability of such equations. Yunus Atalan and Vatan Karakaya [1] studied Hyers-Ulam and Hyers-Ulam-Rassias for the nonlinear Volterra Fredholm integro-differential equation (VFIDE) implementing the fixed point concept. But less number of works have been noticed regarding the stability of integro-differential equation of fractional order. Motivated by Yunus and Vatan work, the main aim of this study is to find out the sufficient conditions for attaining the Hyers Ulam stability of fractional Volterra Fredholm

*Corresponding author
Email addresses: nayan0507@gmail.com (Dipankar Saha), nimaisarkar298@gmail.com (Nimai Sarkar), senmausumi@gmail.com (Mausumi Sen), sahamech90@gmail.com (Subhankar Saha)

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integro-differential equation of the following form:

$$D^{\alpha}v(\tau) = g(\tau) + \int_0^{\tau} H_1(\tau,\sigma)G_1(\sigma, v(\sigma))d\sigma + \int_0^{b} H_2(\tau,\sigma)G_2(\sigma, v(\sigma))d\sigma, 0 < \alpha < 1 \quad (1.1)$$

with the initial condition

$$v^{(j)}(0) = \Delta_j \quad (1.2)$$

for $j = 0, 1, 2, 3, ...n-1$. Here $g(t)$ is the known function, H_1 and H_2 are kernel functions. G_1 and G_2 are unknown functions.

2 Preliminaries

In this section the notion of generalized metric space and the fixed point theorem related to this metric has been discussed.

Definition 2.1. A function $\tilde{d} : Y \times Y \to [0, \infty]$ is called a generalized metric on Y if and only f $\tilde{d}$ satisfies
$(A1)$ $\tilde{d}(\tilde{p}, \tilde{q}) = 0$ if and only if $\tilde{p} = \tilde{q}$;
$(A2)$ $\tilde{d}(\tilde{p}, \tilde{q}) = \tilde{d}(\tilde{q}, \tilde{p})$ for all $\tilde{p}, \tilde{q} \in Y$;
$(A3)$ $\tilde{d}(\tilde{p}, \tilde{r}) \leq \tilde{d}(\tilde{p}, \tilde{q}) + \tilde{d}(\tilde{q}, \tilde{r})$ $\forall$ $\tilde{p}, \tilde{q}, \tilde{r} \in Y$.

Theorem 2.2. [1] Let $(Y, \tilde{d})$ be a generalized complete metric space. Also let $\varphi : Y \to Y$ be a strictly contractive operator with the Lipschitz constant $\lambda < 1$. If there is a nonnegative integer p such that $\tilde{d}(\varphi^{p+1}v, \varphi^{p}v) < \infty$ for some $v \in Y$, then

(i) the sequence $\{\varphi^n v\}$ converges to a unique fixed point v_0 of φ in $V = \{\tilde{v} \in Y : \tilde{d}(\varphi^p v, \tilde{v}) < \infty\}$

(ii) If $v \in Y$, then $\tilde{d}(v, v_0) \leq \frac{1}{1-\lambda}\tilde{d}(\varphi v, v)$

Definition 2.3. The Riemann Liouville integral operator of order γ is defined as
$I^{\gamma}p(\tau) = \frac{1}{\Gamma(\gamma)} \int_0^{\tau} \frac{p(\sigma)}{(\tau-\sigma)^{1-\gamma}} d\sigma, \gamma > 0$ where $\Gamma(\cdot)$ symbolizes the Gamma function provided the integral exists.

Definition 2.4. The Caputo fractional derivative of $p(t)$ of order γ is defined as
$D^{\gamma}p(\tau) = \frac{1}{\Gamma(n-\gamma)} \int_0^{\tau} (\tau - \sigma)^{n-\gamma-1} p^n(\sigma) d\sigma,$

for $n - 1 \leq \gamma \leq n$, $n \in N$, $\tau > 0$.

Lemma 2.5. If $v_0(\tau) \in C(I, R)$, then $v(\tau) \in C(I, R_+)$ is a solution of the problem (1.1)-(1.2) iff v satisfies the following equation

$$\begin{aligned} v(\tau) = \quad & \Sigma_{k=0}^{n-1} v^k(0^+)\frac{\tau^k}{k} + \int_0^{\tau} (\tau - \tau^*)^{q-1} \left(g(\zeta) + \int_0^{\tau^*} H_1(\tau^*, \sigma)G_1(\sigma, v(\sigma))d\sigma \right) d\tau^* + \\ & \int_0^{\tau} (\tau - \tau^*)^{q-1} \left(\int_0^{b} H_2(\tau^*, \sigma)G_2(\sigma, v(\sigma))d\sigma \right) d\tau^* \end{aligned} \quad (2.1)$$

Theorem 2.6. The Laplace transform of the Caputo derivative is defined as

$$\mathcal{L}[D^{\beta}f(\tau)] = s^{\beta}F(\sigma) - \sum_{k=0}^{n-1} \sigma^{\beta-k-1} f^k(0) \quad (2.2)$$

for $n - 1 < \beta \leq n$.

3 Main results

To derive the theoretical findings the following hypothesis have been considered.
(I) $G_1(\tau, v(\tau))$ and $G_2(\tau, v(\tau))$ are Lipschitz functions with Lipschitz constants $L_1(>0)$ and $L_2(>0)$ respectively i.e.,

$$|G_1(\tau, v(\tau)) - G_1(\tau, w(\tau))| \leq L_1|v - w| \tag{3.1}$$

$$|G_2(\tau, v(\tau)) - G_2(\tau, w(\tau))| \leq L_2|v - w| \tag{3.2}$$

and also $G_1(\tau, 0) = G_2(\tau, 0) = 0$, $\forall\ \tau \in I$.
(II) The kernel functions $H_1(\tau, \sigma)$ and $H_2(\tau, \sigma)$ are continuous on $I \times I$, and consequently bounded by $M_1(>0)$ and $M_2(>0)$ in $I \times I$.
(III)There exist two positive continuous functions $M_1^\star$, $M_2^\star$: $I \times I \to R_+$ such that
$M_1^\star = \sup_{\tau\in I} \int_0^\tau |H_1(\tau, \sigma)|d\sigma < \infty$,
$M_2^\star = \sup_{\tau\in I} \int_0^\tau |H_2(\tau, \sigma)|d\sigma < \infty$.

3.1 Hyer Ulam Stability

Definition 3.1. If $\forall\epsilon > 0$ and each continuously differentiable function $v(\tau)$ satisfying

$$|D^\alpha v(\tau) - g(\tau) - \int_0^\tau H_1(\tau, \sigma)G_1(\sigma, v(\sigma))d\sigma - \int_0^b H_2(\tau, \sigma)G_2(\sigma, v(\sigma))d\sigma| \leq \epsilon, \tag{3.3}$$

$\forall\ \tau \in I$, there exists a solution $v_0(\tau)$ of the Volterra Fredholm fractional integro-differential equation and a constant $K > 0$ (independent of $v(\tau)$ and $v_0(\tau)$) with
$|v(\tau) - v_0(\tau)| \leq K\epsilon$, for all $\tau \in I$, then the equation (1.1) is said to be Hyers-Ulam stable on I.
Moreover, if $v^{(k)}(0) = v_0^{(k)}(0)$, $k = 0, 1, 2, ...m - 1$ equation (1.1) is Hyers-Ulam stable with initial conditions.

Remark 1: A function $v \in C'(I, R)$ is a solution of the inequality (3.3) if there exists a function $q_v \in C(I, R)$(which depends on v) such that
(i) $|q_v(\tau)| \leq \epsilon$, $\tau \in I$;
(ii) $D^\alpha v(\tau) = g(\tau) + \int_0^\tau H_1(\tau, \sigma)G_1(\sigma, v(\sigma))d\sigma + \int_0^b H_2(\tau, \sigma)G_2(\sigma, v(\sigma))d\sigma + q_v(\tau)$.

Theorem 3.2. If $v \in C'(I, R)$ satisfies the inequality (3.3), then v satisfies the following inequality: $|v(\tau) - \Sigma_{k=0}^{n-1} v^k(0^+)\frac{\tau^k}{k} - \frac{1}{\Gamma(\beta)} \int_0^\tau (\tau - \tau^*)^{\beta-1} \Big(g(\tau^*) - \int_0^{\tau^*} H_1(\tau^*, \sigma)G_1(\sigma, v(\sigma))d\sigma -$

$$\int_0^b H_2(\tau^*, \sigma)G_2(\sigma, v(\sigma))d\sigma\Big)d\tau^*|$$

$$\leq \epsilon\tau, \tau \in I.$$

Proof . Indeed, if $v \in C'(I, R)$ satisfies the inequality (3.3), then by Remark 1, we have
$D^\beta v(\tau) = g(\tau) + \int_0^\tau H_1(\tau, \sigma)G_1(\sigma, v(\sigma))d\sigma + \int_0^b H_2(\tau, \sigma)G_2(\sigma, v(\sigma))d\sigma + q_v(\tau)$.
Thus yields
$|v(\tau) - \Sigma_{k=0}^{n-1} v^k(0^+)\frac{\tau^k}{k} - \frac{1}{\Gamma(\beta)} \int_0^\tau (\tau - \tau^*)^{\beta-1}(g(\tau^*) - \int_0^{\tau^*} H_1(\tau^*, \sigma)G_1(\sigma, v(\sigma))d\sigma$
$- \int_0^b H_2(\tau^*, \sigma)G_2(\sigma, v(\sigma))d\sigma)d\tau^*|$
$\leq \int_0^\tau |q_v(\sigma)|d\sigma$
$\leq \epsilon\tau$. $\square$

Theorem 3.3. Let $I = [0, b]$ be a non-degenerated interval, L_1, L_2, M_1 and M_2 be nonnegative constants such that

$$\frac{M_1L_1 + (\beta+1)M_2L_2}{\Gamma(\beta+2)} b^{\beta+1} < 1 \tag{3.4}$$

Suppose g is continuous function and $F_1, F_2 : I \times R \to R$ are continuous functions which satisfy the Lipchitz conditions with respect to the second argument and also the kernels k_1, k_2 are bounded by M_1 and M_2 respectively . If for $\epsilon \geq 0$ and $\forall \tau \in I$, a continuously differentiable function $v : I \to R$ satisfy

$$|D^\beta v(\tau) - g(\tau) - \int_0^\tau H_1(\tau,\sigma)G_1(\sigma, v(\sigma))d\sigma - \int_0^b H_2(\tau,\sigma)G_2(\sigma, v(\sigma))d\sigma| \leq \epsilon \tag{3.5}$$

then there exists a unique continuous function $v_0 : I \to R$ given by

$$v_0(\tau) = \Sigma_{k=0}^{n-1} v^k(0^+)\frac{\tau^k}{k} + \frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\tau^*)^{\beta-1}\Big(g(\tau^*) + \int_0^{\tau^*} H_1(\tau^*,\sigma)G_1(\sigma, v(\sigma))d\sigma + \int_0^b H_2(\tau^*,\sigma)G_2(\sigma, v(\sigma))d\sigma\Big)d\tau^* \tag{3.6}$$

and also the following inequality holds good.

$$|v(\tau) - v_0(\tau)| \leq \frac{b\epsilon}{1 - \frac{M_1L_1+(\beta+1)M_2L_2}{\Gamma(\beta+2)} b^{\beta+1}}, \forall \tau \in I. \tag{3.7}$$

Proof . Suppose Y is the set of all real valued continuous functions on closed and bounded interval I. Now $\forall\, u, w \in Y$, a metric on Y is defined by
$\tilde{d}(u, w) = \inf\{N \in [0, \infty] : |v(\tau) - w(\tau)| \leq N \forall \tau \in I\}$. $(Y, \tilde{d})$ is a complete generalized metric space, see [1].

Now the operator $\varphi : Y \to Y$ is considered which is defined as:
$$(\varphi v)(\tau) = \Sigma_{k=0}^{n-1} v^k(0^+)\tfrac{\tau^k}{k} + \tfrac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\tau^*)^{q-1}\Big(g(\tau^*) + \int_0^{\tau^*} H_1(\tau^*,\sigma)G_1(\sigma, v(\sigma))d\sigma + \int_0^b H_2(\tau^*,\sigma)G_2(\sigma, v(\sigma))d\sigma\Big)d\tau^*$$
Suppose $N(v, w)$ be an arbitrary constant such that $\tilde{d}(v, w) \leq N(v, w)$. Now for $v_1(\tau)$, $v_2(\tau)$,

$$|(\varphi v)(\tau)-(\varphi w)(\tau)| \leq \frac{1}{\Gamma(q)}\int_0^{\tau}(\tau-\tau^*)^{q-1}[\int_0^{\tau^*}|H_1(\tau^*,\sigma)|\left(|G_1(\sigma,v(\sigma))-G_1(\sigma,w(\sigma))|\right)d\sigma$$
$$+\int_0^{b}|H_2(\tau^*,\sigma)|\left(|G_1(\sigma,v(\sigma))-G_1(\sigma,w(\sigma))|\right)d\sigma]$$
$$\leq \frac{M_1}{\Gamma(q)}\int_0^{\tau}(\tau-\tau^*)^{q-1}[\int_0^{\tau^*}L_1|v(\sigma)-w(\sigma)|d\sigma]d\tau^*$$
$$+\frac{M_2}{\Gamma(q)}\int_0^{\tau}(\tau-\tau^*)^{q-1}[\int_0^{b}L_2|v(\sigma)-w(\sigma)|d\sigma]d\tau^* \tag{3.8}$$
$$\leq \frac{M_1L_1}{\Gamma(q)}\int_0^{\tau}(\tau-\tau^*)^{q-1}[\int_0^{\tau^*}N(v,w)d\sigma]d\tau^*$$
$$+\frac{M_2L_2}{\Gamma(q)}\int_0^{\tau}(\tau-\tau^*)^{q-1}[\int_0^{b}N(v,w)d\sigma]d\tau^* \tag{3.9}$$
$$\leq \frac{M_1L_1+(\alpha+1)M_2L_2}{\Gamma(q+2)}b^{\alpha+1}N(v,w)$$

Thus by the hypothesis, φ is strictly contractive on the considered space Y. Now for an arbitrary element v_0 in Y, there exists a constant $\tilde{N}\in(0,\infty)\forall\tau\in I$ such that

$$|(\varphi v_0)(\tau)-(v_0)(\tau)| \leq |\Sigma_{k=0}^{n-1}v^k(0^+)\frac{\tau^k}{k}$$
$$+\frac{1}{\Gamma(q)}\int_0^{\tau}(\tau-\tau^*)^{q-1}\left(g(\tau^*)+\int_0^{\tau^*}k_1(\tau^*,\sigma)G_1(\sigma,v(\sigma))d\sigma+\int_0^{b}k_2(\tau^*,\sigma)G_2(\sigma,v(\sigma))d\sigma\right)d\tau^*-v_0(\tau)|$$
$$\leq \tilde{N}$$

. Hence, from the above inequality it is inferred that

$$d(\varphi v_0,v_0)\leq\infty$$

.
Now, using Theorem 2.2, there exists $v_1\in C([0,b],R)$ such that $\varphi^n v_0\to v_1$ in $(Y,\tilde{d})$ as $n\to\infty$ and $\varphi v_1=v_1$. Thus the fixed point v_0 is the desired solution of the considered equation.
Since $\tilde{d}$ is a metric, $v_0: I\to R$ is the unique continuous solution such that

$$v_0(\tau)=\Sigma_{k=0}^{n-1}v^k(0^+)\frac{\tau^k}{k}+\frac{1}{\Gamma(q)}\int_0^{\tau}(\tau-\tau^*)^{q-1}\Big(g(\tau^*)+\int_0^{\tau^*}H_1(\tau^*,\sigma)G_1(\sigma,v(\sigma))d\sigma+\int_0^{b}H_2(\tau^*,\sigma)G_2(\sigma,v(\sigma))d\sigma\Big)d\tau^*$$

Now by the hypothesis (3.5), it yields

$$\tilde{d}(v,\varphi v)\leq b\epsilon \tag{3.10}$$

Linking the Theorem 2.2 with the above equality (3.10), yields the following

$$\begin{aligned}\tilde{d}(v, v_0) &\leq \frac{1}{1 - \frac{M_1L_1 + (\alpha+1)M_2L_2}{\Gamma(q+2)} b^{\alpha+1}} \tilde{d}(v, \varphi v) \\ &\leq \frac{b\epsilon}{1 - \frac{M_1L_1 + (\alpha+1)M_2L_2}{\Gamma(q+2)} b^{\alpha+1}}.\end{aligned} \tag{3.11}$$

□

4 Numerical Scheme

4.1 Laplace Decomposition Method

$$D^{\alpha}v(\tau) = g(\tau) + \int_0^{\tau} H_1(\tau,\sigma)G_1(\sigma, v(\sigma))d\sigma + \int_0^{b} H_2(\tau,\sigma)G_2(\sigma, v(\sigma))d\sigma$$

Applying laplace transform to both sides, it is estimated that

$$\mathcal{L}(D^{\alpha}v(\tau)) = \mathcal{L}(g(\tau)) + \mathcal{L}\left(\int_0^{\tau} H_1(\tau,\sigma)G_1(\sigma, v(\sigma))d\sigma\right) + \mathcal{L}\left(\int_0^{b} H_2(\tau,\sigma)G_2(\sigma, v(\sigma))d\sigma\right) \tag{4.1}$$

Utilizing the differentiation property of laplace transform, it is obtained that

$$s^{\alpha}\mathcal{L}(v(\tau)) - c = \mathcal{L}(g(\tau) + \mathcal{L}\left(\int_0^{\tau} H_1(\tau,\sigma)G_1(\sigma, v(\sigma))d\sigma\right) + \mathcal{L}\left(\int_0^{b} H_2(\tau,\sigma)G_2(\sigma, v(\sigma))d\sigma\right) \tag{4.2}$$

where c is given by
$c = \sum_{k=0}^{m-1} \sigma^{\alpha-k-1} v^k(0)$.
Thus, the above equation is equivalent to

$$\mathcal{L}(v(\tau)) = \frac{c}{\sigma^{\alpha}} + \frac{1}{\sigma^{\alpha}}\mathcal{L}(g(\tau)) + \frac{1}{\sigma^{\alpha}}\mathcal{L}\left(\int_0^{\tau} H_1(\tau,\sigma)G_1(\sigma, v(\sigma))d\sigma\right) + \frac{1}{\sigma^{\alpha}}\mathcal{L}\left(\int_0^{b} H_2(\tau,\sigma)G_2(\sigma, v(\sigma))d\sigma\right) \tag{4.3}$$

The second step in Laplace decomposition method is that solution is expressed as an infinite series

$$v(\tau) = \sum_{n=0}^{\infty} v_n \tag{4.4}$$

Moreover, the nonlinear function is decomposed as

$$G_1(\tau, v(\tau)) = \sum_{n=0}^{\infty} A_n \tag{4.5}$$

and

$$G_2(\tau, v(\tau)) = \sum_{n=0}^{\infty} B_n \tag{4.6}$$

where A_n and B_n are Adomian polynomials are given as

$$A_n(\tau) = \frac{1}{n!}\left[\frac{d^n}{dp^n}\left[G_1(\tau, \sum_{i=0}^{\infty} p^i v_i)\right]_{p=0}\right. \tag{4.7}$$

$$B_n(\tau) = \frac{1}{n!}\Big[\frac{d^n}{dp^n}\left[G_2(\tau, \sum_{i=0}^{\infty} p^i v_i)\right]_p \tag{4.8}$$

$$\mathcal{L}(\sum_{n=0}^{\infty} v_n) = \frac{c}{\sigma^\alpha} + \frac{1}{\sigma^\alpha}\mathcal{L}(g(\tau)) + \frac{1}{s^\alpha}\mathcal{L}(\int_0^\tau H_1(\tau,\sigma)\sum_{n=0}^{\infty} A_n d\sigma) + \frac{1}{\sigma^\alpha}\mathcal{L}(\int_0^b H_2(\tau,\sigma)\sum_{n=0}^{\infty} B_n d\sigma) \tag{4.9}$$

Comparing both sides yields the iterative algorithm

$$\mathcal{L}(v_0) = \frac{c}{\sigma^\alpha} + \frac{1}{\sigma^\alpha}\mathcal{L}(g(\tau)) \tag{4.10}$$

$$\mathcal{L}(v_1) = \frac{1}{\sigma^\alpha}\mathcal{L}(\int_0^\tau H_1(\tau,\sigma)A_0 d\sigma) + \frac{1}{\sigma^\alpha}\mathcal{L}(\int_0^b H_2(\tau,\sigma)B_0 d\sigma) \tag{4.11}$$

$$\mathcal{L}(v_2) = \frac{1}{\sigma^\alpha}\mathcal{L}(\int_0^\tau H_1(\tau,\sigma)A_1 d\sigma) + \frac{1}{\sigma^\alpha}\mathcal{L}(\int_0^b H_2(\tau,\sigma)B_1 d\sigma) \tag{4.12}$$

Thus considering the equations, the recursion relation is as follows

$$\mathcal{L}(v_{k+1}) = \frac{1}{\sigma^\alpha}\mathcal{L}(\int_0^\tau H_1(\tau,\sigma)A_k d\sigma) + \frac{1}{\sigma^\alpha}\mathcal{L}(\int_0^b H_2(\tau,\sigma)B_k d\sigma), k \geq 0 \tag{4.13}$$

Finally by the inverse laplace transform method, the required iterative scheme for estimation of $v_0(\tau), v_1(\tau), ..$ are as follows

$$v_0(\tau) = \mathcal{L}^{-1}\left[\frac{1}{\sigma^\alpha}\mathcal{L}(g(\tau))\right], \tag{4.14}$$

and

$$v_{k+1}(\tau) = \mathcal{L}^{-1}\left[\frac{1}{\sigma^\alpha}\mathcal{L}(\int_0^\tau H_1(\tau,\sigma)A_k d\sigma)\right] + \mathcal{L}^{-1}\left[\frac{1}{\sigma^\alpha}\mathcal{L}(\int_0^b H_2(\tau,\sigma)B_k d\sigma)\right] \tag{4.15}$$

Thus the modified iterated scheme is as follows:

$$v_0(\tau) = \mathcal{L}^{-1}\left[\frac{1}{\sigma^\alpha}\mathcal{L}(g(\tau))\right], \tag{4.16}$$

$$v_1(\tau) = \mathcal{L}^{-1}\left[\frac{1}{\sigma^\alpha}\mathcal{L}(\int_0^\tau H_1(\tau,\sigma)A_0 d\sigma)\right] + \mathcal{L}^{-1}\left[\frac{1}{\sigma^\alpha}\mathcal{L}(\int_0^b H_2(\tau,\sigma)B_0 d\sigma)\right] \tag{4.17}$$

$$v_{k+1}(\tau) = \mathcal{L}^{-1}\left[\frac{1}{\sigma^\alpha}\mathcal{L}(\int_0^\tau H_1(\tau,\sigma)A_k d\sigma)\right] + \mathcal{L}^{-1}\left[\frac{1}{\sigma^\alpha}\mathcal{L}(\int_0^b H_2(\tau,\sigma)B_k d\sigma)\right] \tag{4.18}$$

5 Convergence Result

In this section we analyse a theoretical result in support of the convergence criterion.

Theorem 5.1. Suppose that (A1)-(A3), hold. Then the series solution given by $v(\tau) = \Sigma_{i=0}^{\infty} v_i(\tau)$, and $\|v_1\| < \infty$ obtained by the mth-order deformation converges to the exact solution of the fractional Volterra-Fredholm integro-differential equation (1)-(2) if $\frac{M_1^\star L_1 + M_2^\star L_2}{\Gamma(\alpha+1)} < 1$ holds.

Proof . Denote as $(C[0,b],\cdot)$ the Banach space of all continuous functions on I, with for all τ in I.
First, we define the sequence of partial sums as $s_n = \sum_{i=0}^{n} v_i(\tau)$.
Let s_n and s_m be arbitrary partial sums with $n \geq m$.
We claim that $s_n = \Sigma_{i=0}^{n} v_i(\tau)$ is a Cauchy sequence in $(C[0,b],\cdot)$.
Now we proceed as follows

$$\|s_n - s_m\|_\infty = \max_{\forall\tau\in I} |s_n - s_m|$$

$$= \max_{\forall\tau\in I} |\Sigma_{i=0}^{n} v_i(\tau) - \Sigma_{i=0}^{m} v_i(\tau)|$$

$$= \max_{\forall\tau\in I} |\Sigma_{i=m+1}^{n} v_i(\tau)|$$

$$= \max_{\forall\tau\in I} |\Sigma_{i=m+1}^{n} \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}(\int_0^t H_1(\tau,\sigma)A_{i-1}ds) + \frac{1}{\sigma^\alpha}\mathcal{L}(\int_0^b H_2(\tau,\sigma)B_{i-1}d\sigma)\right]|$$

$$= \max_{\forall\tau\in I} |\mathcal{L}^{-1}\left[\frac{1}{\sigma^\alpha}\mathcal{L}(\int_0^\tau H_1(\tau,\sigma)\Sigma_{i=m}^{n-1}A_{i-1}d\sigma) + \frac{1}{\sigma^\alpha}\mathcal{L}(\int_0^b H_2(\tau,\sigma)\Sigma_{i=m}^{n-1}B_{i-1}d\sigma)\right]|.$$

From (4.5) and (4.6), we have

$$\Sigma_{i=m}^{n-1}A_{i-1} = G_1(\tau,s_{n-1}) - G_1(\tau,s_{m-1}),$$

$$\Sigma_{i=m}^{n-1}B_{i-1} = G_2(\tau,s_{n-1}) - G_2(\tau,s_{m-1}).$$

Thus,

$$\|s_n - s_m\|$$

$$= \max_{\forall\tau\in I} |\mathcal{L}^{-1}\Big[\frac{1}{\sigma^\alpha}\mathcal{L}(\int_0^\tau H_1(\tau,\sigma)(G_1(\tau,s_{n-1}) - G_1(\tau,s_{m-1}))d\sigma) + \frac{1}{s^\alpha}\mathcal{L}(\int_0^b H_2(\tau,\sigma)(G_2(\tau,s_{n-1}) - G_2(\tau,s_{m-1}))d\sigma)\Big]|$$

$$= \max_{\forall\tau\in I} \Big(\mathcal{L}^{-1}\Big[\frac{1}{\sigma^\alpha}\mathcal{L}(\int_0^\tau |H_1(\tau,\sigma)||G_1(\tau,s_{n-1}) - G_1(\tau,s_{m-1})|d\sigma) + \frac{1}{\sigma^\alpha}\mathcal{L}(\int_0^b |H_2(\tau,\sigma)||G_2(\tau,s_{n-1}) - G_2(\tau,s_{m-1})|d\sigma)\Big]\Big)$$

$$\leq \mathcal{L}^{-1}\Big[\frac{1}{\sigma^\alpha}\mathcal{L}(\int_0^\tau |H_1(\tau,\sigma)||G_1(\tau,s_{n-1}) - G_1(\tau,s_{m-1})|d\sigma) + \frac{1}{\sigma^\alpha}\mathcal{L}(\int_0^b |H_2(\tau,\sigma)||G_2(\tau,s_{n-1}) - G_2(\tau,s_{m-1})|d\sigma)\Big]$$

$$\leq \mathcal{L}^{-1}\left[\frac{1}{\sigma^\alpha}\mathcal{L}(M_1^\star L_1\|s_{n-1} - s_{m-1}\|_\infty) + \frac{1}{\sigma^\alpha}\mathcal{L}(M_2^\star L_2\|s_{n-1} - s_{m-1}\|_\infty)\right]$$

$$\leq \mathcal{L}^{-1}\left[\frac{1}{\sigma^{\alpha+1}}M_1^\star L_1\|s_{n-1} - s_{m-1}\|_\infty + \frac{1}{\sigma^{\alpha+1}}M_2^\star L_2\|s_{n-1} - s_{m-1}\|_\infty\right]$$

$$\leq \frac{\tau^\alpha}{\Gamma(\alpha+1)}(M_1^\star L_1 + M_2^\star L_2)\|s_{n-1} - s_{m-1}\|_\infty$$

Thus,

$$\|s_n - s_m\|_\infty \leq \frac{M_1^\star L_1 + M_2^\star L_2}{\Gamma(\alpha+1)}\|s_{n-1} - s_{m-1}\|_\infty$$

Further,

$$\|s_n - s_m\|_\infty \leq \lambda\|s_{n-1} - s_{m-1}\|_\infty.$$

Let $n = m + 1$ then

$$\|s_n - s_m\|_\infty \leq \lambda\|s_m - s_{m-1}\|_\infty \leq \lambda^2\|s_{m-1} - s_{m-2}\|_\infty ... \lambda^m\|s_1 - s_0\|_\infty.$$

$\|s_n - s_m\|_\infty \leq \|s_{m+1} - s_m\|_\infty + \|s_{m+2} - s_{m+1}\|_\infty + ... + \|s_n - s_{n-1}\|_\infty$

$\leq [1 + \lambda + \lambda^2 + ... + \lambda^{n-m-1}]\|s_1 - s_0\|_\infty$

$\leq \lambda^m(\frac{1-\lambda^{n-m}}{1-\lambda})\|v_1\|_\infty.$

Since $0 < \lambda < 1$, $(1 - \lambda^{n-m}) < 1$. But $\|v_1(t)\| < \infty$. Thus, $\|s_n - s_m\|_\infty \to 0$ as $m \to \infty$. Hence it is concluded that s_n is a Cauchy sequence in $C[0, b]$. Finally, $v = \lim_{n\to\infty} v_n$ i.e., the series is convergent.

□

6 Numerical Example

6.1 Example 1

$$D^{1/2}v(\tau) = 1 + 3\tau^2 + \tau^3 + \int_0^\tau (2 + \tau + 2s)v(\sigma)d\sigma, \qquad \tau \in [0, 1/4] \tag{6.1}$$

with the initial condition $v(0) = 0$.
Here, $\beta = 1/2$, $g(\tau) = 1 + 3\tau^2 + \tau^3$, $H_1(\tau, \sigma) = 2 + \tau + 2\sigma$, $G_1(\tau, v(\tau)) = v(\tau)$, $G_2(\tau, v(\tau)) = 0$.
Also the conditions (I) and (II) are satisfied and the inequality (3.4) holds. Thus, by Theorem 3.3, the equation (6.1) is Ulam stable and has a unique solution.
Applying Laplace transform to both sides, it is obtained that

$$\mathcal{L}[D^{1/2}v(\tau)] = \mathcal{L}[1 + 3\tau^2 + \tau^3] + \mathcal{L}[\int_0^\tau (2 + \tau + 2\sigma)v(\sigma)d\sigma] \tag{6.2}$$

Now, by the linearity of laplace transform and the initial condition

$$s^{1/2}\mathcal{L}[v(\tau)] = \mathcal{L}[1 + 3\tau^2 + \tau^3] + \mathcal{L}[\int_0^\tau (2 + \tau + 2\sigma)v(\sigma)d\sigma] \tag{6.3}$$

$$\mathcal{L}[v(\tau)] = \frac{1}{\sigma^{1/2}}\{\mathcal{L}[1 + 3\tau^2 + \tau^3] + \mathcal{L}[\int_0^\tau (2 + \tau + 2\sigma)v(\sigma)d\sigma]\}$$

Substituting (4.4) into the above equation, it is obtained that

$$\mathcal{L}[\sum_{n=0}^{\infty} v_n(\tau)] = \frac{1}{\sigma^{1/2}}\{\mathcal{L}[1 + 3\tau^2 + \tau^3] + \mathcal{L}[\int_0^\tau (2 + \tau + 2\sigma)\sum_{n=0}^{\infty} v_n(\sigma)d\sigma]\} \tag{6.4}$$

Comparing both sides of equation (6.4), it is estimated that

$$\mathcal{L}[v_0(\tau)] = \frac{1}{\sigma^{1/2}}\{\mathcal{L}[1 + 3\tau^2 + \tau^3]\} \tag{6.5}$$

$$\mathcal{L}[v_{n+1}(\tau)] = \frac{1}{\sigma^{1/2}}\{\mathcal{L}[\int_0^\tau (2 + \tau + 2\sigma)v_n(\sigma)d\sigma]\} \tag{6.6}$$

Now applying inverse laplace transform to all the above equations, it is obtained that

$$v_0(\tau) = \mathcal{L}^{-1}\{\frac{1}{\sigma^{1/2}}\{\mathcal{L}[1 + 3\tau^2 + \tau^3]\}\} \tag{6.7}$$

$$v_{n+1}(\tau) = \mathcal{L}^{-1}\{\frac{1}{\sigma^{1/2}}\{\mathcal{L}[\int_0^t (2+\tau+2\sigma)v_n(\sigma)d\sigma]\} \tag{6.8}$$

Thus it is estimated that

$$v_0(\tau) = \frac{2\tau^{1/2}}{\sqrt{\pi}} + \frac{16}{5}\frac{\tau^{5/2}}{\sqrt{\pi}} + \frac{32}{35}\frac{\tau^{7/2}}{\sqrt{\pi}} \tag{6.9}$$

$$v_1(\tau) = \frac{\tau^2}{8} + \frac{11}{96}\tau^3 + \frac{37}{792}\tau^4 + \frac{27}{320}\tau^5 + \frac{29}{1920}\tau^6 \tag{6.10}$$

Similarly the other terms i.e., $v_2(\tau)$, $v_3(\tau)$ and $v_4(\tau)$ of the approximate solution can be obtained.

6.2 Example 2

$$D^{3/2}v(\tau) = \tau^2 + 3\tau^3 + \frac{1}{5}\int_0^\tau (\tau\sigma+1)v(\sigma)d\sigma + \frac{1}{6}\int_0^1 (2+\tau+\sigma^2)v(\sigma)d\sigma \tag{6.11}$$

with the initial condition $v(0) = 0$ and $v'(0) = 2$.
Applying Laplace transform to both sides, it is obtained that

$$\mathcal{L}[D^{3/2}v(\tau)] = \mathcal{L}[\tau^2+3\tau^3] + \mathcal{L}[\frac{1}{5}\int_0^t (\tau\sigma+1)v(\sigma)d\sigma] + \mathcal{L}[\frac{1}{6}\int_0^1 (2+\tau+\sigma^2)v(\sigma)d\sigma] \tag{6.12}$$

Now by the linearity of laplace transform and the initial conditions

$$\sigma^{3/2}\mathcal{L}[v(\tau)] - 2\sigma^{-1/2} = \mathcal{L}[\tau^2+3\tau^3] + \mathcal{L}[\frac{1}{5}\int_0^\tau (\tau\sigma+1)v(\sigma)d\sigma] + \mathcal{L}[\frac{1}{6}\int_0^1 (2+\tau+\sigma^2)v(\sigma)d\sigma] \tag{6.13}$$

$$\mathcal{L}[v(\tau)] = \frac{2}{\sigma^2} + \frac{1}{\sigma^{3/2}}\{\mathcal{L}[\tau^2+3\tau^3] + \mathcal{L}[\frac{1}{5}\int_0^\tau (\tau\sigma+1)v(\sigma)d\sigma] + \mathcal{L}[\frac{1}{6}\int_0^1 (2+\tau+\sigma^2)v(\sigma)d\sigma]\}$$

Substituting (4.4) into the above equation, it is obtained that

$$\mathcal{L}[\sum_{n=0}^\infty v_n(\tau)] = \frac{2}{\sigma^2} + \frac{1}{\sigma^{3/2}}\{\mathcal{L}[\tau^2+3\tau^3] + \mathcal{L}[\frac{1}{5}\int_0^\tau (\tau\sigma+1)\sum_{n=0}^\infty v_n(\sigma)d\sigma] + \mathcal{L}[\frac{1}{6}\int_0^1 (2+\tau+\sigma^2)\sum_{n=0}^\infty v_n(\sigma)d\sigma]\}$$

Comparing both sides of equation (6.24), it is estimated that

$$\mathcal{L}[v_0(\tau)] = \frac{2}{\sigma^2} + \frac{1}{\sigma^{3/2}}\{\mathcal{L}[\tau^2+3\tau^3]\} \tag{6.14}$$

$$\mathcal{L}[v_0(\tau)] = \mathcal{L}[\frac{1}{5}\int_0^\tau (\tau\sigma+1)v_0(\sigma)d\sigma] + \mathcal{L}[\frac{1}{6}\int_0^1 (2+\tau+\sigma^2)v_0(\sigma)d\sigma] \tag{6.15}$$

$$\mathcal{L}[v_{n+1}(\tau)] = \mathcal{L}[\frac{1}{5}\int_0^\tau (\tau\sigma+1)v_n(\sigma)d\sigma] + \mathcal{L}[\frac{1}{6}\int_0^1 (2+\tau+\sigma^2)v_n(\sigma)d\sigma] \tag{6.16}$$

Now applying inverse laplace transform to all the above equations, it is obtained that

$$v_0(\tau) = \mathcal{L}^{-1}\{\frac{2}{\sigma^2}\} + \mathcal{L}^{-1}\{\frac{1}{\sigma^{3/2}}\{\mathcal{L}[\tau^2+3\tau^3]\}\} \tag{6.17}$$

$$v_1(\tau) = \mathcal{L}^{-1}\{\frac{1}{\sigma^{3/2}}\mathcal{L}[\frac{1}{5}\int_0^\tau (\tau\sigma+1)v_0(\sigma)d\sigma] + \mathcal{L}[\frac{1}{6}\int_0^1 (2+\tau+\sigma^2)v_0(\sigma)d\sigma]\} \tag{6.18}$$

$$v_{n+1}(\tau) = \mathcal{L}^{-1}\{\frac{1}{\sigma^{3/2}}\mathcal{L}[\frac{1}{5}\int_0^\tau (\tau\sigma+1)v_n(\sigma)d\sigma] + \mathcal{L}[\frac{1}{6}\int_0^1 (2+\tau+\sigma^2)v_n(\sigma)d\sigma]\} \tag{6.19}$$

Thus it is estimated that

$$v_0(\tau) = 2\tau + \frac{32}{105\sqrt{\pi}}\tau^{7/2} + \frac{64}{105\sqrt{\pi}}\tau^{9/2} \tag{6.20}$$

6.3 Example 3

$$D^{1/2}v(\tau) = \frac{2\tau^{1/2}}{\sqrt{\pi}} + \frac{3\tau\sqrt{\pi}}{4} - \frac{9}{10} + \int_0^1 v(\sigma)d\sigma \tag{6.21}$$

with the initial condition $v(0) = 0$ and the exact solution given by $v(\tau) = \tau^{3/2} + \tau$. Here, $\beta = 1/2$, $g(\tau) = \frac{2\tau^{1/2}}{\sqrt{\pi}} + \frac{3\tau\sqrt{\pi}}{4} - \frac{9}{10}$, $H_2(\tau,\sigma) = 1$, $G_1(\tau, v(\tau)) = 0$, $G_2(\tau, v(\tau)) = v(\tau)$.
Also the conditions (I) and (II) are satisfied and the inequality (3.4) holds. Thus by Theorem 3.3, the equation (6.3) is Ulam stable and has a unique solution.
Applying Laplace transform to both sides, it is obtained that

$$\mathcal{L}[D^{1/2}v(\tau)] = \mathcal{L}[\frac{2\tau^{1/2}}{\sqrt{\pi}}] + \mathcal{L}[\frac{3\tau\sqrt{\pi}}{4}] - \mathcal{L}[\frac{9}{10}] + \mathcal{L}(\int_0^\tau v(\sigma)d\sigma) \tag{6.22}$$

Now by the linearity of laplace transform and the initial condition

$$\begin{aligned} \sigma^{1/2}\mathcal{L}[v(\tau)] =& \mathcal{L}[\frac{2\tau^{1/2}}{\sqrt{\pi}}] + \mathcal{L}[\frac{3\tau\sqrt{\pi}}{4}] - \mathcal{L}[\frac{9}{10}] + \mathcal{L}[\int_0^1 v(\sigma)d\sigma] \\ \mathcal{L}[v(\tau)] =& \frac{1}{\sigma^{1/2}}\mathcal{L}[\frac{2\tau^{1/2}}{\sqrt{\pi}}] + \frac{1}{\sigma^{1/2}}\mathcal{L}[\frac{3\tau\sqrt{\pi}}{4}] - \frac{1}{\sigma^{1/2}}\mathcal{L}[\frac{9}{10}] + \frac{1}{\sigma^{1/2}}\mathcal{L}[\int_0^1 v(\sigma)d\sigma] \end{aligned} \tag{6.23}$$

Substituting (4.4) into the above equation, it is obtained that

$$\mathcal{L}[\sum_{n=0}^{\infty} v_n(\tau)] = \frac{1}{\sigma^{1/2}}\mathcal{L}[\frac{2\tau^{1/2}}{\sqrt{\pi}}] + \frac{1}{\sigma^{1/2}}\mathcal{L}[\frac{3\tau\sqrt{\pi}}{4}] - \frac{1}{\sigma^{1/2}}\mathcal{L}[\frac{9}{10}] + \frac{1}{\sigma^{1/2}}\mathcal{L}[\int_0^1 \sum_{n=0}^{\infty} v_n(\sigma)d\sigma] \tag{6.24}$$

Now applying inverse laplace transform to all the above equations, it is obtained that

$$v_0(\tau) = \mathcal{L}^{-1}\{\frac{1}{\sigma^{1/2}}\mathcal{L}[\frac{2\tau^{1/2}}{\sqrt{\pi}}]\} + \mathcal{L}^{-1}\{\frac{1}{\sigma^{1/2}}\mathcal{L}[\frac{3\tau\sqrt{\pi}}{4}]\} - \mathcal{L}^{-1}\{\frac{1}{\sigma^{1/2}}\mathcal{L}[\frac{9}{10}]\} \tag{6.25}$$

$$v_1(\tau) = \mathcal{L}^{-1}\{\frac{1}{\sigma^{1/2}}\mathcal{L}[\int_0^1 v_0(\sigma)d\sigma]\} \tag{6.26}$$

$$v_{n+1}(\tau) = \mathcal{L}^{-1}\{\frac{1}{\sigma^{1/2}}\mathcal{L}[\int_0^1 v_n(\sigma)d\sigma]\} \tag{6.27}$$

Thus it is estimated that

$$v_0(\tau) = \tau + \tau^{\frac{3}{2}} - \frac{9}{5\sqrt{\pi}}\tau^{1/2} \tag{6.28}$$

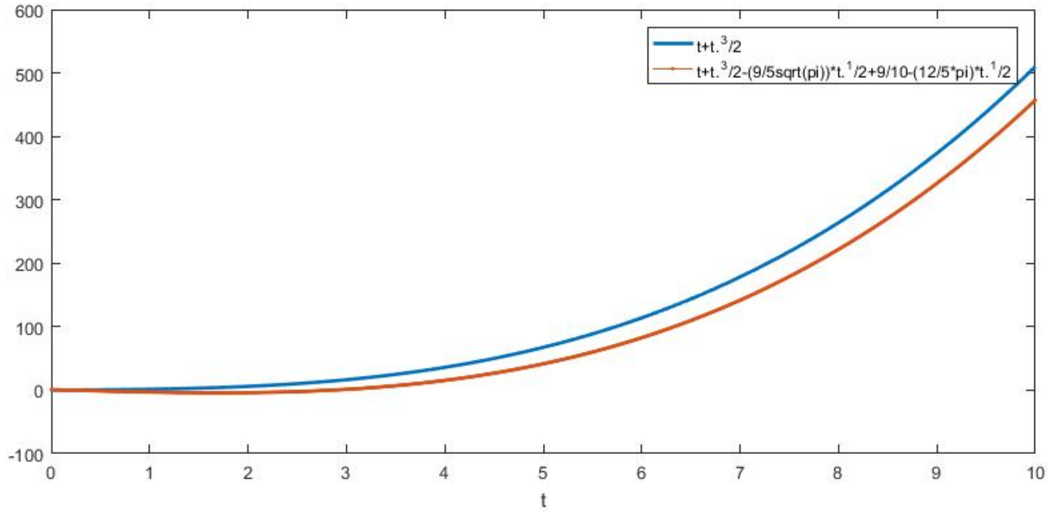

Figure 1:

$$v_1(\tau) = \frac{9}{10} - \frac{12}{5\pi}\tau^{1/2} \tag{6.29}$$

Now $\tilde{v}(\tau) = v_0(\tau) + v_1(\tau) = \tau + \tau^{\frac{3}{2}} - \frac{9}{5\sqrt{\pi}}\tau^{1/2} + \frac{9}{10} - \frac{12}{5\pi}\tau^{1/2}$ is the approximate closed form solution of the above equation. Also the exact and approximate solution of the considered problem is shown in Figure 1.

7 Conclusion

Here sufficient criteria for Ulam stability and existence of the solution of the initial value problem has been derived. Decomposition method has been proposed to find out the solution of the considered nonlinear problem. Finally, some examples have been showcased to ensure the validity of the derived results.

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