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Viscosity approximation method for monotone operators in Hadamard spaces

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Abstract

In this article, we suggest and analyze a viscosity approximation method to a zero of a monotone operator in the setting of Hadamard spaces. We derive the convergence of sequences generated by the proposed viscosity methods under some suitable assumptions. Also, some applications to solve the variational inequality, optimization and fixed point problems are given on Hadamard spaces.

Keywords: Hadamard space, proximal point algorithm, Viscosity approximation method, Monotone operators, Convex minimization
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1 Introduction

One of the most important problems in monotone operator theory is finding a zero of a monotone operator, in the other words:

$$\text{finding} \quad x \in D(A), \quad \text{such that } 0 \in A(x), \tag{1.1}$$

where A is a monotone operator on the metric space X. The solution of the problem (1.1) has many applicatins in convex programming, variational inequalities, split feasibility problem and minimization problem.

The proximal point algorithm is the most popular method for approximating a zero of a monotone operator A, which was introduced in Hilbert space H by martinet [19] and Rockafellar [23], as follows:

$$\begin{cases} x_{n-1} - x_n \in \lambda_n A(x_n), \\ x_0 \in H, \end{cases} \tag{1.2}$$

where (λ_n) is a sequence of positive real numbers. In fact, Rockafellar [23] showed the weak convergence of the sequence generated by the proximal point algorithm to a zero of the maximal monotone operator in Hilbert spaces. For getting

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to the strong convergence, Kamimura and Takahashi [12] proposed Halpern-type proximal point algorithm

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n) J^A_{\lambda_n} x_n, \\ x_0,\ u \in X. \end{cases} \tag{1.3}$$

where $\alpha_n \in [0,1]$, (λ_n) is a sequence of positive real numbers and $J^A_\lambda = (1+\lambda A)^{-1}$, $\lambda > 0$ is the resolvent of the monotone operator A on the Hilbert space H. Takahashi [24] proposed viscosity approximation methods for resolvents of accretive operators, which is an extension of the sequence generated by (1.3), in Banach spaces as follows;

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) J^A_{\lambda_n} x_n \tag{1.4}$$

where f is a contractive mapping. Very recently, Khatibzadeh and Ranjbar [15] established some properties of a monotone operator and its resolvent operator in CAT(0) spaces. They generallized the proximal point algorithm, in the case of monotone operators, to Hadamard spaces. This generallization is an extension of the work of Bacak [3], (also, see [21]). Also in [22] they extended the work of Kamimura and Takahashi [12] to Hadamard spaces.

In this paper, we consider viscosity approximation methods for resolvents of monotone operators in Hadamard spaces. Strong convergence of viscosity approximation methods to a zero of a monotone opertaor is established and some applications of the main result in the convex minimization problems and fixed point theorem are peresented. The results peresented in this paper generalize the work of Takahashi [24] and Ranjbar and Khatibzadeh [22] to Hadamard spaces.

The paper has been organized as follows.

In Section 2, we give some preliminaries in CAT(0) spaces and some lemmas that we need in the sequel. In Section 3, we propose the viscosity approximation methods for resolvents of monotone operators in Hadamard spaces and establish strong convergence of the proposed sequence to a zero of a monotone operator which is a solution of a varitional inequality. Section 4, is devoted to applications of the main result in the convex minimization problems and fixed point theorem.

2 Preliminaries

Let (X,d) be a metric space and $x,y \in X$. A geodesic path joining x to y is an isometry γ defined on a closed interval $[0,l] \subset \mathbb{R}$ to X such that $\gamma(0) = x$, $\gamma(l) = y$ and $d(\gamma(t),\gamma(t')) = |t-t'|$ for all $t,t' \in [0,l]$. In particular $d(x,y) = l$. A geodesic space is a metric space (X,d) such that there is a geodesic between each pair of its points. If such a geodesic is unique for each two points, we call X a uniquely geodesic space. The image of a geodesic path is called a geodesic segment joining x and y. When it is unique, this geodesic segment is denoted by $[x,y]$. For all x and y which belong to uniquely geodesic space X, we write $tx \oplus (1-t)y$ for the unique point z in the geodesic segment joining x and y such that $d(z,x) = (1-t)d(x,y)$ and $d(z,y) = td(x,y)$.

Set $[x,y] = \{tx \oplus (1-t)y\ :\ t \in [0,1]\}$, a subset C of X is called convex if $[x,y] \subseteq C$ for all $x,y \in C$.

Definition 2.1. A geodesic space (X,d) is a CAT(0) space if and only if for all $x,y,z \in X$ and all $0 < t < 1$, the following CN-inequality is satisfied,

$$d^2(tx \oplus (1-t)y, z) \le td^2(x,z) + (1-t)d^2(y,z) - t(1-t)d^2(x,y).$$

Complete $CAT(0)$ spaces are called *Hadamard* spaces. Hadamard space is a uniquely geodesic space. Some examples of the $CAT(0)$ spaces are pre-Hilbert spaces, Hadamard manifolds, $\mathbb{R}$-trees and many others. It is natural that a large number of notions known in these spaces can be extended to Hadamard spaces. Moreover, the extension of some theories, such as monotone operator theory, fixed point theory and many others, from such spaces to Hadamard spaces is useful and valuable.

To know more equivalent definitions and basic properties of CAT(0) spaces, the reader can see the standard texts such as [4, 6, 7, 9, 11].

Lemma 2.2. [6] Let (X,d) be a $CAT(0)$ space. Then for all $x,y,z \in X$ and $t \in [0,1]$, we have

(i) $d(tx \oplus (1-t)y, z) \le td(x,z) + (1-t)d(y,z)$,

(ii) $d(tx \oplus (1-t)y, tz \oplus (1-t)w) \le td(x,z) + (1-t)d(y,w)$.

Berg and Nikolaev [5] introduced quasilinearization concept in Aleksandrov spaces. Ahmadi Kakavandi and Amini [2] based on the work of Berg and Nikolaev [5] organized the concept of vector-like and inner product-like in $CAT(0)$ spaces as follows.

Let X be a Hadamard space and $(x, y) \in X \times X$. Denote (x, y) by $\overrightarrow{xy}$ and call it a vector. The quasi-inner product on Hadamard spaces is a map from $X \times X$ to $\mathbb{R}$ that is defined as follows

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}[d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)],$$

for all $a, b, c, d \in X$.

It is easily seen that for all $a,\ b, c, d;\ x \in X$, we obtain, $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a,b),\ \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle,\ \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$, and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle$.

Lemma 2.3. [8] Let (X, d) be a $CAT(0)$ space and $x, y, z \in X$. Then for each $\lambda \in [0, 1]$,

$$d^2(\lambda x \oplus (1-\lambda)y, z) \leq \lambda^2 d^2(x,z) + (1-\lambda)^2 d^2(y,z) + 2\lambda(1-\lambda)\langle \overrightarrow{xz}, \overrightarrow{yz} \rangle.$$

The metric space X is satisfied in Cauchy-Schwartz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leqslant d(a,b)d(c,d), \quad \text{for all} \quad a, b, c, d \in X.$$

Berg and Nikolaev [5, Corollary 3] showed that a geodesic connected metric space is a $CAT(0)$ space if and only if it satisfies the Cauchy-Schwartz inequality.

A notion of convergence is defined by Lim [18] in metric spaces that is called $\Delta-$convergence. Ahmadi Kakavandi [1] characterized it as follows:

Let (X, d) be a CAT(0) space, and (x_n) be a sequence in X. Then (x_n) is $\Delta-$convergent to x if and only if

$$\limsup_{n\to\infty} \langle \overrightarrow{xx_n}, \overrightarrow{xy} \rangle \leq 0,$$

for all $y \in X$.

It is well-known that every bounded sequence has a $\Delta-$convergent subsequence in Hadamard spaces. Dual space of a Hadamard space X was introduced by Ahmadi Kakavandi and Amini [2]. They introduced an equivalence relation on $\mathbb{R} \times X \times X$ defined by $(t, a, b) \sim (s, c, d)$ whenever $t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle$ for all $x, y \in X$. The equivalence class of $t\overrightarrow{ab} := (t, a, b)$ is

$$[t\overrightarrow{ab}] = \{(s, c, d) : D((t, a, b), (s, c, d)) = 0\}.$$

They defined the dual space of X by

$$X^* = \{[t\overrightarrow{ab}] : (t, a, b) \in \mathbb{R} \times X \times X\}.$$

Note that X^* acts on $X \times X$ by

$$\langle x^*, \overrightarrow{xy} \rangle = t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle, \quad \text{for all } x^* = [t\overrightarrow{ab}] \in X^*, \quad x, y \in X.$$

Clearly $[\overrightarrow{aa}] = [\overrightarrow{bb}]$ for all $a, b \in X$. For fixed $o \in X$, we denote the zero of dual space X^* by $0 = [\overrightarrow{oo}]$. Thus, for all $\alpha, \beta \in \mathbb{R},\ x^*, y^* \in X^*$ and $x, y \in X$, we can define

$$\langle \alpha x^* + \beta y^*, \overrightarrow{yx} \rangle := \alpha\langle x^*, \overrightarrow{yx} \rangle + \beta\langle y^*, \overrightarrow{yx} \rangle.$$

Definition 2.4. [10] Let X be a Hadamard space with dual X^* and $A : X \to 2^{X^*}$ be a multi-valued operator with domain $\mathbb{D}(A) := \{x \in X : Ax \neq \emptyset\}$, $A^{-1}(x^*) := \{x \in X :\ x^* \in Ax\}$,
$graph(A) := \{(x, x^*) \in X \times X^* : x \in \mathbb{D}(A),\ x^* \in Ax\}$ and range $\mathbb{R}(A) := \bigcup_{x\in X} Ax$.
A is called *monotone* if and only if

$$\langle x^* - y^*, \overrightarrow{yx} \rangle \geq 0,$$

for all $x, y \in \mathbb{D}(A),\ \ x^* \in Ax,\ \ y^* \in Ay$.

Definition 2.5. Let (X, d) be a Hadamard space. The map $T : X \to X$ is said

(i) a *contractive mapping* if there exists $r \in (0, 1)$ such that

$$d(Tx, Ty) \leq rd(x, y);$$

for all $x, y \in X$.

(ii) a *nonexpansive mapping* if

$$d(Tx, Ty) \leq d(x, y);$$

for all $x, y \in X$.

(iii) a *firmly nonexpansive mapping* if

$$d^2(Tx, Ty) \leq \langle \overrightarrow{TxTy}, \overrightarrow{xy} \rangle;$$

for all $x, y \in X$.

Clearly, firmly nonexpansive mappings and contraction mappings are nonexpansive mappings. We denote the set of the fixed points of a mapping $T : X \to X$ by $F(T)$, (i.e. $F(T) = \{x \in X : Tx = x\}$). It is known that if T is a nonexpansive mapping on a subset of the CAT(0) space X, then $F(T)$ is closed and convex.

Definition 2.6. [8] Let C be a nonempty convex subset of a Hadamard space X. It is known [[6], Proposition 2.4] that for any $x \in X$ there exists a unique point $u \in C$ such that $d(x, u) = \min_{y \in C} d(x, y)$. The mapping $P_C : X \to C$ defined by $P_C x = u$ is called the metric projection from X onto C.

Theorem 2.7. [8] Let C be a nonempty convex subset of a CAT(0) space X, $x \in X$ and $u \in C$. Then $u = P_C x$ if and only if $\langle \overrightarrow{xu}, \overrightarrow{yu} \rangle \leq 0 \quad$ for all $y \in C$.

Proposition 2.8. [16, Proposition 3.7] Let K be a closed and convex subset of X, and let $f : K \to X$ be a nonexpansive mapping. If the sequence (x_n) in K Δ-converges to x and $d(x_n, f(x_n)) \to 0$, then $x \in K$ and $f(x) = x$.

Definition 2.9. [15] Let X be a Hadamard space with dual X^* and $A : X \to 2^{X^*}$ be a multi-valued operator and $\lambda > 0$. The resolvent operator of A of order λ is the multi-valued mapping $J_\lambda^A : X \to 2^X$ that defined by $J_\lambda^A(x) := \{z \in X | [\frac{1}{\lambda}\overrightarrow{zx}] \in Az\}$.

Theorem 2.10. [15, Theorem 3.9] Let X be a Hadamard space with dual X^* and $A : X \to 2^{X^*}$ is an operator. Suppose J_λ^A is resolvent operator of A of order λ, then
(1) $\mathbb{R}(J_\lambda^A) \subset \mathbb{D}(A)$ and $F(J_\lambda^A) = A^{-1}(0)$; for every $\lambda > 0$.
(2) If A is a monotone operator then J_λ^A is a single-valued and firmly nonexpansive mapping.
(3) If A is a monotone operator and $0 < \lambda \leq \mu$, then

$$d(x, J_\lambda^A(x)) \leq 2d(x, J_\mu^A(x)), \text{ for all } x \in X.$$

Remark 2.11. [15] It is well-known that if T is a nonexpansive mapping on subset C of $CAT(0)$ space X then $F(T)$ is closed and convex.
Thus, if A is a monotone operator on CAT(0) space X then, by parts (i) and (ii) of Theorem 2.10, $A^{-1}(0)$ is closed and convex.

Lemma 2.12. [14, Lemma 2.8] [13, Lemma 2.5] Let C be a closed and convex subset of complete CAT(0) space X, $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and (x_n) be a bounded sequence in C such that the sequence $(d(x_n; Tx_n))$ converges to zero. Then

$$\limsup_n \langle \overrightarrow{up}, \overrightarrow{x_n p} \rangle \leq 0,$$

where $u \in C$ and p is the nearest point of $F(T)$ to u.

Lemma 2.13. [13, Lemma 2.6] Suppose (X,d) is a metric space and $C \subset X$. Let $(T_n)_{n=1}^{\infty} : C \to C$ be a sequence of nonexpansive mappings with a common fixed point and (x_n) be a bounded sequence such that $\lim_n d(x_n, T_n(x_n)) = 0$. Then

$$\limsup_n \langle \overrightarrow{up}, \overrightarrow{T_n x_n p} \rangle \le \limsup_n \langle \overrightarrow{up}, \overrightarrow{x_n p} \rangle,$$

where $u \in C$ and p is the nearest point of $\bigcap_{n=1}^{\infty} F(T_n)$ to u.

The following well-known lemmas are needed to prove the main result.

Lemma 2.14. [25] Let (s_n) be a sequence of nonnegative real numbers, (α_n) be a sequence of real numbers in $(0,1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and t_n be a sequence of real numbers. Suppose that

$$s_{n+1} \le (1-\alpha_n)s_n + \alpha_n t_n; \quad \text{for all n} \ge 1.$$

If $\limsup_{k\to\infty} t_{m_k} \le 0$ for every subsequence (s_{m_k}) of (s_n) satisfying $\liminf_k (s_{m_k+1} - s_{m_k}) \ge 0$, then $\lim_{n\to\infty} s_n = 0$.

3 Viscosity approximation method for monotone operators

Let X be a Hadamard space with dual X^*. The multi-valued operator $A : X \to 2^{X^*}$ is said to satisfies the range condition if $D(J_\lambda^A) = X$, for every $\lambda > 0$. In a Hilbert space, every maximal monotone operator satisfies the *range condition*. It is well known that the maximal monotone operators in Hadamard manifolds satisfy the range condition, (see [17]). Also in [15], it has been proved that the subdifferential of a convex, proper and lower semi-continuous function and the monotone operator $Az = [\overrightarrow{Tzz}]$, where T is a nonexpansive mapping satisfy the range condition in Hadamard spaces.

Viscosity approximation method for resolvents of monotone operators in Hadamard spaces is the sequence generated by:

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n) J_{\lambda_n}^A x_n, \\ x_1 \in X. \end{cases} \tag{3.1}$$

where A is a multi-valued monotone operator which satisfies the range condition, $(\alpha_n) \in [0,1]$, (λ_n) is a sequence of positive real numbers, and f is a contraction mapping on X.
In the sequel, strong convergence of the sequence generated by (3.1) to a zero of the monotone operator A, which is a solution of a variational inequality, is established in Hadamard spaces.

Theorem 3.1. Let (X,d) be a Hadamard space with dual X^*, $A : X \to 2^{X^*}$ be a multi-valued monotone operators that satisfies the range condition and $A^{-1}(0) \neq \emptyset$. Suppose $f : X \to X$ is a contraction mapping with contractive coefficient $\kappa \in (0, \frac{1}{2})$, $(\lambda_n) \subset (0,\infty)$ and $(\alpha_n) \subset (0,1)$ are sequences that satisfy the conditions
$C1 :$ $\lim_{n\to\infty} \alpha_n = 0$,
$C2 :$ $\sum_{n=1}^{\infty} \alpha_n = \infty$,
$C3 :$ $\lambda_n > \lambda > 0, \quad \forall n > 0$.
Then the sequence generated by (3.1) converges strongly to $p \in A^{-1}(0)$ such that $p = P_{A^{-1}(0)} f(p)$ and p also is the unique solution to the following variational inequality:

$$\langle \overrightarrow{pf(p)}, \overrightarrow{xp} \rangle \ge 0, \quad \forall x \in A^{-1}(0).$$

Proof . By Remark 2.11, $A^{-1}(0) = Fix(J_\lambda^A)$ is a convex and closed subset of X. Since $P_{A^{-1}(0)} f$ is a contraction mapping, by Banach contraction principle, there is a unique point p in $A^{-1}(0)$ such that $p = P_{A^{-1}(0)} f(p)$. First we prove that (x_n) is bounded. By Lemma 2.2, we obtain

$$\begin{aligned} d(x_{n+1}, p) &= d(\alpha_n f(x_n) \oplus (1-\alpha_n) J_{\lambda_n}^A x_n, p) \\ &\le \alpha_n d(f(x_n), p) + (1-\alpha_n) d(J_{\lambda_n}^A x_n, p) \\ &\le \alpha_n (d(f(x_n), f(p)) + d(f(p), p)) + (1-\alpha_n) d(J_{\lambda_n}^A x_n, J_{\lambda_n}^A p) \\ &\le \alpha_n \kappa d(x_n, p) + \alpha_n d(f(p), p) + (1-\alpha_n) d(x_n, p) \\ &= (1 - \alpha_n(1-\kappa)) d(x_n, p) + \alpha_n d(f(p), p) \\ &\le max\{d(x_n, p), \frac{d(f(p), p)}{1-\kappa}\}. \end{aligned}$$

Therefore,

$$d(x_n,p) \leq max\{d(x_1,p), \frac{d(f(p),p)}{1-\kappa}\}, \quad \text{for all } n \geq 1. \tag{3.2}$$

which implies (x_n) is bounded and so are $f(x_n)$ and $(J^A_{\lambda_n}x_n)$.

On the other hand, by Lemma 2.3, we have

$$\begin{aligned}
d^2(x_{n+1},p) &= d^2(\alpha_n f(x_n) \oplus (1-\alpha_n)J^A_{\lambda_n}x_n, p) \\
&\leq \alpha_n^2 d^2(f(x_n),p) + (1-\alpha_n)^2 d^2(J^A_{\lambda_n}x_n,p) + 2\alpha_n(1-\alpha_n)\langle\overrightarrow{f(x_n)p}, \overrightarrow{J^A_{\lambda_n}x_np}\rangle \\
&\leq (1-\alpha_n)d^2(x_n,p) + \alpha_n^2 d^2(f(x_n),p) + 2\alpha_n(1-\alpha_n)\langle\overrightarrow{f(x_n)f(p)}, \overrightarrow{J^A_{\lambda_n}x_np}\rangle \\
&\quad + 2\alpha_n(1-\alpha_n)\langle\overrightarrow{f(p)p}, \overrightarrow{J^A_{\lambda_n}x_np}\rangle \\
&\leq (1-\alpha_n)d^2(x_n,p) + \alpha_n^2 d^2(f(x_n),p) + 2\alpha_n(1-\alpha_n)d(f(x_n),f(p))d(J^A_{\lambda_n}x_n,p) \\
&\quad + 2\alpha_n(1-\alpha_n)\langle\overrightarrow{f(p)p}, \overrightarrow{J^A_{\lambda_n}x_np}\rangle \\
&\leq (1-\alpha_n)d^2(x_n,p) + \alpha_n^2 d^2(f(x_n),p) + 2\kappa\alpha_n(1-\alpha_n)d^2(x_n,p) \\
&\quad + 2\alpha_n(1-\alpha_n)\langle\overrightarrow{f(p)p}, \overrightarrow{J^A_{\lambda_n}x_np}\rangle \\
&\leq (1-\alpha_n)d^2(x_n,p) + \alpha_n^2 d^2(f(x_n),p) + 2\kappa\alpha_n d^2(x_n,p) \\
&\quad + 2\alpha_n(1-\alpha_n)\langle\overrightarrow{f(p)p}, \overrightarrow{J^A_{\lambda_n}x_np}\rangle \\
&\leq (1-\alpha_n(1-2\kappa))d^2(x_n,p) + \alpha_n^2 d^2(f(x_n),p) + 2\alpha_n(1-\alpha_n)\langle\overrightarrow{f(p)p}, \overrightarrow{J^A_{\lambda_n}x_np}\rangle
\end{aligned}$$

which implies,

$$d^2(x_{n+1},p) \leq (1-\alpha_n(1-2\kappa))d^2(x_n,p) + \alpha_n^2 d^2(f(x_n),p) + 2\alpha_n(1-\alpha_n)\langle\overrightarrow{f(p)p}, \overrightarrow{J^A_{\lambda_n}x_np}\rangle$$

Hence, by Lemma 2.14 to show $d(x_{n+1},p) \to 0$, it suffices to show that

$$\limsup_k (\alpha_{n_k} d^2(f(x_{n_k}),p) + 2(1-\alpha_{n_k})\langle\overrightarrow{f(p)p}, \overrightarrow{J^A_{\lambda_{n_k}}x_{n_k}p}\rangle) \leq 0$$

for every subsequence $(d(x_{n_k},p))$ of $(d(x_n,p))$ satisfying

$$\liminf_k (d(x_{n_k+1},p) - d(x_{n_k},p)) \geq 0.$$

For this, suppose $(d(x_{n_k},p))$ is a subsequence of $(d(x_n,p))$ such that

$$\liminf_k (d(x_{n_k+1},p) - d(x_{n_k},p)) \geq 0.$$

Then

$$\begin{aligned}
0 &\leq \liminf_k (d(x_{n_k+1},p) - d(x_{n_k},p)) \\
&= \liminf_k (d(\alpha_{n_k} f(x_{n_k}) \oplus (1-\alpha_{n_k})J^A_{\lambda_{n_k}}x_{n_k}, p) - d(x_{n_k},p)) \\
&\leq \liminf_k (\alpha_{n_k} d(f(x_{n_k}),p) + (1-\alpha_{n_k})d(J^A_{\lambda_{n_k}}x_{n_k},p) - d(x_{n_k},p)) \\
&\leq \liminf_k (d(J^A_{\lambda_{n_k}}x_{n_k},p) - d(x_{n_k},p)) + \limsup_k \alpha_{n_k}(d(f(x_{n_k}),p)d(J^A_{\lambda_{n_k}}x_{n_k},p)) \\
&= \liminf_k (d(J^A_{\lambda_{n_k}}x_{n_k},p) - d(x_{n_k},p)) \\
&\leq \limsup_k (d(J^A_{\lambda_{n_k}}x_{n_k},p) - d(x_{n_k},p)) \\
&\leq \limsup_k (d(x_{n_k},p) - d(x_{n_k},p)) = 0.
\end{aligned}$$

Hence

$$\lim_k (d(J^A_{\lambda_{n_k}}x_{n_k},p) - d(x_{n_k},p)) = 0. \tag{3.3}$$

Firmly nonexpansivity of the resolvent operator follows

$$d^2(x_{n_k}, J_{\lambda^A_{n_k}} x_{n_k}) \leq d^2(x_{n_k}, p) - d^2(J^A_{\lambda_{n_k}} x_{n_k}, p),$$

which by (3.3) and boundedness of (x_n), implies

$$d(x_{n_k}, J^A_{\lambda_{n_k}} x_{n_k}) \to 0. \tag{3.4}$$

On the other hand, Theorem 2.10 and the assumption $C3$ imply

$$d(x_{n_k}, J^A_{\lambda} x_{n_k}) \leq 2d(x_{n_k}, J^A_{\lambda_{n_k}} x_{n_k}).$$

Hence, by 3.4, we get $d(x_{n_k}, J^A_\lambda x_{n_k}) \to 0$. Therefore, by Lemma 2.12, $\limsup_n \langle \overrightarrow{f(p)p}, \overrightarrow{x_{n_k}p} \rangle \leq 0$ and by Lemma 2.13, we obtain

$$\limsup_k \langle \overrightarrow{f(p)p}, \overrightarrow{J^A_{\lambda_{n_k}} x_{n_k} p} \rangle \leq 0. \tag{3.5}$$

This together with the assumption $C1$ imply

$$\limsup_k (\alpha_{n_k} d^2(f(x_{n_k}), p) + 2(1-\alpha_{n_k}) \langle \overrightarrow{f(p)p}, \overrightarrow{J^A_{\lambda_{n_k}} x_{n_k} p} \rangle \leq 0,$$

which is the interestd result. □

4 Applications

One of the most important examples of monotone operators are subdifferentials of proper, convex and lower semicontinuous functions. We approximate a minimizer of a proper, convex and lower semicontinuous function in Hadamard spaces.

Definition 4.1. [2] Let (X, d) be a Hadamard space with dual X^* and let $f : X \to]-\infty, \infty]$ be a proper function with efficient domain $D(f) := \{x \in X : f(x) < \infty\}$. The subdifferential operator $\partial f : X \to 2^{X^*}$ is defined by

$$\partial f = \begin{cases} \{x^* \in X^* : f(z) - f(x) \geq \langle x^*, \overrightarrow{xz} \rangle; \quad \forall z \in X\}, & x \in D(f) \\ \emptyset, & \text{otherwise.} \end{cases}$$

Theorem 4.2. [2, 15] Suppose $f : X \to]-\infty, \infty]$ be a proper, convex and lower semi-continuous function in a Hadamard space X, then

(i) f attains its minimum at $x \in X$ if and only if $0 \in \partial f(x)$,

(ii) $\partial f : X \to 2^{X^*}$ is a monotone operator,

(iii) for any $y \in X$ and $\alpha > 0$ there exists a unique point $x \in X$ such that $[\alpha \overrightarrow{xy}] \in \partial f(x)$.

Note that the third part of Theorem 4.2 shows that the subdifferential operator of a proper, convex and lower semi-continuous function satisfies the range condition in Hadamard spaces.

Theorem 4.3. Let (X, d) be a Hadamard space with dual X^*, $f : X \to X$ be a contractive mapping with contractive coefficient $k \in (0, \frac{1}{2})$, and $g : X \to (-\infty, +\infty]$ be a proper, convex and lower semicontinuous function such that *Argmin* $g \neq \emptyset$. Suppose that $(\lambda_n) \subset (0, \infty)$ and $(\alpha_n) \subset (0, 1)$ are sequences that satisfy the conditions

$C1:\quad \lim_{n\to\infty} \alpha_n = 0$,
$C2:\quad \sum_{n=1}^\infty \alpha_n = \infty$,
$C3:\quad \lambda_n > \lambda > 0, \quad \forall n > 0.$

Then the sequence generated by

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n) J^{\partial g}_{\lambda_n} x_n \\ x_1 \in X, \end{cases} \tag{4.1}$$

converges strongly to $p \in Argmin\ g$ such that $p = P_{Argmin\ g} f(p)$ and p also is the unique solution to the following variational inequality:

$$\langle \overrightarrow{pf(p)}, \overrightarrow{xp} \rangle \leq 0, \quad \forall x \in Argmin\ g.$$

Proof . Define $A := \partial g$, then the operator A is a monotone operator that satisfies the range condition. Therefore we can use Theorem 3.1 to get the desired result. $\square$

Now, we peresent an example in Hadamard space X. The problem we are setting up is to solve a nonconvex optimization problem by reducing the problem into convex optimization and apply our algorithm to solve such problem in an Hadamard space. For more details, we refer to[20].

Example 4.4. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by:

$$g(x_1, x_2) = 100\left((x_2+1) - (x_1+1)^2\right)^2 + x_1^2.$$

The function g is not convex in the classical sense. We define a metric on $\mathbb{R}^2$ by:

$$d_H(x,y) = \sqrt{(x_1-y_1)^2 + (x_1^2 - x_2 - y_1^2 + y_2)^2},$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then, $(\mathbb{R}^2, d_H)$ is an Hadamard space with the geodesics

$$\gamma_{x,y}(t) = \left((1-t)x_1 + ty_1, ((1-t)x_1 + ty_1)^2 - (1-t)(x_1^2 - x_2) - t(y_1^2 - y_2)\right).$$

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a mapping defined by $f(x_1, x_2) = (1, 1)$. Then, g is convex in $(\mathbb{R}^2, d_H)$ (see[20]). Therefore, the sequence generated by (4.1) takes the following form:

$$\begin{cases} J_{\lambda_n}^{\partial g}(x_n) = Argmin_{y\in X}\left[g(y) + \frac{1}{2\lambda_n} d^2(y, x_n)\right], \\ x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n) J_{\lambda_n}^{\partial g}(x_n) \\ x_1 \in X. \end{cases} \tag{4.2}$$

We choose $\alpha_n = \frac{1}{n+1}$ and $\lambda_n = n$ for all $n \in \mathbb{N}$. Using the algorithm 4.2 with the initial point $x_1 = (0.6, 0.5)$, the conditions of theorem 4.3 are satisfied, and hence, we conclude that the sequence (x_n) generated by 4.2 converges to a solution of the problem (1.1).

Also, we can approximate a fixed point of a nonexpansive mapping by Theorem 3.1. If $T : H \to H$ is a nonexpansive selfmapping on a Hilbert space H, then $I - T$ is a maximal monotone operator on H, where I is the identity mapping and hence it satisfies the range condition. Khatibzade and Ranjbar [15] verify the maximal monotonicity and the range condition for the operator $I - T$ in Hadamard spaces as follows.

Lemma 4.5. [15, Lemma 6.1] Let (X, d) be a CAT(0) space and let $T : X \to X$ be an arbitrary nonexpansive mapping. Then operator $Ax = [\overrightarrow{Txx}]$ is monotone. In the other word for all $x, y \in X$

$$\langle \overrightarrow{Txx} - \overrightarrow{Tyy}, \overrightarrow{yx} \rangle \geq 0.$$

Proposition 4.6. [15, proposition 6.3] Let (X, d) be a Hadamard space and let $T : X \to X$ be an arbitrary nonexpansive mapping. If the monotone operator $Ax = [\overrightarrow{Txx}]$ is maximal, then it satisfies the range condition.

Proposition 4.7. [15, proposition 6.4] Let X be a Hadamard space. For every nonexpansive mapping $T : X \to X$, the operator $Ax = [\overrightarrow{Txx}]$ satisfies the range condition if and only if

$$d^2(\lambda x \oplus (1-\lambda)y, z) = \lambda d^2(x,z) + (1-\lambda)d^2(y,z) - \lambda(1-\lambda)d^2(x,y),$$

for all $x, y, z \in X$.

In the following theorem, which is easily obtained from Theorem 3.1, a fixed point of a nonexpansive mapping is approximated via viscosity approximation method for resolvent of monotone operators in Hadamard spaces.

Theorem 4.8. Let X be a Hadamard space with dual X^* and T be nonexpansive selfmappings on X such that $F(T) \neq \emptyset$. Suppose $f : X \to X$ be a contraction mapping with contraction coefficient $k \in (0, \frac{1}{2})$, $(\lambda_n) \subset (0, \infty)$ and $(\alpha_n) \subset (0, 1)$ are sequences that satisfy the conditions

$C1: \quad \lim_{n\to\infty} \alpha_n = 0,$
$C2: \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$
$C3: \quad \lambda_n > \lambda > 0, \quad \forall n > 0.$

Then the sequence generated by (3.1) with the operator $Az = [\overrightarrow{Tzz}]$ is convergent strongly to $p \in F(T)$ such that $p = P_{F(T)} f(p)$ and p also is the unique solution to the following variational inequality:

$$\langle \overrightarrow{pf(p)}, \overrightarrow{xp} \rangle \geq 0, \quad \forall x \in F(T).$$

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