Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 1105–1115
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.23420.2537

Fekete-Szegö functional for regular functions based on quasi-subordination

Sondekola Rudra Swamy[a], Şahsene Altınkaya[b,*]

[a]Department of Computer Science and Engineering, RV College of Engineering, Bengaluru- 560 059, Karnataka, India

[b]Department of Mathematics, Faculty of Arts-Science, Beykent University, 34500 Istanbul, Turkey

(Communicated by Mugur Alexandru Acu)

Abstract

Considering two special families of regular functions in an open unit disk based on quasi-subordination, we present sharp bounds for initial coefficient estimates and also determine the classical functional of Fekete-Szegö of functions in these families. Further, we discuss subordination and majorization results for the associated families. Few known and several new consequences are established.

Keywords: Regular functions, Fekete-Szegö functional, quasi-subordination, majorization, subordination
2020 MSC: 30C45

1 Introduction

Let A be the family of normalized regular functions in $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$ that have the form

$$s(z) = z + \sum_{k=2}^{\infty} d_k z^k, \tag{1.1}$$

where $\mathbb{C}$ is the set of complex numbers. Let $\mathcal{S}$ be the set of all members in A that are univalent in $\mathfrak{D}$. Let $\eta(z)$ be regular function in $\mathfrak{D}$ with $|\eta(z)| \le 1,\ (z \in \mathfrak{D})$ such that

$$\eta(z) = \mathfrak{R}_0 + \mathfrak{R}_1 z + \mathfrak{R}_2 z^2 + ..., \tag{1.2}$$

where $\mathfrak{R}_0, \mathfrak{R}_1, \mathfrak{R}_2, ...$ are real. Let $\mathfrak{h}(z)$ be regular in $\mathfrak{D}$ with $\mathfrak{h}'(0) > 0, \mathfrak{h}(0) = 1$ and with positive real part such that

$$\mathfrak{h}(z) = 1 + \mathbb{Q}_1 z + \mathbb{Q}_2 z^2 + ..., \tag{1.3}$$

where $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3, ...$ are real and $\mathbb{Q}_1 > 0$. Through out this work we shall assume that the functions η and $\mathfrak{h}$ follow the above conditions unless otherwise mentioned.

It is known that for $s \in \mathcal{S}$ given by (1.1), there holds upper bounds for $|d_3 - \mu d_2^2|$ when μ is real, which are sharp (see[8]). Since then, the estimation of the sharp upper bounds for $|d_3 - \mu d_2^2|$ with μ being an arbitrary real or

*Corresponding author
Email addresses: sondekola.swamy@gmail.com (Sondekola Rudra Swamy), sahsenealtinkaya@gmail.com (Şahsene Altınkaya)

Received: May 2021 *Accepted:* July 2021

complex number for any compact collection $\mathfrak{F}$ of elements in $\mathcal{S}$ is the classical problem of Fekete-Szegö for $\mathfrak{F}$. Several researchers including [2],[4],[5],[6], [13],[18] and [24] have estimated sharp Fekete-Szegö bounds for many subfamilies of $\mathcal{S}$. Additional information about the classical Fekete-Szegö functional linked with q-derivative operator are available in the works of Alsoboh and Darus [1], Elhaddad and Darus [7]. Very interesting resource about Fekete-Szegö inequality linked with the Horadam polynomials may be found in the investigation by Srivastava et al. [19].

We recall the principle of subordination and also the rule of majorization, between two regular functions $s(z)$ and $\nu(z)$ in $\mathfrak{D}$. We say that $s(z)$ is subordinate to $\nu(z)$, written $s(z) \prec \nu(z)$, $z \in \mathfrak{D}$, if there is a regular function $\omega(z)$ in $\mathfrak{D}$, with $|\omega(z)| < 1$ and $\omega(0) = 0$ such that $s(z) = \nu(\omega(z))$. Moreover $s(z) \prec \nu(z)$ is equivalent to $s(0) = \nu(0)$ and $s(\mathfrak{D}) \subset \nu(\mathfrak{D})$, if ν is univalent in $\mathfrak{D}$. We know that $s(z)$ is majorized by $\nu(z)$, written $s(z) \prec\prec \nu(z)$, $z \in \mathfrak{D}$, if there exists a regular function $\eta(z)$, $z \in \mathfrak{D}$ with $|\eta(z)| \leq 1$ such that $s(z) = \eta(z)\nu(z)$.

A new concept called quasi-subordination due to Robertson [17] generalizes both subordination and majorization. For any two regular functions $s(z)$ and $\nu(z)$, $s(z)$ is quasi-subordinate to $\nu(z)$, written as $s(z) \prec_q \nu(z)$, if there exists regular functions ω and η with $\omega(0) = 0$, $|\omega(z)| < 1$ *and* $|\eta(z)| \leq 1$ such that $s(z) = \eta(z)\nu(\omega(z)), z \in \mathfrak{D}$. Note that if $\eta(z) = 1$, then $s(z) = \nu(\omega(z)), z \in \mathfrak{D}$ so that $s(z) \prec \nu(z)$ in $\mathfrak{D}$. Also observe that if $\omega(z) = z$, then $s(z) = \eta(z)\nu(z)$, $z \in \mathfrak{D}$ and hence, $s(z) \prec\prec \nu(z)$ in $\mathfrak{D}$.

In the literature, the estimates on $|d_2|$, $|d_3|$ and the classical Fekete-Szegö inequality were found for regular functions based on quasi-subordination. More studies about these can be found in the works of [3],[9],[11],[12],[15],[16] and [21].

Let Υ be the collection of regular functions in $\mathfrak{D}$ of the form

$$\omega(z) = \omega_1 z + \omega_2 z^2 + \omega_3 z^3 + ... \tag{1.4}$$

satisfying $|\omega(z)| < 1$, $z \in \mathfrak{D}$.

We now state the lemma due to Keogh and Merkes [10], which is used to prove our main result.

Lemma 1.1. If $\omega \in \Upsilon$, then for any $\mu \in \mathbb{C}$, we have $|\omega_1| \leq 1$, $|\omega_2 - \mu\omega_1^2| \leq 1 + (|\mu| - 1)|\omega_1|^2 \leq max\{1, |\mu|\}$. $\omega(z) = z$ or $\omega(z) = z^2$ exhibit the sharpness of the result.

Motivated by the papers [20],[23] and earlier works on quasi-subordination, we now define two new special classes $M_q(\tau, \gamma, \mu, \mathfrak{h})$ and $\mathfrak{B}_q(\tau, \xi, \gamma, \mathfrak{h})$.

Definition 1.2. A function s in A is said to be in $M_q(\tau, \xi, \gamma, \mathfrak{h})$, $0 \leq \xi \leq 1$, $\tau \geq 0$, $\gamma \in \mathbb{C} - \{0\}$, if

$$\frac{1}{\gamma}\left(\frac{\tau z^2 s''(z) + zs'(z)}{\xi s(z) + (1-\xi)z} - 1\right) \prec_q \mathfrak{h}(z) - 1, z \in \mathfrak{D},$$

where $\mathfrak{h}$ is as stated in (1.3).

Clearly, a function s is in $M_q(\tau, \xi, \gamma, \mathfrak{h})$ if and only if there exits a regular function $\eta(z)$ with $|\eta(z)| \leq 1$, $z \in \mathfrak{D}$ such that

$$\frac{\frac{1}{\gamma}\left(\frac{\tau z^2 s''(z)+zs'(z)}{\xi s(z)+(1-\xi)z} - 1\right)}{\eta(z)} \prec \mathfrak{h}(z) - 1, z \in \mathfrak{D},$$

where $\mathfrak{h}$ is as stated in (1.3).

If we set $\eta(z) \equiv 1$, then $M_q(\tau, \xi, \gamma, \mathfrak{h})$ is denoted by $M(\tau, \xi, \gamma, \mathfrak{h})$ satisfying

$$1 + \frac{1}{\gamma}\left(\frac{\tau z^2 s''(z) + zs'(z)}{\xi s(z) + (1-\xi)z} - 1\right) \prec \mathfrak{h}(z), z \in \mathfrak{D}.$$

We note that i) $\tau = 0$, ii) $\xi = 0$ and iii) $\xi = 1$ lead the family $M_q(\tau, \xi, \gamma, \mathfrak{h})$ to the below mentioned subfamilies:

1. $\mathfrak{K}_q(\xi,\ \gamma,\ \mathfrak{h})$ is the set of functions $s \in A$ satisfying

$$\frac{1}{\gamma}\left(\frac{zs'(z)}{\xi s(z) + (1-\xi)z} - 1\right) \prec_q \mathfrak{h}(z) - 1.$$

2. $\mathfrak{L}_q(\tau,\ \gamma,\ \mathfrak{h})$ is the family of functions $s \in A$ satisfying

$$\frac{1}{\gamma}\left(s'(z) + \tau z s''(z) - 1\right) \prec_q) \prec_q \mathfrak{h}(z) - 1.$$

3. $\mathfrak{M}_q(\tau, \gamma, \mathfrak{h})$ is another family of functions $s \in A$ satisfying

$$\frac{1}{\gamma}\left(\tau\left(\frac{zs''(z)}{s(z)}\right)+\left(\frac{zs'(z)}{s(z)}\right)-1\right) \prec_q \mathfrak{h}(z)-1.$$

Definition 1.3. A function $s \in A$ having the power series (1.1) is said to be in the family $\mathfrak{B}_q(\alpha, \xi, \gamma, \mathfrak{h})$, $0 \leq \gamma \leq 1$, $\alpha \geq 1$ and $\xi \in \mathbb{C} - \{0\}$, if

$$\frac{1}{\gamma}\left(\frac{z[s'(z)]^{\alpha}}{\xi s(z)+(1-\xi)z}-1\right) \prec_q \mathfrak{h}(z)-1, z \in \mathfrak{D},$$

where $\mathfrak{h}$ is as stated in (1.3).

Clearly, a function s is in $\mathfrak{B}_q(\alpha, \xi, \gamma, \mathfrak{h})$ if and only if there exits a regular function $\eta(z)$ with $|\eta(z)| \leq 1$, $z \in \mathfrak{D}$ such that

$$\frac{\frac{1}{\gamma}\left(\frac{z[s'(z)]^{\alpha}}{\xi s(z)+(1-\xi)z}-1\right)}{\eta(z)} \prec \mathfrak{h}(z)-1, z \in \mathfrak{D},$$

where $\mathfrak{h}$ is as stated in (1.3).

If we set $\eta(z) \equiv 1$, then $\mathfrak{B}_q(\alpha, \xi, \gamma, \mathfrak{h})$ is denoted by $\mathfrak{B}(\alpha, \xi, \gamma, \mathfrak{h})$ satisfying

$$1+\frac{1}{\gamma}\left(\frac{z[s'(z)]^{\alpha}}{\xi s(z)+(1-\xi)z}-1\right) \prec \mathfrak{h}(z), z \in \mathfrak{D}.$$

We note that i) $\alpha = 1$, ii) $\xi = 0$ and iii) $\xi = 1$ lead the family $\mathfrak{B}_q(\alpha, \xi, \gamma, \mathfrak{h})$ to the below mentioned subfamilies:

1. $\mathfrak{G}_q(\xi, \gamma, \mathfrak{h})$ is the set of functions $s \in A$ satisfying

$$\frac{1}{\gamma}\left(\frac{zs'(z)}{\xi s(z)+(1-\xi)z}-1\right) \prec_q \mathfrak{h}(z)-1, z \in \mathfrak{D}.$$

2. $\mathfrak{P}_q(\alpha, \gamma, \mathfrak{h})$ is the set of functions $s \in A$ satisfying

$$[s'(z)]^{\alpha} \prec_q \mathfrak{h}(z)-1.$$

3. $\mathfrak{N}_q(\alpha, \gamma, \mathfrak{h})$ is the class of functions $s \in A$ satisfying

$$\frac{z[s'(z)]^{\alpha}}{s(z)} \prec_q, \mathfrak{h}(z)-1.$$

Two families $M_q(\tau, \xi, \gamma, \mathfrak{h})$ and $\mathfrak{B}_q(\tau, \xi, \gamma, \mathfrak{h})$ are of special interest. In view of this, we deem it worth while to note the relevance of these with subclasses defined above as well as some known ones. Indeed we have $i)M_q(0, \xi, \gamma, \mathfrak{h}) = \mathfrak{K}_q(\xi, \gamma, \mathfrak{h})$, $ii)M_q(\tau, 0, \gamma, \mathfrak{h}) = \mathfrak{L}_q(\tau, \gamma, \mathfrak{h})$ [22], $iii)M_q(\tau, 1, \gamma, \mathfrak{h}) = \mathfrak{M}_q(\tau, \gamma, \mathfrak{h})$, $iv)M_q(0, 1, 1, \mathfrak{h}) = S_q(\mathfrak{h})$[15] , $v)M_q(1, 0, \gamma, \mathfrak{h}) = P_q(\mathfrak{h})$ [9], $vi)\mathfrak{B}_q(0, \xi, \gamma, \mathfrak{h}) = \mathfrak{G}_q(\xi, \gamma, \mathfrak{h})$, $vii)\mathfrak{B}_q(\alpha, 0, \gamma, \mathfrak{h}) = \mathfrak{P}_q(\alpha, \gamma, \mathfrak{h})$ and $viii)\mathfrak{B}_q(\alpha, 1, \gamma, \mathfrak{h}) = \mathfrak{N}_q(\alpha, \gamma, \mathfrak{h})$.

2 Fekete-Szegö results for the class $M_q(\tau, \xi, \gamma, \mathfrak{h})$

In this section, we derive the estimates for $|d_2|$ and Fekete-Szegö functional $|d_3 - \mu d_2^2|$ for elements in $M_q(\tau, \xi, \gamma, \mathfrak{h})$. Few known and several new consequences of this result are pointed out.

Theorem 2.1. Let $0 \leq \xi \leq 1, \tau \geq 0$ and $\gamma \in \mathbb{C} - \{0\}$. If the function $s \in M_q(\tau, \xi, \gamma, \mathfrak{h})$, then

$$|d_2| \leq \frac{|\gamma|\mathbb{Q}_1}{2(\tau+1)-\xi} \tag{2.1}$$

and for any complex number $\mu \in \mathbb{C}$,

$$|d_3 - \mu d_2^2| \leq \frac{|\gamma|\mathbb{Q}_1}{3(2\tau+1)-\xi} max\left(1, \left|\mathfrak{I}\mathbb{Q}_1 - \frac{\mathbb{Q}_2}{\mathbb{Q}_1}\right|\right), \tag{2.2}$$

where

$$\mathfrak{I} = \gamma\left(\frac{\mu(3(2\tau+1)-\xi)}{(2(\tau+1)-\xi)^2} - \frac{\xi}{2(\tau+1)-\xi}\right). \tag{2.3}$$

Proof . Let $s \in M_q(\tau,\xi,\gamma,\mathfrak{h})$. Then there exists $\omega(z)$, a Schwarz function and $\eta(z)$, a regular function, such that

$$\frac{1}{\gamma}\left(\frac{\tau z^2 s''(z)+zs'(z)}{\xi s(z)+(1-\xi)z}-1\right)=\eta(z)(\mathfrak{h}(\omega(z))-1),\ z\in\mathfrak{D}. \tag{2.4}$$

Series expansions of s and its successive derivatives from (1.1) gives

$$\begin{aligned}&\frac{1}{\gamma}\left(\frac{\tau z^2 s''(z)+zs'(z)}{\xi s(z)+(1-\xi)z}-1\right)=\\&\quad\frac{1}{\gamma}\left[(2(\tau+1)-\xi)d_2 z+\left((3(2\tau+1)-\xi)d_3-(2(\tau+1)-\xi)\xi d_2^2\right)z^2+\ldots\right].\end{aligned} \tag{2.5}$$

Similarly from (1.2), (1.3) and (1.4), we obtain

$$\eta(z)(\mathfrak{h}(\omega(z))-1)=\mathfrak{R}_0\mathbb{Q}_1\omega_1 z+\left[\mathfrak{R}_1\mathbb{Q}_1\omega_1+\mathfrak{R}_0(\mathbb{Q}_1\omega_2+\mathbb{Q}_2\omega_1^2)\right]z^2+\ldots. \tag{2.6}$$

Using (2.5) and (2.6) in (2.4), we get

$$d_2=\frac{\gamma\mathfrak{R}_0\mathbb{Q}_1\omega_1}{2(\tau+1)-\xi} \tag{2.7}$$

and

$$d_3=\frac{\gamma\mathbb{Q}_1}{3(2\tau+1)-\xi}\left[\mathfrak{R}_1\omega_1+\mathfrak{R}_0\left\{\omega_2+\left(\frac{\xi\gamma\mathfrak{R}_0\mathbb{Q}_1}{2(\tau+1)-\xi}+\frac{\mathbb{Q}_2}{\mathbb{Q}_1}\right)\omega_1^2\right\}\right]. \tag{2.8}$$

Thus, for any $\mu\in\mathbb{C}$, we get from (2.7) and (2.8)

$$d_3-\mu d_2^2=\frac{\gamma\mathbb{Q}_1}{3(2\tau+1)-\xi}\left[\mathfrak{R}_1\omega_1+\left(\omega_2+\frac{\mathbb{Q}_2}{\mathbb{Q}_1}\omega_1^2\right)\mathfrak{R}_0-\mathfrak{J}\mathbb{Q}_1\mathfrak{R}_0^2\omega_1^2\right], \tag{2.9}$$

where $\mathfrak{J}$ is as stated in (2.3).

Since $\eta(z)$ is a regular function bounded by one in $\mathfrak{D}$, we have (see [14],p.172)

$$|\mathfrak{R}_0|\leq 1 \quad\text{and}\quad \mathfrak{R}_1=(1-\mathfrak{R}_0^2)x \qquad x\leq 1. \tag{2.10}$$

The assertion (2.1) follows from (2.7) using (2.10) and Lemma 1.1. From (2.9) and (2.10), we obtain

$$d_3-\mu d_2^2=\frac{\gamma\mathbb{Q}_1}{3(2\tau+1)-\xi}\left[x\omega_1+\left(\omega_2+\frac{\mathbb{Q}_2}{\mathbb{Q}_1}\omega_1^2\right)\mathfrak{R}_0-(\mathfrak{J}\mathbb{Q}_1\omega_1^2+x\omega_1)\mathfrak{R}_0^2\right]. \tag{2.11}$$

If $\mathfrak{R}_0=0$, then (2.11) yields

$$|d_3-\mu d_2^2|\leq\frac{|\gamma|\mathbb{Q}_1}{3(2\tau+1)-\xi}. \tag{2.12}$$

On the other side, if $\mathfrak{R}_0\neq 0$, we define a function

$$L(\mathfrak{R}_0)=x\omega_1+\left(\omega_2+\frac{\mathbb{Q}_2}{\mathbb{Q}_1}\omega_1^2\right)R_0-(\mathfrak{J}\mathbb{Q}_1\omega_1^2+x\omega_1)\mathfrak{R}_0^2. \tag{2.13}$$

The equation (2.13) is a quadratic in $\mathfrak{R}_0$ and hence regular in $|\mathfrak{R}_0|\leq 1$. Clearly, $|L(\mathfrak{R}_0)|$ attains its maximum value at $\mathfrak{R}_0=e^{i\theta}$, $0\leq\theta\leq 2\pi$. Thus,

$$\begin{aligned}max|L(\mathfrak{R}_0)|&=\max_{0\leq\theta\leq 2\pi}|L(e^{i\theta})|=|L(1)|\\&=|\omega_2-\left(\mathfrak{J}\mathbb{Q}_1-\frac{\mathbb{Q}_2}{\mathbb{Q}_1}\right)\omega_1^2|.\end{aligned}$$

Therefore, it follows from (2.11) that

$$|d_3-\mu d_2^2|\leq\frac{|\gamma|\mathbb{Q}_1}{3(2\tau+1)-\xi}\left|\omega_2-\left(\mathfrak{J}\mathbb{Q}_1-\frac{\mathbb{Q}_2}{\mathbb{Q}_1}\right)\omega_1^2\right|. \tag{2.14}$$

By virtue of Lemma 1.1, we obtain

$$|d_3-\mu d_2^2|\leq\frac{|\gamma|\mathbb{Q}_1}{3(2\tau+1)-\xi}max\left(1,\left|\mathfrak{J}\mathbb{Q}_1-\frac{\mathbb{Q}_2}{\mathbb{Q}_1}\right|\right). \tag{2.15}$$

The assertion (2.2) now follows from (2.12) and (2.15). This ends the proof. □

Taking $\tau=0$ in Theorem 2.1, we arrive at the following outcome.

Corollary 2.2. Let $0 \leq \xi \leq 1$ *and* $\gamma \in \mathbb{C} - \{0\}$. If the function $s \in \mathfrak{K}_q(\xi, \gamma, \mathfrak{h})$, then $|d_2| \leq \frac{|\gamma|\mathbb{Q}_1}{2-\xi}$ and for some $\mu \in \mathbb{C}$, $|d_3 - \mu d_2^2| \leq \frac{|\gamma|\mathbb{Q}_1}{3-\xi} max\left(1, \left|\gamma\left(\frac{(\mu-\xi)(2-\xi)+\mu}{(2-\xi)^2}\right)\mathbb{Q}_1 - \frac{\mathbb{Q}_2}{\mathbb{Q}_1}\right|\right)$.

Remark 2.3. For $\xi = 1$ and $\gamma = 1$, Corollary 2.2 reduces to Corollary 2.2 of [15].

We conclude the following result for the class $\mathfrak{L}_q(\tau, \gamma, \mathfrak{h})$, on putting $\xi = 0$ in Theorem 2.1.

Corollary 2.4. [22] Let $\gamma \in \mathbb{C} - \{0\}$ and $\tau \geq 0$. If the function $s \in \mathfrak{L}_q(\tau, \gamma, \mathfrak{h})$, then $|d_2| \leq \frac{|\gamma|\mathbb{Q}_1}{2(\tau+1)}$ and for some $\mu \in \mathbb{C}$, $|d_3 - \mu d_2^2| \leq \frac{|\gamma|\mathbb{Q}_1}{3(2\tau+1)} max\left(1, \left|\gamma\left(\frac{3\mu(2\tau+1)}{4(\tau+1)^2}\right)\mathbb{Q}_1 - \frac{\mathbb{Q}_2}{\mathbb{Q}_1}\right|\right)$.

Remark 2.5. For $\tau = 1$, Corollary 2.4 reduces to Corollary 1 of [9].

Allowing $\xi = 1$ in Theorem 2.1, we have the below outcome.

Corollary 2.6. Let $\gamma \in \mathbb{C} - \{0\}$ and $\tau \geq 0$. If the function $s \in \mathfrak{M}_q(\tau, \gamma, \mathfrak{h})$, then $|d_2| \leq \frac{|\gamma|\mathbb{Q}_1}{2\tau+1}$ and for some $\mu \in \mathbb{C}$, $|d_3 - \mu d_2^2| \leq \frac{|\gamma|\mathbb{Q}_1}{2(3\tau+1)} max\left(1, \left|\gamma\left(\frac{(2\tau+1)(2\mu-1)+2\tau\mu}{(2\tau+1)^2}\right)\mathbb{Q}_1 - \frac{\mathbb{Q}_2}{\mathbb{Q}_1}\right|\right)$.

Our next result is based on majorization.

Theorem 2.7. Let $\gamma \in \mathbb{C} - \{0\}, 0 \leq \xi \leq 1$ and $\tau \geq 0$. If the function $s \in A$ fulfills

$$\frac{1}{\gamma}\left(\frac{\tau z^2 s''(z) + z s'(z)}{\xi s(z) + (1-\xi)z} - 1\right) \prec\prec (\mathfrak{h}(z) - 1), z \in \mathfrak{D}, \tag{2.16}$$

then

$$|d_2| \leq \frac{|\gamma|\mathbb{Q}_1}{2(\tau+1) - \xi} \tag{2.17}$$

and for any complex number μ,

$$|d_3 - \mu d_2^2| \leq \frac{|\gamma|\mathbb{Q}_1}{3(2\tau+1) - \xi} max\left(1, \left|\mathfrak{J}\mathbb{Q}_1 - \frac{\mathbb{Q}_2}{\mathbb{Q}_1}\right|\right), \tag{2.18}$$

where $\mathfrak{J}$ is as stated by (2.3).

Proof . Assume that (2.16) holds. There exists a holomorphic function $\eta(z)$, from the principle of majorization, such that

$$\frac{1}{\gamma}\left(\frac{\tau z^2 s''(z) + z s'(z)}{\xi s(z) + (1-\xi)z} - 1\right) = \eta(z)(\mathfrak{h}(z) - 1).$$

Following the proof of Theorem 2.1, we obtain the desired results (2.17) and (2.18), by setting $\omega(z) = z$ (so that $\omega_1 = 1, \omega_n = 0, n \geq 2$). This completes the proof. □ Our next result is associated with $M(\tau, \xi, \gamma, \mathfrak{h})$

Theorem 2.8. Let $0 \leq \xi \leq 1, \tau \geq 0$ and $\gamma \in \mathbb{C} - \{0\}$. If $s \in M(\tau, \xi, \gamma, \mathfrak{h})$, then

$$|d_2| \leq \frac{|\gamma|\mathbb{Q}_1}{2(\tau+1) - \xi}$$

and for any $\mu \in \mathbb{C}$,

$$|d_3 - \mu d_2^2| \leq \frac{|\gamma|\mathbb{Q}_1}{3(2\tau+1) - \xi} max\left(1, \left|\mathfrak{J}\mathbb{Q}_1 - \frac{\mathbb{Q}_2}{\mathbb{Q}_1}\right|\right),$$

where $\mathfrak{J}$ is as stated in (2.3).

Proof . Let $s \in M(\tau, \xi, \gamma, \mathfrak{h})$. Taking $\eta(z) = 1$, $z \in \mathfrak{D}$, we get $\mathfrak{R}_0 = 1, \mathfrak{R}_n = 0, n \in N$ and by following the proof of Theorem 2.1, we attain the desired results, which ends the proof. □

We now settle the bounds of $|d_3 - \mu d_2^2|$ for real γ and μ, when $s \in M_q(\tau, \xi, \gamma, \mathfrak{h})$.

Theorem 2.9. Let $0 \leq \xi \leq 1$, $\tau \geq 0$. If the function $s \in M_q(\tau, \xi, \gamma, \mathfrak{h})$, then for real γ and μ, we have

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{|\gamma|\mathbb{Q}_1}{3(2\tau+1)-\xi}\left[\gamma\left(\frac{\xi}{2(\tau+1)-\xi} - \frac{\mu(3(2\tau+1)-\xi)}{(2(\tau+1)-\xi)^2}\right)\mathbb{Q}_1 + \frac{\mathbb{Q}_2}{\mathbb{Q}_1}\right] & (\mu \leq \rho) \\ \frac{|\gamma|\mathbb{Q}_1}{3(2\tau+1)-\xi} & (\rho \leq \mu \leq \rho + 2\sigma) \\ -\frac{|\gamma|\mathbb{Q}_1}{3(2\tau+1)-\xi}\left[\gamma\left(\frac{\xi}{2(\tau+1)-\xi} - \frac{\mu(3(2\tau+1)-\xi)}{(2(\tau+1)-\xi)^2}\right)\mathbb{Q}_1 + \frac{\mathbb{Q}_2}{\mathbb{Q}_1}\right] & (\mu \geq \rho + 2\sigma) \end{cases} \tag{2.19}$$

where

$$\rho = \frac{\xi(2(\tau+1)-\xi)}{3(2\tau+1)-\xi} - \frac{(2(\tau+1)-\xi)^2}{\gamma(3(2\tau+1)-\xi)}\left(\frac{1}{\mathbb{Q}_1} - \frac{\mathbb{Q}_2}{\mathbb{Q}_1^2}\right) \tag{2.20}$$

and

$$\sigma = \frac{(2(\tau+1)-\xi)^2}{\gamma(3(2\tau+1)-\xi)\mathbb{Q}_1}. \tag{2.21}$$

Proof . Let μ and γ be the real values. Then (2.19) can be obtained from (2.2) and (2.3), respectively, under the below cases:

$$\mathfrak{J}\mathbb{Q}_1 - \frac{\mathbb{Q}_2}{\mathbb{Q}_1} \leq -1, -1 \leq \mathfrak{J}\mathbb{Q}_1 - \frac{\mathbb{Q}_2}{\mathbb{Q}_1} \leq 1 \text{ and } \mathfrak{J}\mathbb{Q}_1 - \frac{\mathbb{Q}_2}{\mathbb{Q}_1} \geq 1,$$

where $\mathfrak{J}$ is as stated in (2.3). We also note the following:
Equality holds

(i) for $\mu < \rho$ or $\mu > \rho + 2\sigma$ if and only if $\eta(z) = 1$ and $w(z) = z$ or one of its rotations.

(ii) for $\rho < \mu < \rho + 2\sigma$ if and only if $\eta(z) = 1$ and $w(z) = z^2$ or one of its rotations.

(iii) for $\mu = \rho + 2\sigma$ if and only if $\eta(z) = 1$ and $w(z) = -\frac{z(z+\theta)}{1+\theta z}, 0 \leq \theta \leq 1$, or one of its rotation, while for $\mu = \rho$, the equality holds if and only if $\eta(z) = 1$ and $w(z) = \frac{z(z+\theta)}{1+\theta z}, 0 \leq \theta \leq 1$, or one of its rotations.

□ The second part of assertion in (2.19) for real values of μ and γ is improved further as below:

Theorem 2.10. Let $0 \leq \xi \leq 1, \tau \geq 0$. If the function $s \in M_q(\tau, \xi, \gamma, \mathfrak{h})$, then for real γ and μ, we have

$$|d_3 - \mu d_2^2| + (\mu - \rho)|d_2|^2 \leq \frac{|\gamma|\mathbb{Q}_1}{3(2\tau+1)-\xi} \quad (\rho \leq \mu \leq \rho + \sigma) \tag{2.22}$$

and

$$|d_3 - \mu d_2^2| + (\rho + 2\sigma - \mu)|d_2^2| \leq \frac{|\gamma|\mathbb{Q}_1}{3(2\tau+1)-\xi} \quad (\rho + \sigma \leq \mu \leq \rho + 2\sigma), \tag{2.23}$$

where ρ_1 and σ are given by (2.20) and (2.21), respectively.

Proof . Let $s \in M_q(\tau, \xi, \gamma, \mathfrak{h})$. For real μ satisfying $\rho \leq \mu \leq \rho_1 + \sigma$ and using (2.7) and (2.14), we get

$$\begin{aligned} |d_3 - \mu d_2^2| &+ (\mu - \rho)|d_2|^2 \\ &\leq \frac{|\gamma|\mathbb{Q}_1}{3(2\tau+1)-\xi}\left[|w_2| - \frac{|\gamma|\mathbb{Q}_1(3(2\tau+1)-\xi)}{(2(\tau+1)-\xi)^2}(\mu - \rho - \sigma)|w_1|^2 \right. \\ &\left. + \frac{|\gamma|\mathbb{Q}_1(3(2\tau+1)-\xi)}{(2(\tau+1)-\xi)^2}(\mu - \rho)|w_1|^2\right]. \end{aligned}$$

Therefore, by using Lemma 1.1, we obtain

$$|d_3 - \mu d_2^2| + (\mu - \rho)|d_2|^2 \leq \frac{|\gamma|\mathbb{Q}_1}{3(2\tau+1)-\xi}\left[1 - |w_1|^2 + |w_1|^2\right],$$

which yields the assertion (2.22).

If $\rho + \sigma \le \mu \le \rho + 2\sigma$, then again from (2.7), (2.14) and Lemma 1.1, we have

$$\begin{aligned} |d_3 - \mu d_2^2| &+ (\rho + 2\sigma - \mu)|d_2|^2 \\ &\le \frac{|\gamma|\mathbb{Q}_1}{3(2\tau+1)-\xi}\left[|w_2| + \frac{|\gamma|\mathbb{Q}_1(3(2\tau+1)-\xi)}{(2(\tau+1)-\xi)^2}(\rho+2\sigma-\mu)|w_1|^2 \right. \\ &\left. + \frac{|\gamma|\mathbb{Q}_1(3(2\tau+1)-\xi)}{(2(\tau+1)-\xi)^2}(\mu-\rho)|w_1|^2\right] \\ &\le \frac{|\gamma|\mathbb{Q}_1}{3(2\tau+1)-\xi}\left[1 - |w_1|^2 + |w_1|^2\right], \end{aligned}$$

which estimates (2.23). □

Remark 2.11. Numerous consequences of Theorem 2.7 to Theorem 2.10 can be obtained for families $\mathfrak{K}_q(\xi,\gamma,\mathfrak{h})$, $\mathfrak{L}_q(\tau,\gamma,\mathfrak{h})$ and $\mathfrak{M}_q(\tau,\gamma,\mathfrak{h})$ by taking i) $\tau = 0$, ii) $\xi = 0$ and iii) $\xi = 1$, respectively.

3 Fekete-Szegö results for the class $\mathfrak{B}_q(\alpha,\xi,\gamma,\mathfrak{h})$

In this section, we derive the estimates for $|d_2|$ and Fekete-Szegö functional $|d_3 - \mu d_2^2|$ for elements in $\mathfrak{B}_q(\alpha,\xi,\gamma,\mathfrak{h})$. Few known and many new consequences of this result are pointed out. A result based on majorization is stated in Corollary 3.7. Another result associated with $\mathfrak{B}(\alpha,\xi,\gamma,\mathfrak{h})$ is stated in Corollary 3.8. We also state the bounds of $|d_3 - \mu d_2^2|$ for real γ and μ, when a function $s \in \mathfrak{B}_q(\alpha,\xi,\gamma,\mathfrak{h})$ in Corollary 3.9. Proofs of these corollaries are omitted due to Section 2.

Theorem 3.1. Let $0 \le \xi \le 1, \alpha \ge 1$ and $\gamma \in \mathbb{C} - \{0\}$. If the function $s \in \mathfrak{B}_q(\alpha,\xi,\gamma,\mathfrak{h})$, then

$$|d_2| \le \frac{|\gamma|\mathbb{Q}_1}{2\alpha - \xi} \tag{3.1}$$

and for any complex number $\mu \in \mathbb{C}$,

$$|d_3 - \mu d_2^2| \le \frac{|\gamma|\mathbb{Q}_1}{3\alpha - \xi} max\left(1, \left|\mathfrak{J}_1\mathbb{Q}_1 - \frac{\mathbb{Q}_2}{\mathbb{Q}_1}\right|\right), \tag{3.2}$$

where

$$\mathfrak{J}_1 = \gamma\left(\frac{\xi^2 - 2\alpha\xi + 2\alpha(\alpha-1) + \mu(3\alpha-\xi)}{(2\alpha-\xi)^2}\right). \tag{3.3}$$

Proof . Let $s \in \mathfrak{B}_q(\alpha,\xi,\gamma,\mathfrak{h})$. Then there exists $\omega(z)$, a Schwarz function and $\eta(z)$, a regular function such that

$$\frac{1}{\gamma}\left(\frac{z[s'(z)]^\alpha}{\xi s(z) + (1-\xi)z} - 1\right) = \eta(z)(\mathfrak{h}(\omega(z)) - 1). \tag{3.4}$$

Series expansions of s and its successive derivatives from (1.1) gives

$$\begin{aligned} &\frac{1}{\gamma}\left(\frac{z[s'(z)]^\alpha}{\xi s(z) + (1-\xi)z} - 1\right) = \\ &\quad \frac{1}{\gamma}\left[(2\alpha-\xi)d_2 z + \left((3\alpha-\xi)d_3 + (2\alpha(\alpha-1) - (2\alpha-\xi)\xi)d_2^2\right)z^2 + ...\right]. \end{aligned} \tag{3.5}$$

Similarly from (1.2), (1.3) and (1.4), we obtain

$$\eta(z)(\mathfrak{h}(\omega(z)) - 1) = \mathfrak{R}_0\mathbb{Q}_1\omega_1 z + \left[\mathfrak{R}_1\mathbb{Q}_1\omega_1 + \mathfrak{R}_0(\mathbb{Q}_1\omega_2 + \mathbb{Q}_2\omega_1^2)\right]z^2 + \dots . \tag{3.6}$$

Using (3.5) and (3.6) in (3.4), we get

$$d_2 = \frac{\gamma\mathfrak{R}_0\mathbb{Q}_1\omega_1}{2\alpha - \xi} \tag{3.7}$$

and

$$d_3 - \mu d_2^2 = \frac{\gamma\mathbb{Q}_1}{3\alpha-\xi}\left[\mathfrak{R}_1\omega_1 + \left(\omega_2 + \frac{\mathbb{Q}_2}{\mathbb{Q}_1}\omega_1^2\right)\mathfrak{R}_0 - \mathfrak{J}_1\mathbb{Q}_1\mathfrak{R}_0^2\omega_1^2\right], \tag{3.8}$$

where $\mu \in \mathbb{C}$ and $\mathfrak{J}_1$ is as stated in (3.3).

Since $\eta(z)$ is a regular function bounded by one in $\mathfrak{D}$, we have (see [14],p.172)

$$|\mathfrak{R}_0| \leq 1 \quad \text{and} \quad \mathfrak{R}_1 = (1-\mathfrak{R}_0^2)x \qquad x \leq 1. \tag{3.9}$$

The assertion (3.1) follows from (3.7) using (3.9) and Lemma 1.1. From (3.8) and (3.9), we obtain

$$d_3 - \mu d_2^2 = \frac{\gamma \mathbb{Q}_1}{3\alpha - \xi}\left[x\omega_1 + \left(\omega_2 + \frac{\mathbb{Q}_2}{\mathbb{Q}_1}\omega_1^2\right)\mathfrak{R}_0 - (\mathfrak{J}_1\mathbb{Q}_1\omega_1^2 + x\omega_1)\mathfrak{R}_0^2\right]. \tag{3.10}$$

If $\mathfrak{R}_0 = 0$, then (3.10) yields

$$|d_3 - \mu d_2^2| \leq \frac{|\gamma|\mathbb{Q}_1}{3\alpha - \xi}. \tag{3.11}$$

On the other side, if $\mathfrak{R}_0 \neq 0$, we define a function

$$L_1(\mathfrak{R}_0) = x\omega_1 + \left(\omega_2 + \frac{\mathbb{Q}_2}{\mathbb{Q}_1}\omega_1^2\right)R_0 - (\mathfrak{J}_1\mathbb{Q}_1\omega_1^2 + x\omega_1)\mathfrak{R}_0^2. \tag{3.12}$$

The equation (3.12) is a quadratic in $\mathfrak{R}_0$ and hence regular in $|\mathfrak{R}_0| \leq 1$. Clearly $|L_1(\mathfrak{R}_0)|$ attains its maximum value at $\mathfrak{R}_0 = e^{i\theta}$, $0 \leq \theta \leq 2\pi$. Thus

$$max|L_1(\mathfrak{R}_0)| = \max_{0\leq\theta\leq 2\pi}|L_1(e^{i\theta})| = |L_1(1)| = \left|\omega_2 - \left(\mathfrak{J}_1\mathbb{Q}_1 - \frac{\mathbb{Q}_2}{\mathbb{Q}_1}\right)\omega_1^2\right|.$$

Therefore, it follows from (3.10) that

$$|d_3 - \mu d_2^2| \leq \frac{|\gamma|\mathbb{Q}_1}{3\alpha - \xi}\left|\omega_2 - \left(\mathfrak{J}_1\mathbb{Q}_1 - \frac{\mathbb{Q}_2}{\mathbb{Q}_1}\right)\omega_1^2\right|. \tag{3.13}$$

By virtue of Lemma 1.1, we obtain

$$|d_3 - \mu d_2^2| \leq \frac{|\gamma|\mathbb{Q}_1}{3\alpha - \xi} max\left(1, \left|\mathfrak{J}_1\mathbb{Q}_1 - \frac{\mathbb{Q}_2}{\mathbb{Q}_1}\right|\right). \tag{3.14}$$

The assertion (3.2) now follows from (3.11) and (3.14). This ends the proof. □ Taking $\alpha = 1$ in Theorem 3.1, we obtain the below outcome.

Corollary 3.2. Let $0 \leq \xi \leq 1$ *and* $\gamma \in \mathbb{C} - \{0\}$. If the function $s \in \mathfrak{G}_q(\xi, \gamma, \mathfrak{h})$, then $|d_2| \leq \frac{|\gamma|\mathbb{Q}_1}{2-\xi}$ and for some $\mu \in \mathbb{C}$, $|d_3 - \mu d_2^2| \leq \frac{|\gamma|\mathbb{Q}_1}{3-\xi} max\left(1, \left|\gamma\left(\frac{(\mu-\xi)(2-\xi)+\mu}{(2-\xi)^2}\right)\mathbb{Q}_1 - \frac{\mathbb{Q}_2}{\mathbb{Q}_1}\right|\right)$.

Remark 3.3. For $\xi = 1$ and $\gamma = 1$, Corollary 3.2 reduces to Corollary 2.2 of [15].

We conclude the following outcome for the class $\mathfrak{P}_q(\tau, \gamma, \mathfrak{h})$ on putting $\xi = 0$ in Theorem 3.1.

Corollary 3.4. [22] Let $\gamma \in \mathbb{C} - \{0\}$ and $\alpha \geq 1$.If the function $s \in \mathfrak{P}_q(\alpha, \gamma, \mathfrak{h})$, then $|d_2| \leq \frac{|\gamma|\mathbb{Q}_1}{2\alpha}$ and for some $\mu \in \mathbb{C}$, $|d_3 - \mu d_2^2| \leq \frac{|\gamma|\mathbb{Q}_1}{3\alpha} max\left(1, \left|\gamma\left(\frac{3\mu+2(\alpha-1)}{4\alpha}\right)\mathbb{Q}_1 - \frac{\mathbb{Q}_2}{\mathbb{Q}_1}\right|\right)$.

Remark 3.5. For $\alpha = 1$, Corollary 3.4 reduces to Corollary 1 of [9].

Allowing $\xi = 1$ in Theorem 3.1, we have the following outcome.

Corollary 3.6. Let $\gamma \in \mathbb{C} - \{0\}$ and $\alpha \geq 1$. If the function $s \in \mathfrak{M}_q(\alpha, \gamma, \mathfrak{h})$, then $|d_2| \leq \frac{|\gamma|\mathbb{Q}_1}{2\alpha-1}$ and for some $\mu \in \mathbb{C}$, $|d_3 - \mu d_2^2| \leq \frac{|\gamma|\mathbb{Q}_1}{3\alpha-1} max\left(1, \left|\gamma\left(\frac{(2\alpha^2-4\alpha+1)+\mu(3\alpha-1)}{(2\alpha-1)^2}\right)\mathbb{Q}_1 - \frac{\mathbb{Q}_2}{\mathbb{Q}_1}\right|\right)$.

Our next outcome is based on majorization.

Corollary 3.7. Let $\gamma \in \mathbb{C} - \{0\}, 0 \le \xi \le 1$ and $\alpha \ge 1$. If the function $s \in A$ satisfies

$$\frac{1}{\gamma}\left(\frac{z[s'(z)]^{\alpha}}{\xi s(z) + (1-\xi)z} - 1\right) \prec\prec (\mathfrak{h}(z) - 1), z \in \mathfrak{D},$$

then

$$|d_2| \le \frac{|\gamma|\mathbb{Q}_1}{2\alpha - \xi}$$

and for any complex number μ,

$$|d_3 - \mu d_2^2| \le \frac{|\gamma|\mathbb{Q}_1}{3\alpha - \xi} max\left(1, \left|\mathfrak{J}_1\mathbb{Q}_1 - \frac{\mathbb{Q}_2}{\mathbb{Q}_1}\right|\right),$$

where $\mathfrak{J}_1$ is as stated by (3.3).

Our next result is associated with $\mathfrak{B}(\alpha, \xi, \gamma, \mathfrak{h})$

Corollary 3.8. Let $0 \le \xi \le 1, \alpha \ge 1$ and $\gamma \in \mathbb{C} - \{0\}$. If $s \in \mathfrak{B}(\alpha, \xi, \gamma, \mathfrak{h})$, then

$$|d_2| \le \frac{|\gamma|\mathbb{Q}_1}{2\alpha - \xi}$$

and for any $\mu \in \mathbb{C}$,

$$|d_3 - \mu d_2^2| \le \frac{|\gamma|\mathbb{Q}_1}{3\alpha - \xi} max\left(1, \left|\mathfrak{J}_1\mathbb{Q}_1 - \frac{\mathbb{Q}_2}{\mathbb{Q}_1}\right|\right),$$

where $\mathfrak{J}_1$ is as stated in (3.3).

The following outcomes are eligible for real values of γ and μ.

Corollary 3.9. Let $0 \le \xi \le 1$, $\alpha \ge 1$. If the function $s \in \mathfrak{B}_q(\alpha, \xi, \gamma, \mathfrak{h})$, then for real γ and μ, we have

$$|d_3 - \mu d_2^2| \le \begin{cases} \frac{|\gamma|\mathbb{Q}_1}{3\alpha-\xi}\left[\gamma\left(\frac{\xi}{2\alpha-\xi} - \frac{\mu(3\alpha-\xi)}{(2\alpha-\xi)^2}\right)\mathbb{Q}_1 + \frac{\mathbb{Q}_2}{\mathbb{Q}_1}\right] & (\mu \le \rho_1) \\ \frac{|\gamma|\mathbb{Q}_1}{3\alpha-\xi} & (\rho_1 \le \mu \le \rho_1 + 2\sigma_1) \\ -\frac{|\gamma|\mathbb{Q}_1}{3\alpha-\xi}\left[\gamma\left(\frac{\xi}{2\alpha-\xi} - \frac{\mu(3\alpha-\xi)}{(2\alpha-\xi)^2}\right)\mathbb{Q}_1 + \frac{\mathbb{Q}_2}{\mathbb{Q}_1}\right] & (\mu \ge \rho_1 + 2\sigma_1) \end{cases} \tag{3.15}$$

where

$$\rho_1 = \frac{\xi(2\alpha - \xi)}{3\alpha - \xi} - \frac{(2\alpha - \xi)^2}{\gamma(3\alpha - \xi)}\left(\frac{1}{\mathbb{Q}_1} - \frac{\mathbb{Q}_2}{\mathbb{Q}_1^2}\right) \tag{3.16}$$

and

$$\sigma_1 = \frac{(2\alpha - \xi)^2}{\gamma(3\alpha - \xi)\mathbb{Q}_1}. \tag{3.17}$$

The second part of assertion in (3.15) for real values of μ and γ is improved further as below:

Corollary 3.10. Let $0 \le \xi \le 1, \alpha \ge 1$. If the function $s \in \mathfrak{B}_q(\alpha, \xi, \gamma, \mathfrak{h})$, then for real γ and μ, we have

$$|d_3 - \mu d_2^2| + (\mu - \rho_1)|d_2|^2 \le \frac{|\gamma|\mathbb{Q}_1}{3\alpha - \xi} \quad (\rho_1 \le \mu \le \rho_1 + \sigma_1)$$

and

$$|d_3 - \mu d_2^2| + (\rho_1 + 2\sigma_1 - \mu)|d_2^2| \le \frac{|\gamma|\mathbb{Q}_1}{3\alpha - \xi} \quad (\rho_1 + \sigma_1 \le \mu \le \rho_1 + 2\sigma_1),$$

where ρ_1 and σ_1 are given by (3.16) and (3.17), respectively.

Remark 3.11. Numerous consequences of Theorem 3.7 to Theorem 3.10 can be obtained for families $\mathfrak{G}_q(\xi, \gamma, \mathfrak{h})$, $\mathfrak{P}_q(\alpha, \gamma, \mathfrak{h})$ and $\mathfrak{N}_q(\alpha, \gamma, \mathfrak{h})$ by taking i) $\alpha = 1$, ii) $\xi = 0$ and iii) $\xi = 1$, respectively.

4 Conclusion

Two special families of regular functions based on quasi-subordination is initiated and explored. Upper bounds for $|d_2|$ and the celebrated Fekete-Szegö functional have been fixed for each of the presented families. Through our main result, we have highlighted many interesting new consequences.

References

[1] A. Alsoboh and M. Darus, *On Fekete-Szegö problem associated with q-derivative operator*, IOP Conf. Ser. J. Phys. Conf. Ser. **1212** (2019), 1–7.

[2] OP. Ahuja and M. Jahangiri, *Fekete-Szegö problem for a unified class of analytic functions*, Pan Amer. Math. J. **7** (1997), 67–78.

[3] R. Bharavi Sharma and K. Rajya Laxmi, *Fekete-Szegö inequalities for some subclasses of bi-univalent functions through quasi-subordination*, Asian Eur. J. Math. **13** (2020), 1–16.

[4] M. Çaglar and H. Orhan, *Fekete–Szegö problem for certain subclasses of analytic functions defined by the combination of differential operators*, Bol. Soc. Mat. Mex. **27** (2021), no. 2, 1–12.

[5] N.E. Cho and S. Owa, *On the Fekete-Szegö problem for strongly α-logarithmic quasiconvex functions*, Southeast Asian Bull. Math. **28** (2004), no. 3, 421–430.

[6] E. Deniz and H. Orhan, *The Fekete-Szegö problem for a generalized subclass of analytic functions*, Kyungpook Math. J. **50** (2010), no. 1, 37–47.

[7] S. Elhaddad and M. Darus, *On Fekete-Szegö problems for a certain subclass defined by q-analogue of Ruscheweyh operator*, IOP Con. Series: J. Phys. Conf. Ser. **1212** (2019), 1–6.

[8] M. Fekete and G. Szegö, *Eine Bemerkung Über ungerade schlichte Funktionen*, J. London Math. Soc. **8** (1933), no. 2, 85–89.

[9] S. Kant and P.P. Vyas, *Sharp bounds of Fekete-Szegö functional for quasi-subordination class*, Acta. Univ. Sapientiae Math. **11** (2019), 87–98.

[10] F.R. Keogh and E.P. Merkes, *A coefficient inequality for certain classes of analytic functions*, Proc. Amer. Math. Soc. **20** (1969), no. 1, 8–12.

[11] N. Magesh, V.K. Balaji and J. Yamini, *Certain subclasses of bistarlike and biconvex functions based on quasi-subordination*, Abstr. Appl. Anal. **2016** (2016), 1–6.

[12] N. Magesh, V.K. Balaji and C. Abirami, *Fekete-Szegö inequalities for certain subclasses of starlike and convex functions of complex order associated with quasi-subordination*, Khayyam J. Math. **2** (2016), no. 2, 112–119.

[13] M.H. Mohd and M. Darus, *Fekete-Szegö problems for Quasi-Subordination classes*, Abst. Appl. Anal. **2012** (2012), 1–14.

[14] Z. Nehari, *Conformal mapping*, Dover, New York, 1975.

[15] T. Panigrahi and R.K. Raina, *Fekete-Szegö coefficient functional for quasi-subordination class*, Afr. Mat. **28** (2017), no. 5, 707–716.

[16] F.Y. Ren, S. Owa and S. Fukui, *Some Inequalities on quasi-subordinate functions*, Bull. Aust. Math. Soc. **43** (1991), no. 2, 317–324.

[17] M.S. Robertson, *Quasi-subordination and coefficient conjecture*, Bull. Amer. Math. Soc. **76** (1970), no. 1, 1–9.

[18] H.M. Srivastava, A.K. Mishra and M.K. Das, *The Fekete-Szegö problem for a subclass of close-to-convex functions*, Complex Var. Theory Appl. **44** (2001), no. 2, 145–163.

[19] H.M. Srivastava, Ş Altınkaya and S. Yalçın, *Certain subclasses of bi-univalent functions associated with the Horadam polynomials*, Iran J. Sci. Technol. Trans. Sci. **43** (2019), no. 4, 1873-–1879.

[20] S.R. Swamy and Y. Sailaja, *Horadam polynomial coefficient estimates for two families of holomorphic and bi-univalent functions*, Int. J. Math. Trends Technol. **66** (2020), no. 8, 131–138.

[21] S.R. Swamy and Y. Sailaja, *Sharp bounds of Fekete-Szegö functional for quasi-subordination class*, Int. J. Math. Trends Technol. **66** (2020), no. 11, 87–94.

[22] S.R. Swamy and Y. Sailaja, *On the Fekete-Szegö coefficient functional for quasi-subordination class*, Palestine J. Math. **10** (2021), no. 2, 1-7.

[23] S.R. Swamy, *Ruscheweyh derivative and a new generalized Multiplier differential operator*, Ann. Pure Appl. Math. **10** (2015), no. 2, 229–238.

[24] L. Taishun and X. Qinghua, *Fekete and Szegö inequality for a subclass of starlike mappings of order α on the bounded starlike circular domain in* $\mathbb{C}^n$, Acta Math. Sci. **37** (2017), no. 3, 722–731.