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Positive periodic solutions for a neutral differential equation with iterative terms arising in biology and population dynamics

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Abstract

In this article, a class of first order neutral delay differential equations with iterative terms is investigated. The proofs of the existence of positive periodic solutions rely on the Krasnoselskii's fixed point theorem together with the Green's functions method. Furthermore, by the aid of the Banach fixed point theorem and under an extra condition, we establish the existence, uniqueness and stability results. We provide an example to show the accuracy of the conditions of the obtained findings which extend and generalize earlier ones in the literature.

Keywords: Neutral functional-differential equations, Continuous dependence, Fixed point theorems, Periodic solutions
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1 Introduction

Recently, a great attention has been devoted to investigate neutral differential equations where the highest order derivatives occur with delays. These equations arise in various applied sciences such as population dynamic, biology, hematology, physics, economics, chemistry, and so forth. For any further and detailed information, we refer the interested reader to [2, 7, 14, 16, 18] and the references cited therein. Among the papers that dealt with the existence of periodic solutions for first order neutral differential equations, we cite some of them which are relevant to what we are discussing in this work.

In 1991, Serra [18] used the Mawhin's coincidence degree theory to investigate the existence of periodic solutions for the following neutral differential equation:

$$\frac{d}{dt}\left[x\left(t\right)-cx\left(t-\tau\right)\right]=f\left(t,x\left(t\right)\right),$$

with $t\in\mathbb{R}$, $c\in\mathbb{R}$, $\tau\in\left]0,2\pi\right[$, $f:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is Caratheodory function and $2\pi-$periodic with respect to the time.

In 2008, by means of the Krasnoselskii's fixed point theorem, Luo et al. [16] gave the sufficient conditions that guarantee the existence of positive periodic solutions of the following neutral differential equation:

$$\frac{d}{dt}\left[x\left(t\right)-cx\left(t-\tau\left(t\right)\right)\right]=-a\left(t\right)x\left(t\right)+f\left(t,x\left(t-\tau\left(t\right)\right)\right),$$

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where $a \in C(\mathbb{R}, (0, \infty))$, $\tau \in C(\mathbb{R}, \mathbb{R})$ and $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ are common periodic functions with respect to the time variable.

In 2016, Candan [7] employed the same aforementioned fixed point theorem to discuss the positive periodic solutions for the following neutral differential equation:

$$\frac{d}{dt}[x(t) - P(t)x(t-\tau)] = -a(t)x(t) + f(t, x(t-\tau)),$$

here $a \in C(\mathbb{R}, (0, \infty))$, $P \in C^1(\mathbb{R}, \mathbb{R})$ and $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ are common periodic functions with respect to the time variable and $\tau > 0$.

In the present work, we give new sufficient criteria for the existence, uniqueness and continuous dependence of positive periodic solutions for the following first order neutral differential equation with iterative terms:

$$\frac{d}{dt}[x(t) - cx(t-\tau(t))] = -a(t)x(t) + f(t, x(t-\tau(t))) - H\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right), \tag{1.1}$$

where the iterate $x^{[n]}(t)$ stands for x composed with itself n times, i.e. $x^{[2]}(t) = x(x(t)), ..., x^{[n]}(t) = x^{[n-1]}(x(t))$, $c \in (0, 1)$, $a, \tau \in C(\mathbb{R}, (0, \infty))$, $f \in \mathcal{C}([0, T] \times \mathbb{R}, (0, \infty))$ and $H \in \mathcal{C}([0, T] \times \mathbb{R}^n, (0, \infty))$ are $T-$periodic functions with respect to the time variable. Furtheremore, the functions $f(t, x)$ and $H(t, x_1, x_2, ..., x_n)$ are supposed globally Lipschitz in x and $x_1, ..., x_n$ respectively, i.e. there exist a positive constant k and n positive constants $l_1, l_2, ..., l_n$ such that

$$|f(t, x) - f(t, y)| \leq k\|x - y\|, \tag{1.2}$$

and

$$|H(t, x_1, x_2, ..., x_n) - H(t, y_1, y_2, ..., y_n)| \leq \sum_{i=1}^{n} l_i \|x_i - y_i\|, \tag{1.3}$$

It is noteworthy that equation (1.1) is a first order iterative differential equation and such equations are used to model a variety of phenomena observed in an extremely wide range of areas, including life sciences (see, e.g., [1, 6, 19], and references cited therein). For instance, it can model many biological and ecological equations such as: Neutral Mackey-Glass models with harvesting, Neutral Wazewska-Lasota model with harvesting, Neutral Nicholson's blowflies model with harvesting and Neutral housеflies model with harvesting.

So, to put forward a more meaningful and realistic model that can describe a biological phenomenon and contain the minimal basic biological information about it, we assume that the production f (flux or recruitment) term incorporates a time-varying delay and also we take into account a harvesting strategy H with time and state dependent delays that lead to the appearance of the iterates $x^{[i]}(t)$.

The strong interest in this work is motivated by the fact that the harvesting of individuals provides a good description of the population dynamics and plays a prominent role in getting a better understanding of its effects on the management of biological resources. Moreover, in many biological and vital phenomena, the delays are generally depending on both the time and the state variable that can give the iterations in the model such as in infectious diseases spread, blood cell production and insect population growth.

We now outline some key features of our work by the following items:

- It should be pointed out that equation (1.1) is more general than those investigated in [6, 7, 16, 18]. Furthermore, to the best of our knowledge, there are no published papers that address this problem with a time varying delay and an iterative harvesting term that involves implicitly $(n-1)$ time and state dependent delays.

- Despite that recent years have gradually witnessed an unprecedented interest towards such kind of equations (we can mention, for instance, [1], [3]-[6], [8]-[13], [15], [17], [19]-[21]), the investigation in this direction remains scarce and their theory has not yet been developed enough. So, it is our belief that our work is of significance because it contributes in the literature of this emergent theory.

- There is no doubt that the harvesting strategy plays a crucial role in the population dynamics and it is also quite normal that it involves many delays. So, on account of these facts, we attempt to understand the effect of the harvesting strategy on the population dynamics by adding an iterative harvesting term involving implicitly many time and state dependent delays.

- We are interested in the positivity, boundedness and periodicity of solutions which makes our results even more powerful and biologically meaningful. This is due to the fact that the state $x(t)$ in biological phenomena could, for example, stand for an amount of cells, a density, a number of individuals or a size of the population which should be positive and bounded quantities and generally periodic.

The rest of the manuscript is furnished as follows: In Section 2, we start with some preliminaries which will be needed in what follows. In Section 3, by virtue of the Krasnoselskii's fixed point theorem and some Green's function properties, we construct some new results about the existence of positive periodic solutions for equation (1.1). In Section 4, the existence, uniqueness and continuous dependence on parameters of the solutions are established by using Banach fixed point theorem. In Section 5, we exhibit an example to which our key outcomes can be applied. Finally, we conclude the paper with a brief conclusion.

2 Mathematical background

For $T, M, L > 0$ and $m \geq 0$, let

$$P_T = \{x \in \mathcal{C}(\mathbb{R}, \mathbb{R}), x(t+T) = x(t)\},$$

be a Banach space furnished with the norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,T]} |x(t)|,$$

and

$$P_T(L, m, M) = \{x \in P_T, m \leq x(t) \leq M, |x(t_2) - x(t_1)| \leq L|t_2 - t_1|, \forall t_1, t_2 \in \mathbb{R}\}.$$

Then $P_T(L, m, M)$ is a closed convex and bounded subset of P_T. Since the uniform boundedness and the equicontinuity of $P_T(L, m, M)$ follow from its definition, then it follows from the Arzelà-Ascoli theorem that it is compact.

For convenience, throughout this work, we introduce the following notations:

$$a_1 = \max_{t\in[0,T]} a(t), \ f_1 = \max_{t\in[0,T]} |f(t,0)|, \ H_1 = \max_{t\in[0,T]} |H(t,0,0,...,0)|,$$

$$\eta_1 = \frac{\exp\left(-\int_0^T a(u)\,du\right)}{\exp\left(\int_0^T a(u)\,du\right) - 1}, \ \eta_2 = \frac{\exp\left(\int_0^T a(u)\,du\right)}{\exp\left(\int_0^T a(u)\,du\right) - 1}, \ \Lambda = \sum_{i=1}^{n} l_i \sum_{j=0}^{i-1} L^j.$$

In the sequel we will assume that the following hypotheses are satisfied:
There exists a positive constant $f_0 > 0$ such that

$$f(t,x) \geq f_0, \ \forall t \in \mathbb{R}, \ \forall x \in (0,\infty). \tag{2.1}$$

The following estimates are satisfied:

$$cM + \eta_2 T(kM + f_1) \leq M, \tag{2.2}$$

$$\eta_1 T f_0 - \eta_2 T(M\Lambda + H_1) - cT\eta_2 a_1 M + cm \geq m, \tag{2.3}$$

and

$$\eta_2(2 + a_1 T)(H_1 + f_1 + M(k + \Lambda + ca_1)) + L(1+L)c \leq L. \tag{2.4}$$

Now, we state and prove the following lemma, which we intend to use later.

Lemma 2.1. $x \in P_T(L, m, M) \cap \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ is a solution of (1.1) if and only if $x \in P_T(L, m, M)$ satisfies the following integral equation:

$$x(t) = \int_t^{t+T} G(t,s)\left[f(s, x(s-\tau(s))) - H\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right)\right.$$
$$\left. -ca(s)x(s-\tau(s))\right] ds + cx(t-\tau(t)), \tag{2.5}$$

where

$$G(t,s) = \frac{\exp\left(\int_t^s a(u)\,du\right)}{\exp\left(\int_0^T a(u)\,du\right) - 1}. \tag{2.6}$$

Proof . Let $x \in P_T(L,m,M) \cap \mathcal{C}^1(\mathbb{R},\mathbb{R})$ be a solution of (1.1). Multiplying both sides of (1.1) by $\exp\left(\int_0^t a(u)\,du\right)$, we get

$$\begin{aligned}
&\frac{d}{dt}\left[(x(t) - cx(t-\tau(t)))\exp\left(\int_0^t a(u)\,du\right)\right] \\
&= \left[f(t, x(t-\tau(t))) - H\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right)\right]\exp\left(\int_0^t a(u)\,du\right) \\
&- ca(t)\,x(t-\tau(t))\exp\left(\int_0^t a(u)\,du\right).
\end{aligned}$$

It follows from the periodic properties and the integration from t to $t+T$ that

$$\begin{aligned}
&(x(t) - cx(t-\tau(t)))\left(\exp\left(\int_0^{t+T} a(u)\,du\right) - \exp\left(\int_0^t a(u)\,du\right)\right) \\
&= \int_t^{t+T}\left\{f(s, x(s-\tau(s))) - H\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right)\right. \\
&\left.-c(s)\,a(s)\,x(s-\tau(s))\right\}\exp\left(\int_0^s a(u)\,du\right) ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
x(t) = &\int_t^{t+T} \frac{\exp\left(\int_t^s a(u)\,du\right)}{\exp\left(\int_0^T a(u)\,du\right) - 1}\left\{f(s, x(s-\tau(s))) - H\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right)\right. \\
&\left.-ca(s)\,x(s-\tau(s))\right\} ds + cx(t-\tau(t)).
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.2. [21]If $x, y \in P_T(L,m,M)$, then

$$\left\|x^{[m]} - y^{[m]}\right\| \leq \sum_{j=0}^{m-1} L^j \|x-y\|, \ m = 1, 2, ...$$

Remark 2.3. We have

$$0 < \eta_1 \leq G(t,s) \leq \eta_2, \tag{2.7}$$
$$G(t+T, s+T) = G(t,s), \ \forall t, s \in \mathbb{R}, \tag{2.8}$$
$$\int_{t_1}^{t_1+T} |G(t_2,s) - G(t_1,s)|\,ds \leq Ta_1\eta_2\,|t_2 - t_1|, \tag{2.9}$$

for all $t, s, t_1, t_2 \in \mathbb{R}$ and in view of (1.2), (1.3) and Lemma 2.2 we obtain

$$|f(s, x(s-\tau(s)))| \leq kM + f_1, \tag{2.10}$$

and

$$\left|H\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right)\right| \leq M\Lambda + H_1, \tag{2.11}$$

for all $x \in P_T(L,m,M)$.

3 Main Results

In this section, we will use the Krasnoselskii's fixed point theorem to prove the existence of positive periodic solutions of equation (1.1). For this and, by virtue of Lemma 2.1, we define an operator $N : P_T(L,m,M) \to P_T$ as follows:

$$(Nx)(t) = (F_1x)(t) + (F_2x)(t),$$

where $F_1, F_2 : P_T(L,m,M) \to P_T$ are defined as follows:

$$(F_1x)(t) = \int_t^{t+T} G(t,s)\left[f(s,x(s-\tau(s))) - H\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)\right.$$
$$\left. -ca(s)\,x(s-\tau(s))\right]ds, \tag{3.1}$$

and

$$(F_2x)(t) = cx(t-\tau(t)). \tag{3.2}$$

So, the existence of solutions for equation (1.1) is equivalent whether or not the operator N has a fixed point.

Since $c \in (0,1)$, then F_2 is a contraction. So, to apply Krasnoselskii's fixed point theorem, it suffices to prove that F_1 is continuous and compact and that $F_1x + F_2y \in P_T(L,m,M)$ for all $x,y \in P_T(L,m,M)$.
We start by proving the compactness and the continuity of F_1.

Lemma 3.1. Operator $F_1 : P_T(L,m,M) \to P_T$ is continuous and compact.

Proof . Since $P_T(L,m,M)$ is a compact subset of P_T, then the compactness of F_1 follows immediately from its continuity.
Let us prove that F_1 is continuous. Indeed, for all $x,y \in P_T(L,m,M)$, we have

$$\begin{aligned}|(F_1x)(t) - (F_1y)(t)| \le & \int_t^{t+T} |f(s,x(s-\tau(s))) - f(s,y(s-\tau(s)))|\,G(t,s)\,ds \\ & + \int_t^{t+T} G(t,s)\left|H\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)\right. ds \\ & \left.-H\left(s,y(s),y^{[2]}(s),...,y^{[n]}(s)\right)\right| ds \\ & + \int_t^{t+T} ca(s)\,|x(s-\tau(s)) - y(s-\tau(s))|\,G(t,s)\,ds.\end{aligned}$$

It follows from (1.2), (1.3) and (2.7) that

$$\|F_1x - F_1y\| \le \eta_2 Tk\,\|x-y\| + \eta_2 T\sum_{i=1}^{n} l_i\left\|x^{[i]} - y^{[i]}\right\| + \eta_2 Tca_1\,\|x-y\|.$$

By using Lemma 2.2, we obtain

$$\|F_1x - F_1y\| \le \eta_2 T(k + ca_1 + \Lambda)\,\|x-y\|, \tag{3.3}$$

which establishes that the operator F_1 is Lipschitz continuous and hence continuous. Therefore, F_1 is compact. □

Lemma 3.2. Let $\tau \in P_T(L,m,M)$. If conditions (2.1)-(2.4) hold, then

$$F_1x + F_2y \in P_T(L,m,M),$$

for all $x,y \in P_T(L,m,M)$.

Proof . Let $x,y \in P_T(L,m,M)$. From (2.2), (2.7) and (2.10) we get

$$\begin{aligned}(F_1x)(t) + (F_2y)(t) &\le cy(t-\tau(t)) + \int_t^{t+T} G(t,s)\,f(s,x(s-\tau(s)))\,ds \\ &\le cM + \eta_2 T(kM + f_1) \\ &\le M,\end{aligned}$$

and from (2.1), (2.3), (2.7) and (2.11), we arrive at

$$\begin{aligned}(F_1x)(t) + (F_2y)(t) &\ge \eta_1 Tf_0 - \eta_2 T(M\Lambda + H_1) - cT\eta_2 a_1 M + cm \\ &\ge m.\end{aligned}$$

Consequently,

$$m \leq (Ax)(t) + (By)(t) \leq M, \tag{3.4}$$

for all $x, y \in P_T(L, m, M)$.
Let $\tau \in P_T(L, m, M)$ and $t_1, t_2 \in [0, T]$ with $t_1 < t_2$. For all $x, y \in P_T(L, m, M)$, we have

$$\begin{aligned}
&\left|((F_1x) + (F_2y))(t_2) - ((F_1x) + (F_2y))(t_1)\right| \\
&\leq \left| \int_{t_2}^{t_2+T} G(t_2, s) f(s, x(s - \tau(s))) ds - \int_{t_1}^{t_1+T} G(t_1, s) f(s, x(s - \tau(s))) ds \right| \\
&+ \left| \int_{t_2}^{t_2+T} G(t_2, s) H\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \right. \\
&\left. - \int_{t_1}^{t_1+T} G(t_1, s) H\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \right| \\
&+ \left| \int_{t_2}^{t_2+T} G(t_2, s) ca(s) x(s - \tau(s)) ds - \int_{t_1}^{t_1+T} G(t_1, s) ca(s) x(s - \tau(s)) ds \right| \\
&+ \left| cy(t_2 - \tau(t_2)) - cy(t_1 - \tau(t_1)) \right|.
\end{aligned}$$

By using (2.7), (2.9) and (2.10), we obtain

$$\begin{aligned}
&\left| \int_{t_2}^{t_2+T} G(t_2, s) f(s, x(s - \tau(s))) ds - \int_{t_1}^{t_1+T} G(t_1, s) f(s, x(s - \tau(s))) ds \right| \\
&\leq \left| \int_{t_2}^{t_1} G(t_2, s) f(s, x(s - \tau(s))) ds \right| + \left| \int_{t_1+T}^{t_2+T} G(t_2, s) f(s, x(s - \tau(s))) ds \right| \\
&+ \int_{t_1}^{t_1+T} |G(t_2, s) - G(t_1, s)| f(s, x(s - \tau(s))) ds. \\
&\leq (2 + a_1T) \eta_2 (kM + f_1) |t_2 - t_1|.
\end{aligned} \tag{3.5}$$

From (2.7), (2.9) and (2.11), we get

$$\begin{aligned}
&\left| \int_{t_2}^{t_2+T} G(t_2, s) H\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds - \int_{t_1}^{t_1+T} G(t_1, s) H\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \right| \\
&\leq \left| \int_{t_2}^{t_1} G(t_2, s) H\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \right| + \left| \int_{t_1+T}^{t_2+T} G(t_2, s) H\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \right| \\
&+ \int_{t_1}^{t_1+T} |G(t_2, s) - G(t_1, s)| H\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds, \\
&\leq (2 + a_1T)(M\Lambda + H_1) \eta_2 |t_2 - t_1|.
\end{aligned} \tag{3.6}$$

In view of (2.7) we obtain

$$\begin{aligned}
&\left| \int_{t_2}^{t_2+T} G(t_2, s) ca(s) x(s - \tau(s)) ds - \int_{t_1}^{t_1+T} G(t_1, s) ca(s) x(s - \tau(s)) ds \right| \\
&\leq \left| \int_{t_2}^{t_1} G(t_2, s) ca(s) x(s - \tau(s)) ds \right| + \left| \int_{t_1+T}^{t_2+T} G(t_2, s) ca(s) x(s - \tau(s)) ds \right| \\
&+ \int_{t_1}^{t_1+T} |G(t_2, s) - G(t_1, s)| ca(s) x(s - \tau(s)) ds \\
&\leq (2 + a_1T) \eta_2 cMa_1 |t_2 - t_1|.
\end{aligned} \tag{3.7}$$

Since $\tau \in P_T(L,m,M)$, then

$$\begin{aligned} |cy(t_2-\tau(t_2)) - cy(t_1-\tau(t_1))| &\le c\,|y(t_2-\tau(t_2)) - y(t_1-\tau(t_1))| \\ &\le Lc\,|t_2-t_1+\tau(t_2)-\tau(t_1)| \\ &\le Lc\,(|t_2-t_1| + L\,|t_2-t_1|) \\ &\le L(1+L)\,c\,|t_2-t_1|. \end{aligned} \tag{3.8}$$

Thus, it follows from (2.4) and (3.5)-(3.8) that

$$|((F_1x)+(F_2y))(t_2) - ((F_1x)+(F_2y))(t_1)| \le L\,|t_2-t_1|, \tag{3.9}$$

for all $t_1, t_2 \in \mathbb{R}$ and $x, y \in P_T(L,m,M)$.
According to (3.4) and (3.9) we conclude the desired result. □

Now we are ready to present our first existence theorem.

Theorem 3.3. Let $\tau \in P_T(L,m,M)$. If conditions (2.1)-(2.4) hold, then equation (1.1) has at least one positive periodic solution in $P_T(L,m,M)$.

Proof . Based on Lemmas 3.1 and 3.2, the fact that $P_T(L,m,M)$ is a compact subset of P_T and that F_2 is a contraction, we conclude by the Krasnoselskii's fixed point theorem that there exists at least $x \in P_T(L,m,M)$ satisfies $N(x(t)) = x(t)$. Thanks to Lemma 2.1, x is a solution of equation (1.1). □

4 Existence, uniqueness and stability

4.1 Existence and uniqueness

Theorem 4.1. Let $\tau \in P_T(L,m,M)$. If conditions (2.1)-(2.4) and the following estimate:

$$T\eta_2(k+\Lambda+ca_1)+c<1, \tag{4.1}$$

are fulfilled, then equation (1.1) has a unique positive periodic solution $x \in P_T(L,m,M)$.

Proof . Let $x, y \in P_T(L,m,M)$. By repeating the same steps as those in the proof of Lemma 3.2, we infer that $N(P_T(L,m,M)) \subset P_T(L,m,M)$ and similarly as in the proof of Lemma 3.1, we get

$$\|(Nx)-Ny\| \le (T\eta_2(k+\Lambda+ca_1)+c)\,\|x-y\|.$$

According to (4.1) and the Banach fixed point theorem, N is a contraction and hence N has a unique fixed point which is the unique solution of (1.1). □

4.2 Stability

Remark 4.2. If

$$G_1(t,s) = \frac{\exp\left(\int_t^s \upsilon_1(u)\,du\right)}{\exp\left(\int_0^T \upsilon_1(u)\,du\right)-1},\ G_2(t,s) = \frac{\exp\left(\int_t^s \upsilon_2(u)\,du\right)}{\exp\left(\int_0^T \upsilon_2(u)\,du\right)-1},$$

then

$$\int_t^{t+T} |G_1(t,s)-G_2(t,s)|\,ds \le \mu\,\|\upsilon_1-\upsilon_2\|, \tag{4.2}$$

where

$$\mu = \frac{T^2\exp\left(T\left(\|\upsilon_2\|+\max\left(\|\upsilon_1\|,\|\upsilon_2\|\right)\right)\right)}{\left(\exp\left(\int_0^T \upsilon_1(u)\,du\right)-1\right)\left(\exp\left(\int_0^T \upsilon_2(u)\,du\right)-1\right)} + \frac{T^2\exp\left(T\max\left(\|\upsilon_1\|,\|\upsilon_2\|\right)\right)}{\exp\left(\int_0^T \upsilon_1(u)\,du\right)-1}.$$

Theorem 4.3. The unique solution obtained in Theorem 4.1 depends continuously on functions a, f and H.

Proof . If G_1 and G_2 are given as in Remark 4.2, let

$$x_1(t) = \int_t^{t+T} G_1(t,s)\Big[f_1(s, x_1(s-\tau(s))) - H_1\Big(s, x_1(s), x_1^{[2]}(s), ..., x_1^{[n]}(s)\Big) \\ -c\upsilon_1(s)\, x_1(s-\tau(s))\Big]\, ds + c x_1(t-\tau(t)),$$

and

$$x_2(t) = \int_t^{t+T} G_2(t,s)\Big[f_2(s, x_2(s-\tau(s))) - H_2\Big(s, x_2(s), x_2^{[2]}(s), ..., x_2^{[n]}(s)\Big) \\ -c\upsilon_2(s)\, x_2(s-\tau(s))\Big]\, ds + c x_2(t-\tau(t)),$$

be two different solutions of equation (1.1). We have

$$\begin{aligned} |x_1(t) - x_2(t)| \leq & \int_t^{t+T} |G_1(t,s) f_1(s, x_1(s-\tau(s))) - G_2(t,s) f_2(s, x_2(s-\tau(s)))|\, ds \\ & + \int_t^{t+T} \Big| G_1(t,s) H_1\Big(s, x_1(s), x_1^{[2]}(s), ..., x_1^{[n]}(s)\Big) \\ & - G_2(t,s) H_2\Big(s, x_2(s), x_2^{[2]}(s), ..., x_2^{[n]}(s)\Big)\Big|\, ds \\ & + \int_t^{t+T} |G_1(t,s) c\upsilon_1(s) x_1(s-\tau(s)) - G_2(t,s) c\upsilon_2(s) x_2(s-\tau(s))|\, ds \\ & + |c x_1(t-\tau(t)) - c x_2(t-\tau(t))|. \end{aligned}$$

Using (1.2), (1.3), (2.7), (2.10), (2.11) and (4.2), we get

$$\begin{aligned} & \int_t^{t+T} |G_1(t,s) f_1(s, x_1(s-\tau(s))) - G_2(t,s) f_2(s, x_2(s-\tau(s)))|\, ds \\ & \leq \int_t^{t+T} G_1(t,s) |f_1(s, x_1(s-\tau(s))) - f_2(s, x_1(s-\tau(s)))|\, ds \\ & + \int_t^{t+T} f_2(s, x_1(s-\tau(s))) |G_1(t,s) - G_2(t,s)|\, ds \\ & + \int_t^{t+T} G_2(t,s) |f_2(s, x_1(s-\tau(s))) - f_2(s, x_2(s-\tau(s)))|\, ds \\ & \leq T\eta_2 \|f_1 - f_2\| + \mu(kM + f_1)\|\upsilon_1 - \upsilon_2\| + T\eta_2 k \|x_1 - x_2\|, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} & \int_t^{t+T} \Big| G_1(t,s) H_1\Big(s, x_1(s), x_1^{[2]}(s), ..., x_1^{[n]}(s)\Big) - G_2(t,s) H_2\Big(s, x_2(s), x_2^{[2]}(s), ..., x_2^{[n]}(s)\Big)\Big|\, ds \\ & \leq \int_t^{t+T} G_1(t,s) \Big| H_1\Big(s, x_1(s), x_1^{[2]}(s), ..., x_1^{[n]}(s)\Big) - H_2\Big(s, x_1(s), x_1^{[2]}(s), ..., x_1^{[n]}(s)\Big)\Big|\, ds \\ & + \int_t^{t+T} H_2\Big(s, x_1(s), x_1^{[2]}(s), ..., x_1^{[n]}(s)\Big) |G_1(t,s) - G_2(t,s)|\, ds \\ & + \int_t^{t+T} G_2(t,s) \Big| H_2\Big(s, x_1(s), x_1^{[2]}(s), ..., x_1^{[n]}(s)\Big) - H_2\Big(s, x_2(s), x_2^{[2]}(s), ..., x_2^{[n]}(s)\Big)\Big|\, ds \\ & \leq T\eta_2 \|H_1 - H_2\| + \mu(\Lambda M + H_1)\|\upsilon_1 - \upsilon_2\| + T\eta_2 \Lambda \|x_1 - x_2\|. \end{aligned} \tag{4.4}$$

On the other hand, we have

$$\begin{aligned}
&\int_t^{t+T} |G_1(t,s)\, c\upsilon_1(s)\, x_1(s-\tau(s)) - G_2(t,s)\, c\upsilon_2(s)\, x_2(s-\tau(s))|\, ds \\
&\le \int_t^{t+T} c\upsilon_1(s)\, G_1(t,s)\, |x_1(s-\tau(s)) - x_2(s-\tau(s))|\, ds \\
&+ \int_t^{t+T} c\upsilon_1(s)\, x_2(s-\tau(s))\, |G_1(t,s) - G_2(t,s)|\, ds \\
&+ \int_t^{t+T} cx_2(s-\tau(s))\, G_2(t,s)\, |\upsilon_1(s) - \upsilon_2(s)|\, ds \\
&\le T\eta_2 c\,\|\upsilon_1\|\,\|x_1 - x_2\| + c\,\|\upsilon_1\|\, M\mu\,\|\upsilon_1 - \upsilon_2\| + T\eta_2 cM\,\|\upsilon_1 - \upsilon_2\|,
\end{aligned} \tag{4.5}$$

and

$$|cx_1(t-\tau(t)) - cx_2(t-\tau(t))| \le c\,\|x_1 - x_2\|. \tag{4.6}$$

Thanks to (4.3)-(4.6), we obtain

$$\begin{aligned}
\|x_1 - x_2\| \le\; & T\eta_2\,\|f_1 - f_2\| + \mu(kM + f_1)\,\|\upsilon_1 - \upsilon_2\| + T\eta_2 k\,\|x_1 - x_2\| \\
&+ T\eta_2\,\|H_1 - H_2\| + \mu(\Lambda M + H_1)\,\|\upsilon_1 - \upsilon_2\| + T\eta_2\Lambda\,\|x_1 - x_2\| \\
&+ T\eta_2 c\,\|\upsilon_1\|\,\|x_1 - x_2\| + c\,\|\upsilon_1\|\, M\mu\,\|\upsilon_1 - \upsilon_2\| + T\eta_2 cM\,\|\upsilon_1 - \upsilon_2\| \\
&+ c\,\|x_1 - x_2\|.
\end{aligned}$$

Taking into account (4.1) we get the following estimate:

$$\begin{aligned}
\|x_1 - x_2\| \le\; & \frac{1}{1 - T\eta_2(k + \Lambda + c\upsilon_1)} \left(T\eta_2\,\|f_1 - f_2\| + T\eta_2\,\|H_1 - H_2\|\right. \\
& \left.(\mu(kM + f_1) + \mu(\Lambda M + H_1) + \mu c\,\|\upsilon_1\|\, M + T\eta_2 cM)\,\|\upsilon_1 - \upsilon_2\|\right),
\end{aligned}$$

which completes the proof. □

5 Example

In this section, we are going to perform an illustrating application of our obtained findings.

Example 5.1. Consider the following neutral differential equation with an iterative harvesting term:

$$\begin{aligned}
\frac{d}{dt}\left[x(t) - 0.001x(t-\tau(t))\right] =& -\left(\frac{1}{70} + \frac{1}{70}\sin^4\left(\frac{2\pi}{35}t\right)\right) x(t) \\
&+ \left(\frac{1}{9\pi^3} + \frac{1}{36\pi^3}\sin^2\left(\frac{2\pi}{35}t\right) + \frac{1}{7\pi^3}\sin^2\left(\frac{2\pi}{35}t\right) x(t-\tau(t))\right) \\
&- \left(\frac{1}{9\pi^8}\sin^2\left(\frac{2\pi}{35}t\right) + \frac{1}{20\pi^8}\sin^2\left(\frac{2\pi}{35}t\right) x(t) + \frac{1}{30\pi^8}\sin^2\left(\frac{2\pi}{35}t\right) x^{[2]}(t)\right),
\end{aligned} \tag{5.1}$$

where $m = 0.05$, $M = 1.5$, $L = \pi$, i.e. $P_T(L, m, M) = P_{35}(\pi, 0.05, 1.5)$ and $c = 0.001$.
We have

$$\begin{aligned}
&l_1 = \frac{1}{20\pi^8},\; l_2 = \frac{1}{30\pi^8},\; H_1 = \frac{1}{9\pi^8},\; \Lambda \simeq 1.981\,9 \times 10^{-5}, \\
&f_0 = \frac{1}{9\pi^3},\; f_1 = \frac{5}{36\pi^3},\; k = \frac{1}{7\pi^3},\; a_1 = \frac{1}{35},\; \eta_1 \simeq 0.50856,\; \eta_2 \simeq 2.0114.
\end{aligned}$$

So

$$\begin{aligned}
&cM + \eta_2 T(kM + f_1) = 0.80337 < M = 1.5, \\
&\eta_1 T f_0 - \eta_2 T(M\Lambda + H_1) - cT\eta_2 a_1 M + cm = 0.05.7901 > m = 0.05, \\
&\eta_2(2 + a_1 T)((kM + f_1) + (M\Lambda + H_1) + cMa_1) + L(1 + L)c \simeq 0.082252 \le L = \pi,
\end{aligned}$$

and

$$\eta_2 Tk + \eta_2 T\Lambda + \eta_2 Tca_1 + c \simeq 0.32876 < 1.$$

Then all conditions of Theorems 3.3 and 4.1 are satisfied. Thereby equation (5.1) has one and only one positive periodic solution in $P_{35}(\pi, 0.05, 1.5)$ that depends continuously on a, f and H.

6 Conclusion

This article is concerned with a first order neutral differential equation with iterative terms arising in biology and population dynamics. We have obtained some results regarding the existence, uniqueness and continuous dependence on parameters of positive periodic and bounded solutions for the addressed equation. Our key step in this work was the choice of the space and its subset which have fulfilled all our requirements whether biological or mathematical. The second step is the conversion of the equation (1.1) into an integral one whose solutions were solutions of the proposed equation and vice versa. So, by means of the Krasnoselskii's fixed point theorem as well as some properties of a Green's function, we established the existence of the solutions by defining an integral operator written as a sum of two operators, one of them is a contraction and the other is continuous and compact. While, for the existence and stability of the unique solution, it was handy that we used Banach fixed point theorem.

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