Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 815–827
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2020.20585.2177

A study on dependent impulsive integro-differential evolution equations of general type in Banach space

Abdelati El Allaoui[a,*], Said Melliani[b], Lalla Saadia Chadli[b], JinRong Wang[c]

[a]MISCOM, National School of Applied Sciences, Cadi Ayyad University, Safi, Morocco

[b]Laboratory of Applied Mathematics and Scientific Calculus, Sultan Moulay Slimane University, BP 523, 23000 Beni Mellal, Morocco

[c]Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, China

(Communicated by Javad Damirchi)

Abstract

This paper deals with the study of a coupled system of generalized impulsive integro-differential evolution equations with periodic boundary value. We show the existence and uniqueness of the solution for the proposed problem using Banach fixed point theorem. Another way was used to show the existence result with the aim of breaking out of the widely used Theorems style, we take advantage Monch's fixed point theorem using a non-compactness measure that we introduced. Some examples are given to our obtained results.

Keywords: Boundary conditions, Evolution equations, Integro-differential equation, Existence, Impulses, Measure of noncampactness
2020 MSC: 47J35, 47G20, 34A12, 35R12

1 Introduction

The modeling of several real world problems by evolution equations has pushed researchers, notably mathematicians, to research the development of this field (see [2, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 16, 18, 19]). Different types of integro-different equations that are a branch of evolution equations have been treated by several researchers [1, 17]. We quote that, in [1] the authors discussed with more details the following integro-differential equation

$$\begin{cases} x'(t) = Ax(t) + \int_0^t B(t-s)x(s)ds + \varphi(t, x(t)) \text{ for } t \in [0, a] \text{ and } t \neq t_1 \\ \Delta x\left(t_i\right) = I_i\left(x\left(t_i\right)\right) \text{ for } i = 1, \ldots, p \text{ and } 0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = a \\ x(0) = g(x) \end{cases}$$

where A and B are two closed linear operators. To show the existence of solution for this problem, they used Darbo's fixed point Theorem as a tool.

*Corresponding author
Email addresses: a.elallaoui@uca.ma (Abdelati El Allaoui), said.melliani@gmail.com (Said Melliani), sa.chadli@yahoo.fr (Lalla Saadia Chadli), wjr9669@126.com (JinRong Wang)

Received: June 2020 *Accepted:* August 2020

From above, we were motivated to study the following coupled system of a more general class of impulsive periodic boundary value integro-differential equations:

$$\begin{cases} x'(t) = Ax(t) + \int_0^t B_1(t-\tau)x(\tau)d\tau + \varphi_1(t, x(t), y(t)), & t \in (s_i, t_{i+1}], \quad i = 0, 1, 2, \cdots, m, \ (1) \\ y'(t) = By(t) + \int_0^t B_2(t-\tau)y(\tau)d\tau + \varphi_2(t, x(t), y(t)), & t \in (s_i, t_{i+1}], \quad i = 0, 1, 2, \cdots, m, \ (2) \\ x(t) = T(t-t_i)\psi_{1i}(t, x(t), y(t)), & t \in (t_i, s_i], \quad i = 1, 2, \cdots, m, \\ y(t) = S(t-t_i)\psi_{2i}(t, x(t), y(t)), & t \in (t_i, s_i], \quad i = 1, 2, \cdots, m, \\ x(s_i) + g_1(x, y) = x_i \in X, & i = 1, \cdots, m, \\ y(s_i) + g_2(x, y) = y_i \in X, & i = 1, \cdots, m, \\ x(0) = x(a), \\ y(0) = y(a). \end{cases} \tag{1.1}$$

Provided, the operators $A : D(A) \subset X \longrightarrow X$ and $B : D(B) \subset X \longrightarrow X$ are the infinitesimal generators of a uniformly continuous semigroup $\{T(t), t \geq 0\}$ and $\{S(t), t \geq 0\}$ respectively on a Banach space X provided with a norm $\|.\|$, where they satisfy $\|T(t)\| \leq M_T e^{\omega_T a}$ and $\|S(t)\| \leq M_S e^{\omega_S a}$, B_1 and B_2 are two closed linear operators on X which satisfy $D(A) \subset D(B_1)$ and $D(B) \subset D(B_2)$, and for each $x \in X$ the maps $t \longmapsto B_1(t)x$ and $t \longmapsto B_2(t)x$ are bounded differentiable and the maps $t \longmapsto B_1'(t)x$ and $t \longmapsto B_2'(t)x$ are bounded uniformly continuous on $[0, +\infty)$.

and the fixed points s_i and t_i satisfy

$$0 = s_0 < t_1 \leq s_1 \leq t_2 < \cdots < t_m \leq s_m \leq t_{m+1} = a$$

are pre-fixed numbers, $\varphi_1, \varphi_2 : \ ,(s_i, t_{i+1}] \times X \times X \longrightarrow X$, $\psi_{1i}, \psi_{2i} : \ ,(t_i, s_i] \times X \times X \longrightarrow X$ and $g_1, g_2 : \mathcal{PC}([0,a], X) \times \mathcal{PC}([0,a], X) \longrightarrow X$ are given functions, such that $T(t-t_i)\psi_{1i}(t, x(t), y(t))|_{t=s_i} = x_i - g_1(x, y)$ and $S(t-t_i)\psi_{2i}(t, x(t), y(t))|_{t=s_i} = y_i - g_2(x, y)$; $i = 1, \cdots, m$.

To show the existence of solution for this problem we use Banach and Monch's fixed point theorems and by introducing a measure of noncompactness.

2 Preliminaries

In this section we recall same basic notions used to build our result.

Denote by $\mathcal{B}(Y)$ the set of all bounded subsets of a Banach space Y.

Definition 2.1. We say that $m : \mathcal{B}(Y) \longrightarrow \mathbb{R}^+$ is a measure of noncompactness on Y if the following proprieties are satisfied:

1. $m(A) = 0$ if and only if A is precompact.
2. $m(A) = m(\overline{A})$, for all $A \in \mathcal{B}(Y)$.
3. $m(A \cup B) = \max\{m(A), m(B)\}$, for all A, $B \in \mathcal{B}(Y)$.

We recall the Kuratowski measure of noncompactness defined by

$$m(A) = \inf\left\{\rho > 0 : A \subset \cup_{j=1}^m A_j, \operatorname{diam}(A_j) \leq \rho\right\}, \text{ for } A \in \mathcal{B}(Y).$$

Now, we present the following theorem called Monch's fixed point theorem on which we will be based to show the existence of our solution.

Theorem 2.2. [15] Let Ω be a bounded, closed, and convex subset of Y such that $0 \in \Omega$, $\Lambda : \Omega \longrightarrow \Omega$ is a continuous mapping. Then, Λ has at least a fixed point if $C = \overline{co}(\Lambda(C))$ or $C = \Lambda(C) \cup \{0\} \Rightarrow \overline{C}$ is compact for each $C \subset \Omega$. Where $\overline{co}(\Lambda(C))$ is the closed convex hull of $\Lambda(C)$.

Let

$$L^\infty([0,a]) = \{l : [0,a] \longrightarrow \mathbb{R} : l \text{ is measurable and essentially bounded}\}.$$

With the following norm

$$\|l\|_{L^\infty} = \inf\{\beta > 0 : |l(t)| \leq \beta, \ \text{ a.e. } t \in [0,a]\}$$

$L^\infty([0,a])$ is Banach space.

Definition 2.3. [3] A resolvent operator for the problem

$$\begin{cases} x'(t) = Ax(t) + \int_0^t B(t-\tau)x(\tau)d\tau, & t \in [0, +\infty) \\ x(0) = x_0 \in Y. \end{cases}$$

is a bounded linear operator-valued function $\Gamma(t)$ satisfying the following proprieties:

1. $\Gamma(0) = I_Y$. (I_Y the identity of Y) and there exist two constants $N > 0$, and $b \in \mathbb{R}$, such that $\|\Gamma(t)\| \leq Ne^{bt}$.
2. $t \longrightarrow \Gamma(t)y$ is strongly continuous for each $y \in Y$.
3. $\Gamma(t)$ is bounded for $t \geq 0$. And for $x \in D(A), \Gamma(\cdot)x \in \mathcal{C}(\mathbb{R}_+, D(A)) \cap \mathcal{C}^1(\mathbb{R}_+, Y)$ and satisfying the following propriety
$$\Gamma'(t)x = A\Gamma(t)x + \int_0^t B(t-\tau)\Gamma(\tau)xd\tau = \Gamma(t)Ax + \int_0^t \Gamma(t-\tau)B(\tau)xd\tau; t \in [0, \infty).$$

For more details concerning the basic concepts used in this paper we refer [9].

3 Main result

Firstly, we provide the following result we need:
We define on $\mathcal{B}(X \times X)$ the map $\widehat{m}$ by

$$\widehat{m}(D \times E) = \max\{m(D), m(E)\}, \text{ for, } C \times D \in \mathcal{B}(X \times X) \subset \mathcal{B}(X) \times \mathcal{B}(X).$$

For $D \times E$, $F \times G$, $\in \mathcal{B}(X \times X)$, we have

$$\begin{aligned} &\widehat{m}(D \times E) = 0 \Leftrightarrow m(D) = 0 \,\text{and}\, m(E) = 0 \Leftrightarrow D \times E \,\text{is precompact}, \\ &\widehat{m}(\overline{D \times E}) = \widehat{m}(\overline{D} \times \overline{E}) = \max\{m(\overline{D}), m(\overline{E})\} = \max\{m(D), m(E)\} = \widehat{m}(D \times E), \end{aligned}$$

and

$$\begin{aligned} \widehat{m}\big((D \times E) \cup (F \times G)\big) = \widehat{m}\big((D \cup F) \times (E \cup G)\big) &= \max\{m(D \cup F), m(E \cup G)\} \\ &= \max\{m(D), m(F), m(E), m(F)\} \\ &= \max\{\widehat{m}\big(D \times E\big), \widehat{m}\big(F \times G\big)\}. \end{aligned}$$

So, $\widehat{m}$ is a measure of noncompactness on $X \times X$.

Now, we define the following spaces

$$\begin{aligned} \mathcal{PC}([0,a], X) = \;& \{x : [0,a] \longrightarrow X : x \in \mathcal{C}([0,t_1] \cup (t_i, s_i] \cup (s_i, t_{i+1}], X); \, i = 1, \cdots m, \\ & x(t_i^-), \, x(t_i^+), \, x(s_i^-) \,\text{and}\, x(s_i^+) \,\text{exist, with}\, x(t_i^-) = x(t_i) \,\text{and}\, x(s_i^-) = x(s_i)\big\} \end{aligned}$$

endowed with the norm $\|x\|_{\mathcal{PC}} = \sup_{t \in [0,a]} \|x(t)\|$. And

$$\mathcal{PC}^2 := \mathcal{PC}([0,a], X) \times \mathcal{PC}([0,a], X),$$

which is a Banach space with the following norm

$$\|(x,y)\|_2 = \|x\|_{\mathcal{PC}} + \|y\|_{\mathcal{PC}}, \text{ for } (x,y) \in \mathcal{PC}^2.$$

Firstly, we give the expression of mild solution for the following impulsive integro-differential equation

$$\begin{cases} x'(t) = Ax(t) + \int_0^t B(t-\tau)x(\tau)d\tau + \varphi(t, x(t)), & t \in (s_i, t_{i+1}], \quad i = 0, 1, 2, \cdots, m, \\ x(t) = T(t - t_i)\psi_i(t, x(t)), & t \in (t_i, s_i], \quad i = 1, 2, \cdots, m, \\ x(s_i) + g(x) = x_i \in X, & i = 1, \cdots, m, \\ x(0) = x(a). \end{cases}$$

For $t \in [0, t_1]$, we have

$$\begin{aligned} x(t) &= \Gamma(t)x(0) + \int_0^t \Gamma(t-\tau)\varphi(\tau, x(\tau))d\tau \\ &= \Gamma(t)x(a) + \int_0^t \Gamma(t-\tau)\varphi(\tau, x(\tau))d\tau \\ &= \Gamma(t)\left[\Gamma(a)\big(x_m - g(x)\big) + \int_{s_m}^a \Gamma(a-\tau)\varphi(\tau, x(\tau))d\tau\right] + \int_0^t \Gamma(t-\tau)\varphi(\tau, x(\tau))d\tau \\ &= \Gamma(t)\Gamma(a)\big(x_m - g(x)\big) + \Gamma(t)\int_{s_m}^a \Gamma(a-\tau)\varphi(\tau, x(\tau))d\tau + \int_0^t \Gamma(t-\tau)\varphi(\tau, x(\tau))d\tau \end{aligned}$$

Let Γ_1, Γ_2 the resolvents associated with equations (1) and (2) respectively.

Now, we can define the form of our solution, it's given in the following definition

Definition 3.1. We say that (x, y) is a mild solution of the problem (1.1) if $(x, y) \in \mathcal{PC}^2$ and satisfies the following system

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{cases} \begin{pmatrix} \Gamma(t)_1\Gamma_1(a)\big(x_m - g_1(x,y)\big) + \Gamma_1(t)\int_{s_m}^a \Gamma_1(a-\tau)\varphi_1(\tau, x(\tau), y(\tau))d\tau + \int_0^t \Gamma_1(t-\tau)\varphi_1(\tau, x(\tau), y(\tau))d\tau \\ \Gamma_2(t)\Gamma_2(a)\big(y_m - g_2(x,y)\big) + \Gamma_2(t)\int_{s_m}^a \Gamma_2(a-\tau)\varphi_2(\tau, x(\tau), y(\tau))d\tau + \int_0^t \Gamma_2(t-\tau)\varphi_2(\tau, x(\tau), y(\tau))d\tau \end{pmatrix} & t \in [0, t_1] \\ \begin{pmatrix} \Gamma_1(t)\big(x_i - g_1(x,y)\big) + \int_{s_i}^t \Gamma_1(t-\tau)\varphi_1(\tau, x(\tau), y(\tau))d\tau \\ \Gamma_2(t)\big(y_i - g_2(x,y)\big) + \int_{s_i}^t \Gamma_2(t-\tau)\varphi_2(\tau, x(\tau), y(\tau))d\tau \end{pmatrix} & \text{for } t \in (s_i, t_{i+1}], \quad i = 1, 2, \cdots, m \\ \begin{pmatrix} T(t-t_i)\psi_{1i}(t, x(t), y(t)) \\ S(t-t_i)\psi_{2i}(t, x(t), y(t)) \end{pmatrix} & \text{for } t \in (t_i, s_i], \quad i = 1, 2, \cdots, m \end{cases}$$

Now, we pose the following hypotheses on which our existence result is based.

- $\mathbf{A}_1$ The functions $t \longmapsto \varphi_j(t, x, y)$ and $t \longmapsto \psi_{ji}(t, x, y)$; ; $j = 1, 2$, are measurable on $[0, a]$ for all $(x, y) \in X \times X$, and continuous on $X \times X$ for a.e. t in $(s_i, t_{i+1}]$ and $(t_i, s_i]$, respectively.
- $\mathbf{A}_2$ There exit μ_1, μ_2, ν_{1i}, $\nu_{2i} \in \mathcal{L}^\infty([0, a])$; $i = 1, \cdots, m$, which satisfy

  $$\|\varphi_j(t, x, y)\| \le \mu_j(t)\left(1 + \|x\| + \|y\|\right); \ a.e\ t \in (s_i, t_{i+1}], \text{ and for all } x,\ y \in X;\ j = 1, 2,$$

  and

  $$\|\psi_{ji}(t, x, y)\| \le \nu_{ji}(t)\left(1 + \|x\| + \|y\|\right); \ i = 1 \cdots, m,\ a.e\ t \in (t_i, s_i], \text{ and for all } x,\ y \in X;\ j = 1, 2,$$
- $\mathbf{A}_3$ There exists a constant α_1, $\alpha_2 > 0$, such that

  $$\|g_j(x, y)\| \le \alpha_j\left(1 + \|x\|_{\mathcal{PC}} + \|y\|_{\mathcal{PC}}\right)\ a.e\ t \in [0, a], \text{ and for all } x,\ y \in \mathcal{PC}([0, a], X);\ j = 1, 2.$$
- $\mathbf{A}_4$ For all bounded set $\Theta \subset X \times X$, and $t \in [0, a]$, we have

  $$\widehat{m}(\varphi_j(t, \Theta)) \le \mu_j(t)\widehat{m}(\Theta), \text{ and } \widehat{m}(\psi_{ji}(t, \Theta)) \le \nu_{ji}(t)\widehat{m}(\Theta);\ i = 1, \cdots, m;\ j = 1; 2,$$

  and for all bounded set $\tilde{\Theta} \subset \mathcal{PC}^2$, we have

  $$\widehat{m}(g_j(\tilde{\Theta})) \le \alpha_j \sup_{t \in [0, a]} \widehat{m}(\tilde{\Theta}(t)),\ j = 1, 2,$$

  where $\tilde{\Theta}(t) = \left\{(x(t), y(t)) : (x, y) \in \mathcal{PC}^2\right\}$, for all $t \in [0, a]$.
- $\mathbf{H}_1$ The functions $\varphi_j \in \mathcal{C}([0, a] \times X \times X, X)$, $\psi_{ji} \in \mathcal{C}([s_i, t_i] \times X \times X, X)$; $i = 1, \cdots, m$; $j = 1, 2$, and g_1, g_2 are continuous.

H$_2$ There exist constants L_{φ_j}, $L_{\psi_{ji}}$, $L_{g_j} > 0$; $j = 1, 2$, $i = 1, \cdots, m$, such that, for $j = 1; 2$

$$\|\varphi_j(t, x_1, y_1) - \varphi_j(t, x_2, y_2)\| \leq L_{\varphi_j} (\|x_1 - x_2\| + \|y_1 - y_2\|), \text{ for each } t \in [s_i, t_{i+1}];\ i = 0, \cdots, m;,\ x_j,\ y_j \in X,$$
$$\|\psi_{ji}(t, x_1, y_1) - \psi_{ji}(t, x_2, y_2)\| \leq L_{\psi_{ji}} (\|x_1 - x_2\| + \|y_1 - y_2\|), \text{ for each } t \in [t_i, s_i], i = 1, \cdots, m,\ x_j,\ y_j \in X,$$
$$\|g_j(x_1, y_1) - g_j(x_2, y_2)\| \leq L_{g_j} (\|x_1 - x_2\|_{\mathcal{PC}} + \|y_1 - y_2\|_{\mathcal{PC}}), \text{ for each } x_j,\ y_j \in \mathcal{PC}([0, a], X).$$

To reduce the form of mathematical expressions, we use the following notations:

$$\begin{aligned}
&\lambda_j = \|\mu_j\|_{L^\infty},\ \sigma_j = \max_{i=1,\cdots,m} \|\nu_{ji}\|_{L^\infty},\ N_j = \sup_{t\in[0,a]} \|\Gamma_j(t)\|_{\mathcal{B}},\ j = 1, 2 \\
&r_1 = \frac{N_1 (N_1\|x_m\| + N_1\alpha_1 + aN_1\lambda_1 + a\lambda_1) + N_2 (N_2\|y_m\| + N_2\alpha_2 + aN_2\lambda_2 + a\lambda_2)}{1 - \delta_1}, \\
&r_2 = \frac{N_1 (\max_i \|x_i\| + \alpha_1 + a\lambda_1) + N_2 (\max_i \|y_i\| + \alpha_2 + a\lambda_2)}{1 - \big(N_1 (\alpha_1 + a\lambda_1) + N_2 (\alpha_2 + a\lambda_2)\big)}, \\
&r_3 = \frac{\delta_2}{1 - \delta_2}, \\
&\delta_1 = N_1 (N_1\alpha_1 + aN_1\lambda_1 + a\lambda_1) + N_2 (N_2\alpha_2 + aN_2\lambda_2 + a\lambda_2), \\
&\delta_2 = \sigma_1 M_T e^{\omega_T a} + \sigma_2 M_S e^{\omega_S a},\ \delta = \max\{\delta_1, \delta_2\} \\
&\kappa_{11} = N_1 \left(L_{g_1} N_1 + L_{\varphi_1} N_1 \max_{i=1,\cdots,m} (t_{i+1} - s_i) + L_{\varphi_1} t_1 \right), \\
&\kappa_{12} = N_2 \left(L_{g_2} N_2 + L_{\varphi_2} N_2 \max_{i=1,\cdots,m} (t_{i+1} - s_i) + L_{\varphi_2} t_1 \right), \\
&\kappa_{21} = \max_{i=1,\cdots,m} L_{\psi_{1i}} M_T e^{\omega_T a},\ \text{and } \kappa_{22} = \max_{i=1,\cdots,m} L_{\psi_{2i}} M_S e^{\omega_S a}.
\end{aligned}$$

After provided assumptions, now we are in a position to present our first existence result based on Banach's fixed point theorem.

Theorem 3.2. Let assumptions **H**$_1$ and **H**$_2$ be satisfied. Suppose also that

$$\kappa := \max\{\kappa_{11} + \kappa_{12}, \kappa_{21} + \kappa_{22}\} < 1.$$

Then, the problem (1.1) has a unique mild solution on $[0, a]$.

Proof . We define on $\mathcal{PC}^2$ the following operator

$$(\Lambda(x, y))(t) = \big(\Lambda_1(x, y)(t), \Lambda_2(x, y)(t)\big), \tag{3.1}$$

where

$$\Lambda_1(x, y)(t) = \begin{cases} \Gamma(t)_1 \Gamma_1(a)\big(x_m - g_1(x, y)\big) + \Gamma_1(t) \int_{s_m}^a \Gamma_1(a - \tau)\varphi_1(\tau, x(\tau), y(\tau))d\tau + \int_0^t \Gamma_1(t - \tau)\varphi_1(\tau, x(\tau), y(\tau))d\tau, & t \in [0, t_1] \\ \Gamma_1(t)\big(x_i - g_1(x, y)\big) + \int_{s_i}^t \Gamma_1(t - \tau)\varphi_1(\tau, x(\tau), y(\tau))d\tau,\ t \in (s_i, t_{i+1}], & i = 1, 2, \cdots, m \\ T(t - t_i)\psi_{1i}(t, x(t), y(t)), \quad \text{for } t \in (t_i, s_i], & i = 1, 2, \cdots, m \end{cases}$$

and

$$\Lambda_2(x, y)(t) = \begin{cases} \Gamma_2(t)\Gamma_2(a)\big(y_m - g_2(x, y)\big) + \Gamma_2(t) \int_{s_m}^a \Gamma_2(a - \tau)\varphi_2(\tau, x(\tau), y(\tau))d\tau + \int_0^t \Gamma_2(t - \tau)\varphi_2(\tau, x(\tau), y(\tau))d\tau, & t \in [0, t_1] \\ \Gamma_2(t)\big(y_i - g_2(x, y)\big) + \int_{s_i}^t \Gamma_2(t - \tau)\varphi_2(\tau, x(\tau), y(\tau))d\tau, \quad \text{for } t \in (s_i, t_{i+1}], & i = 1, 2, \cdots, m \\ S(t - t_i)\psi_{2i}(t, x(t), y(t)), \quad \text{for } t \in (t_i, s_i], & i = 1, 2, \cdots, m \end{cases}$$

Let (x_1, y_1), $(x_2, y_2) \in \mathcal{PC}^2$, we discuss all possible cases.

Case 1: For $t \in [0, t_1]$, we have

$$\begin{aligned}
\|\Lambda_1(x_1,y_1)(t) - \Lambda_1(x_2,y_2)(t)\| &\leq \|\mathit{\Gamma}_1(t)\mathit{\Gamma}_1(a)g_1(x_1,y_1) - \mathit{\Gamma}_1(t)\mathit{\Gamma}_1(a)g_1(x_2,y_2)\| \\
&\quad + \|\mathit{\Gamma}_1(t)\|_{\mathcal{B}} \int_{s_m}^{a} \|\mathit{\Gamma}_1(a-\tau)\|_{\mathcal{B}} \|\varphi_1(\tau, x_1(\tau), y_1(\tau)) - \varphi_1(\tau, x_2(\tau), y_2(\tau))\| d\tau \\
&\quad + \int_0^t \|\mathit{\Gamma}_1(t-\tau)\|_{\mathcal{B}} \|\varphi_1(\tau, x_1(\tau), y_1(\tau)) - \varphi_1(\tau, x_2(\tau), y_2(\tau))\| d\tau \\
&\leq L_{g_1} N_1^2 \left(\|x_1 - x_2\|_{\mathcal{PC}} + \|y_1 - y_2\|_{\mathcal{PC}}\right) \\
&\quad + L_{\varphi_1} N_1^2 \int_{s_m}^{a} \left(\|x_1(\tau) - x_2(\tau)\| + \|y_1(\tau) - y_2(\tau)\|\right) d\tau \\
&\quad + L_{\varphi_1} N_1 \int_0^t \left(\|x_1(\tau) - x_2(\tau)\| + \|y_1(\tau) - y_2(\tau)\|\right) d\tau \\
&\leq N_1 \left(L_{g_1} N_1 + L_{\varphi_1} N_1 (a - s_m) + L_{\varphi_1} t_1\right) \left(\|x_1 - x_2\|_{\mathcal{PC}} + \|y_1 - y_2\|_{\mathcal{PC}}\right) \\
&\leq N_1 \left(L_{g_1} N_1 + L_{\varphi_1} N_1 (a - s_m) + L_{\varphi_1} t_1\right) \|(x_1,y_1) - (x_2,y_2)\|_2 \\
&\leq \kappa_{11} \|(x_1,y_1) - (x_2,y_2)\|_2
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\|\Lambda_2(x_1,y_1)(t) - \Lambda_2(x_2,y_2)(t)\| &\leq N_2 \left(L_{g_2} N_2 + L_{\varphi_2} N_2 (a - s_m) + L_{\varphi_2} t_1\right) \|(x_1,y_1) - (x_2,y_2)\|_2 \\
&\leq \kappa_{12} \|(x_1,y_1) - (x_2,y_2)\|_2
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
\|\Lambda(x_1,y_1)(t) - \Lambda(x_2,y_2)(t)\| &= \|\Lambda_1(x_1,y_1)(t) - \Lambda_1(x_2,y_2)(t)\| + \|\Lambda_2(x_1,y_1)(t) - \Lambda_2(x_2,y_2)(t)\| \\
&\leq (\kappa_{11} + \kappa_{12}) \|(x_1,y_1) - (x_2,y_2)\|_2
\end{aligned}$$

Case 2: For $t \in (s_i, t_{i+1}]$; $i = 1, \cdots, m$, we have

$$\begin{aligned}
\|\Lambda_1(x_1,y_1)(t) - \Lambda_1(x_2,y_2)(t)\| &\leq \|\mathit{\Gamma}_1(t)\|_{\mathcal{B}} \|g_1(x_1,y_1) - g_1(x_2,y_2)\| \\
&\quad + \int_{s_i}^{t} \|\mathit{\Gamma}_1(t-\tau)\|_{\mathcal{B}} \|\varphi_1(\tau, x_1(\tau), y_1(\tau)) - \varphi_1(\tau, x_2(\tau), y_2(\tau))\| d\tau \\
&\leq L_{g_1} N_1 \left(\|x_1 - x_2\|_{\mathcal{PC}} + \|y_1 - y_2\|_{\mathcal{PC}}\right) \\
&\quad + L_{\varphi_1} N_1 \int_{s_i}^{t} \left(\|x_1(\tau) - x_2(\tau)\| + \|y_1(\tau) - y_2(\tau)\|\right) d\tau \\
&\leq N_1 \left(L_{g_1} + L_{\varphi_1} \max_{i=1,\cdots,m} (t_{i+1} - s_i)\right) \|(x_1,y_1) - (x_2,y_2)\|_2 \\
&\leq \kappa_{11} \|(x_1,y_1) - (x_2,y_2)\|_2
\end{aligned}$$

Likewise, we get

$$\begin{aligned}
\|\Lambda_2(x_1,y_1)(t) - \Lambda_2(x_2,y_2)(t)\| &\leq N_2 \left(L_{g_2} + L_{\varphi_2} \max_{i=1,\cdots,m} (t_{i+1} - s_i)\right) \|(x_1,y_1) - (x_2,y_2)\|_2 \\
&\leq \kappa_{12} \|(x_1,y_1) - (x_2,y_2)\|_2
\end{aligned}$$

Hence,

$$\|\Lambda(x_1,y_1)(t) - \Lambda(x_2,y_2)(t)\| \leq (\kappa_{11} + \kappa_{12}) \|(x_1,y_1) - (x_2,y_2)\|_2$$

Case 3: For $t \in (t_i, s_i]$; $i = 1, \cdots, m$, we have

$$\begin{aligned}
\|\Lambda_1(x_1,y_1)(t) - \Lambda_1(x_2,y_2)(t)\| &\leq \|T(t - t_i)\| \|\psi_{1i}(t, x_1(t), y_1(t)) - \psi_{1i}(t, x_2(t), y_2(t))\| \\
&\leq L_{\psi_{1i}} M_T e^{\omega_T a} \|(x_1,y_1) - (x_2,y_2)\|_2 \\
&\leq \kappa_{21} \|(x_1,y_1) - (x_2,y_2)\|_2
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
\|\Lambda_2(x_1,y_1)(t) - \Lambda_2(x_2,y_2)(t)\| &\leq L_{\psi_{2i}} M_S e^{\omega_S a} \|(x_1,y_1) - (x_2,y_2)\|_2 \\
&\leq \kappa_{22} \|(x_1,y_1) - (x_2,y_2)\|_2
\end{aligned}$$

Then, we have

$$\|\Lambda(x_1,y_1)(t) - \Lambda(x_2,y_2)(t)\| \leq (\kappa_{21} + \kappa_{22}) \|(x_1,y_1) - (x_2,y_2)\|_2$$

Finally, we get the following inequality

$$\|\Lambda(x_1,y_1)-\Lambda(x_2,y_2)\|_2 \leq \max\{\kappa_{11}+\kappa_{12},\kappa_{21}+\kappa_{22}\}\|(x_1,y_1)-(x_2,y_2)\|_2$$

Therefore, Λ is a contraction. So, according to Banach fixed point theorem, problem (1.1) has a unique mild solution. □ Using Monch's fixed point theorem, we present the second result of existence as follows:

Theorem 3.3. Suppose that assumptions $\mathbf{A}_1$-$\mathbf{A}_4$ are satisfied, in addition

$$\delta < 1. \tag{3.2}$$

Then problem (1.1) has at least one mild solution on $[0,a]$.

Proof . To proof this result we transform our problem into fixed point, for this we consider the operator $\Lambda : \mathcal{PC}^2 \longrightarrow \mathcal{PC}^2$ defined in (3.1), and we define the ball $B_r := \left\{(x,y)\in\mathcal{PC}^2 : \|(x,y)\|_2 \leq r\right\}$, where

$$r \geq \max\{r_1,r_2,r_3\}.$$

Firstly, we prove that Λ is defined from B_r into itself. Indeed:

Case 1: For $(x,y)\in B_r$, and $t\in[0,t_1]$, we have

$$\begin{aligned}\|\Lambda_1(x,y)(t)\| &\leq \|\varGamma_1(t)\|_{\mathcal{B}}\|\varGamma_1(a)\|_{\mathcal{B}}\big(\|x_m\|+\|g_1(x,y)\|\big)+\|\varGamma_1(t)\|_{\mathcal{B}}\int_{s_m}^{a}\|\varGamma_1(a-\tau)\|_{\mathcal{B}}\|\varphi_1(\tau,x(\tau),y(\tau))\|d\tau\\ &\quad+\int_0^t\|\varGamma_1(t-\tau)\|_{\mathcal{B}}\|\varphi_1(\tau,x(\tau),y(\tau))\|d\tau\\ &\leq N_1^2\big(\|x_m\|+\alpha_1(1+\|x\|_{\mathcal{PC}}+\|y\|_{\mathcal{PC}})\big)+aN_1^2\lambda_1(1+\|x\|_{\mathcal{PC}}+\|y\|_{\mathcal{PC}})+aN_1\lambda_1(1+\|x\|_{\mathcal{PC}}+\|y\|_{\mathcal{PC}})\\ &\leq N_1\left(N_1\|x_m\|+N_1\alpha_1+aN_1\lambda_1+a\lambda_1\right)+N_1\left(N_1\alpha_1+aN_1\lambda_1+a\lambda_1\right)r\end{aligned}$$

Similarly, we get

$$\|\Lambda_2(x,y)(t)\| \leq N_2\left(N_2\|y_m\|+N_2\alpha_2+aN_2\lambda_2+a\lambda_2\right)+N_2\left(N_2\alpha_2+aN_2\lambda_2+a\lambda_2\right)r$$

Then,

$$\begin{aligned}\|\Lambda(x,y)\|_2 &= \|\Lambda_1(x,y)\|_2+\|\Lambda_2(x,y)\|_2\\ &\leq N_1\left(N_1\|x_m\|+N_1\alpha_1+aN_1\lambda_1+a\lambda_1\right)\\ &\quad+N_2\left(N_2\|y_m\|+N_2\alpha_2+aN_2\lambda_2+a\lambda_2\right)\\ &\quad+\left[N_1\left(N_1\alpha_1+aN_1\lambda_1+a\lambda_1\right)+N_2\left(N_2\alpha_2+aN_2\lambda_2+a\lambda_2\right)\right]r\\ &\leq (1-\delta_1)r_1+\delta_1 r\\ &\leq r\end{aligned}$$

Case 2: For $(x,y)\in B_r$, and $t\in(s_i,t_{i+1}]$; $i=1,\cdots,m$, we have

$$\begin{aligned}\|\Lambda_1(x,y)(t)\| &\leq \|\varGamma_1(t)\|_{\mathcal{B}}\big(\|x_i\|+\|g_1(x,y)\|\big)+\int_{s_i}^{t}\|\varGamma_1(t-\tau)\|_{\mathcal{B}}\|\varphi_1(\tau,x(\tau),y(\tau))\|d\tau\\ &\leq N_1\big(\|x_i\|+\alpha_1(1+\|x\|_{\mathcal{PC}}+\|y\|_{\mathcal{PC}})\big)+aN_1\lambda_1(1+\|x\|_{\mathcal{PC}}+\|y\|_{\mathcal{PC}})\\ &\leq N_1\left(\|x_i\|+\alpha_1+a\lambda_1\right)+N_1\left(\alpha_1+a\lambda_1\right)r\end{aligned}$$

In the same way, we get

$$\|\Lambda_2(x,y)(t)\| \leq N_2\left(\|y_i\|+\alpha_2+a\lambda_2\right)+N_2\left(\alpha_2+a\lambda_2\right)r.$$

Therefore,

$$\begin{aligned}\|\Lambda(x,y)\| &\leq N_1\left(\|x_i\|+\alpha_1+a\lambda_1\right)+N_2\left(\|y_i\|+\alpha_2+a\lambda_2\right)+\left[N_1\left(\alpha_1+a\lambda_1\right)+N_2\left(\alpha_2+a\lambda_2\right)\right]r\\ &\leq \big(1-\left[N_1\left(\alpha_1+a\lambda_1\right)+N_2\left(\alpha_2+a\lambda_2\right)\right]\big)r_2+\left[N_1\left(\alpha_1+a\lambda_1\right)+N_2\left(\alpha_2+a\lambda_2\right)\right]r\\ &\leq r.\end{aligned}$$

Case 3: For $(x,y) \in B_r$, and $t \in (t_i, s_i]$; $i = 1, \cdots, m$, we have

$$\begin{aligned}\|\Lambda_1(x,y)(t)\| &\leq \|T(t-t_i)\|\|\psi_{1i}(t,x(t),y(t))\| \\ &\leq \sigma_1 M_T e^{\omega_T a}(1+r)\end{aligned}$$

Similarly, we obtain

$$\|\Lambda_2(x,y)(t)\| \leq \sigma_2 M_S e^{\omega_S a}(1+r)$$

Then,

$$\begin{aligned}\|\Lambda(x,y)\|_2 &\leq \left(\sigma_1 M_T e^{\omega_T a} + \sigma_2 M_S e^{\omega_S a}\right)(1+r) \\ &\leq (1-\delta_2)r_3 + \delta_2 r \\ &\leq r.\end{aligned}$$

which shows that Λ is defined from B_r into itself.

The rest of proof will be done in four steps by discussing all cases in each step.

Step 1: Λ is continuous:

Let $(x_n, y_n)_{n\geq 0} \subset B_r$ be a sequence, such that $\lim_{n\to+\infty}(x_n, y_n) = (x,y)$ in B_r.

Clearly, we have $\|(x_n,y_n)-(x,y)\|_2 = \|x_n - x\|_{\mathcal{PC}} + \|y_n - y\|_{\mathcal{PC}}$ which implies that

$$\lim_{n\to+\infty}(x_n,y_n) = (x,y) \text{ in } B_r \text{ if and only if } \lim_{n\to+\infty} x_n = x \text{ and } \lim_{n\to+\infty} y_n = y \text{ in } \{x \in \mathcal{PC}([0,a],X) : \|x\|_{\mathcal{PC}} \leq r\}.$$

Case 1: For $t \in [0, t_1]$, we have

$$\begin{aligned}\|\Lambda_1(x_n,y_n)(t) - \Lambda_1(x,y)(t)\| &\leq \|\mathit{\Gamma}_1(t)\|_{\mathcal{B}}\|\mathit{\Gamma}_1(a)\|_{\mathcal{B}}\|g_1(x_n,y_n) - g_1(x,y)\| \\ &\quad + \|\mathit{\Gamma}_1(t)\|_{\mathcal{B}}\int_{s_m}^a \|\mathit{\Gamma}_1(a-\tau)\|_{\mathcal{B}}\|\varphi_1(\tau,x_n(\tau),y_n(\tau)) - \varphi_1(\tau,x(\tau),y(\tau))\|d\tau \\ &\quad + \int_0^t \|\mathit{\Gamma}_1(t-\tau)\|_{\mathcal{B}}\|\varphi_1(\tau,x_n(\tau),y_n(\tau)) - \varphi_1(\tau,x(\tau),y(\tau))\|d\tau \\ &\leq N_1^2\|g_1(x_n,y_n) - g_1(x,y)\| \\ &\quad + N_1^2\int_{s_m}^a \|\varphi_1(\tau,x_n(\tau),y_n(\tau)) - \varphi_1(\tau,x(\tau),y(\tau))\|d\tau \\ &\quad + N_1\int_0^t \|\varphi_1(\tau,x_n(\tau),y_n(\tau)) - \varphi_1(\tau,x(\tau),y(\tau))\|d\tau\end{aligned}$$

And

$$\begin{aligned}\|\Lambda_2(x_n,y_n)(t) - \Lambda_2(x,y)(t)\| &\leq N_2^2\|g_2(x_n,y_n) - g_2(x,y)\| \\ &\quad + N_2^2\int_{s_m}^a \|\varphi_2(\tau,x_n(\tau),y_n(\tau)) - \varphi_2(\tau,x(\tau),y(\tau))\|d\tau \\ &\quad + N_2\int_0^t \|\varphi_2(\tau,x_n(\tau),y_n(\tau)) - \varphi_2(\tau,x(\tau),y(\tau))\|d\tau\end{aligned}$$

Case 2: For $t \in (s_i, t_{i+1}]$; $i = 1, \cdots, m$, we have

$$\begin{aligned}\|\Lambda_1(x_n,y_n)(t) - \Lambda_1(x,y)(t)\| &\leq \|\mathit{\Gamma}_1(t)\|_{\mathcal{B}}\|g_1(x_n,y_n) - g_1(x,y)\| \\ &\quad + \int_{s_i}^t \|\mathit{\Gamma}_1(t-\tau)\|_{\mathcal{B}}\|\varphi_1(\tau,x_n(\tau),y_n(\tau)) - \varphi_1(\tau,x(\tau),y(\tau))\|d\tau \\ &\leq N_1\|g_1(x_n,y_n) - g_1(x,y)\| \\ &\quad + N_1\int_{s_i}^t \|\varphi_1(\tau,x_n(\tau),y_n(\tau)) - \varphi_1(\tau,x(\tau),y(\tau))\|d\tau\end{aligned}$$

Similarly, we get

$$\|\Lambda_2(x_n,y_n)(t) - \Lambda_2(x,y)(t)\| \leq N_2\|g_2(x_n,y_n) - g_2(x,y)\| + N_2\int_{s_i}^t \|\varphi_2(\tau,x_n(\tau),y_n(\tau)) - \varphi_2(\tau,x(\tau),y(\tau))\|d\tau$$

Case 3: For $t \in (t_i, s_i]$; $i = 1, \cdots, m$, we have

$$\begin{aligned}\|\Lambda_1(x_n, y_n)(t) - \Lambda_1(x, y)(t)\| &\leq \|T(t-t_i)\| \|\psi_{1i}(t, x_n(t), y_n(t)) - \psi_{1i}(t, x(t), y(t))\| \\ &\leq M_T e^{\omega_T a} \|\psi_{1i}(t, x_n(t), y_n(t)) - \psi_{1i}(t, x(t), y(t))\|\end{aligned}$$

And

$$\|\Lambda_2(x_n, y_n)(t) - \Lambda_2(x, y)(t)\| \leq M_S e^{\omega_S a} \|\psi_{2i}(t, x_n(t), y_n(t)) - \psi_{2i}(t, x(t), y(t))\|$$

We know that, φ_j, ψ_{ji} and g_j; $j = 1, 2$; $i = 1, \cdots, m$ are continuous, then according to Lebesgue-dominated convergence theorem, we get from each previous step

$$\lim_{n\to+\infty} \|\Lambda_1(x_n, y_n) - \Lambda_1(x, y)\|_2 = 0 \text{ and } \lim_{n\to+\infty} \|\Lambda_2(x_n, y_n) - \Lambda_2(x, y)\|_2 = 0,$$

and since from (3.1) we have

$$\lim_{n\to+\infty} \|\Lambda(x_n, y_n) - \Lambda(x, y)\|_2 = 0$$
$$\Updownarrow$$
$$\left(\lim_{n\to+\infty} \|\Lambda_1(x_n, y_n) - \Lambda_1(x, y)\| = 0 \quad \text{and} \quad \lim_{n\to+\infty} \|\Lambda_2(x_n, y_n) - \Lambda_2(x, y)\| = 0\right).$$

So, we deduce that $\lim_{n\to+\infty} \|\Lambda(x_n, y_n) - \Lambda(x, y)\|_2 = 0$.

Step 2: $\Lambda(B_r)$ is bounded. Indeed:

We have Λ is defined on B_r into itself. So, $\Lambda(B_r) \subset B_r$ which prove that $\Lambda(B_r)$ is bounded.

Step 3: Λ is equicontinuous.

Case 1: For $(x, y) \in B_r$ and $0 \leq \tau_1 < \tau_2 \leq t_1$, we have

$$\begin{aligned}\|\Lambda_1(x, y)(\tau_2) - \Lambda_1(x, y)(\tau_1)\| &\leq \|\Gamma_1(a)\|_{\mathcal{B}} \big(\|x_m\| + \|g_1(x, y)\|\big) \|\Gamma_1(\tau_2) - \Gamma_1(\tau_1)\|_{\mathcal{B}} \\ &\quad + \|\Gamma_1(\tau_2) - \Gamma_1(\tau_1)\|_{\mathcal{B}} \int_{s_m}^{a} \|\Gamma_1(a-\tau)\|_{\mathcal{B}} \|\varphi_1(\tau, x(\tau), y(\tau))\| d\tau \\ &\quad + \int_0^{\tau_1} \|\Gamma_1(\tau_2-\tau) - \Gamma_1(\tau_1-\tau)\|_{\mathcal{B}} \|\varphi_1(\tau, x(\tau), y(\tau))\| d\tau \\ &\quad + \int_{\tau_1}^{\tau_2} \|\Gamma_1(\tau_2-\tau)\|_{\mathcal{B}} \|\varphi_1(\tau, x(\tau), y(\tau))\| d\tau \\ &\leq N_1 \big[\big(\|x_m\| + \alpha_1(1+r)\big) + (a-s_m)\lambda_1(1+r)\big] \|\Gamma_1(\tau_2) - \Gamma_1(\tau_1)\|_{\mathcal{B}} \\ &\quad + \lambda_1(1+r) \int_0^{\tau_1} \|\Gamma_1(\tau_2-\tau) - \Gamma_1(\tau_1-\tau)\|_{\mathcal{B}} d\tau + N_1\lambda_1(1+r)(\tau_2-\tau_1)\end{aligned}$$

In the same manner, we get

$$\begin{aligned}\|\Lambda_2(x, y)(\tau_2) - \Lambda_2(x, y)(\tau_1)\| &\leq N_2 \big[\big(\|y_m\| + \alpha_2(1+r)\big) + (a-s_m)\lambda_2(1+r)\big] \|\Gamma_2(\tau_2) - \Gamma_2(\tau_1)\|_{\mathcal{B}} \\ &\quad + \lambda_2(1+r) \int_0^{\tau_1} \|\Gamma_2(\tau_2-\tau) - \Gamma_2(\tau_1-\tau)\|_{\mathcal{B}} d\tau + N_2\lambda_2(1+r)(\tau_2-\tau_1)\end{aligned}$$

Case 2: For $(x, y) \in B_r$ and $s_i < \tau_1 < \tau_2 \leq t_{i+1}$; $i = 1, \cdots, m$, we have

$$\begin{aligned}\|\Lambda_1(x, y)(\tau_2) - \Lambda_1(x, y)(\tau_1)\| &\leq \|\Gamma_1(\tau_2) - \Gamma_1(\tau_1)\|_{\mathcal{B}} \big(\|x_i\| + \|g_1(x, y)\|\big) \\ &\quad + \int_{s_i}^{\tau_1} \|\Gamma_1(\tau_2-\tau) - \Gamma_1(\tau_1-\tau)\|_{\mathcal{B}} \|\varphi_1(\tau, x(\tau), y(\tau))\| d\tau \\ &\quad + \int_{\tau_1}^{\tau_2} \|\Gamma_1(\tau_2-\tau)\|_{\mathcal{B}} \|\varphi_1(\tau, x(\tau), y(\tau))\| d\tau \\ &\leq \big(\|x_i\| + \alpha_1(1+r)\big) \|\Gamma_1(\tau_2) - \Gamma_1(\tau_1)\|_{\mathcal{B}} \\ &\quad + \lambda_1(1+r) \int_{s_i}^{\tau_1} \|\Gamma_1(\tau_2-\tau) - \Gamma_1(\tau_1-\tau)\|_{\mathcal{B}} d\tau \\ &\quad + N_1\lambda_1(1+r)(\tau_2-\tau_1)\end{aligned}$$

And

$$\begin{aligned}\|\Lambda_2(x, y)(\tau_2) - \Lambda_2(x, y)(\tau_1)\| &\leq \big(\|y_i\| + \alpha_2(1+r)\big) \|\Gamma_2(\tau_2) - \Gamma_2(\tau_1)\|_{\mathcal{B}} \\ &\quad + \lambda_2(1+r) \int_{s_i}^{\tau_1} \|\Gamma_2(\tau_2-\tau) - \Gamma_2(\tau_1-\tau)\|_{\mathcal{B}} d\tau \\ &\quad + N_2\lambda_2(1+r)(\tau_2-\tau_1)\end{aligned}$$

Case 3: For $(x,y)\in B_r$ and $t_i<\tau_1<\tau_2\le s_i$; $i=1,\cdots,m$, we have

$$\begin{aligned}\|\Lambda_1(x,y)(\tau_2)-\Lambda_1(x,y)(\tau_1)\| &\le \|T(\tau_1-t_i)\|\|T(\tau_2-\tau_1)\psi_{1i}(\tau_2,x(\tau_2),y(\tau_2))-\psi_{1i}(\tau_1,x(\tau_1),y(\tau_1))\|\\ &\le M_Te^{\omega_T a}\|T(\tau_2-\tau_1)\psi_{1i}(\tau_2,x(\tau_2),y(\tau_2))-\psi_{1i}(\tau_1,x(\tau_1),y(\tau_1))\|\end{aligned}$$

Similarly, we obtain

$$\|\Lambda_2(x,y)(\tau_2)-\Lambda_2(x,y)(\tau_1)\|\le M_Se^{\omega_S a}\|S(\tau_2-\tau_1)\psi_{2i}(\tau_2,x(\tau_2),y(\tau_2))-\psi_{2i}(\tau_1,x(\tau_1),y(\tau_1))\|$$

In all previous cases, we have

$$\|\Lambda(x,y)(\tau_2)-\Lambda(x,y)(\tau_1)\|=\|\Lambda_1(x,y)(\tau_2)-\Lambda_1(x,y)(\tau_1)\|+\|\Lambda_2(x,y)(\tau_2)-\Lambda_2(x,y)(\tau_1)\|\to 0 \text{ as } \tau_1\to\tau_2.$$

This allows us to conclude that Λ is equicontinuous.

Step 4: Let $C\subset B_r$ be a non empty subset, such that

$$C\subset\overline{\Lambda(C)}\cup\{(0,0)\}=(\overline{\Lambda_1(C)}\times\overline{\Lambda_2(C)})\cup\{(0,0)\}=(\overline{\Lambda_1(C)}\cup\{0\})\times\overline{\Lambda_2(C)}\cup\{0\}).$$

Clearly, it is bounded and equicontinuous.
Consider the function l defined by

$$l(t)=\widehat{m}\left(C(t)\right),\ t\in[0,a],$$

which is continuous.

Case 1: For $t\in[0,t_1]$, We have

$$l(t)=\widehat{m}\left(C(t)\right)\le\widehat{m}\left(\overline{\Lambda(C)}(t)\cup\{(0,0)\}\right)\le\widehat{m}\left(\overline{\Lambda(C)}(t)\right)=\widehat{m}\left(\Lambda(C)(t)\right)=\max\{m\left(\Lambda_1(C)(t)\right),m\left(\Lambda_2(C)(t)\right)\}.$$

Since, we have

$$\begin{aligned}m\left(\Lambda_1(C)(t)\right) &\le \|\mathit{\Gamma}_1(t)\|_{\mathcal{B}}\|\mathit{\Gamma}_1(a)\|_{\mathcal{B}}\alpha_1\sup_{t\in[0,a]}\widehat{m}\big(C(t)\big)\\ &\quad+\|\mathit{\Gamma}_1(t)\|_{\mathcal{B}}\int_{s_m}^{a}\|\mathit{\Gamma}_1(a-\tau)\|_{\mathcal{B}}\lambda_1\widehat{m}\big(C(\tau)\big)d\tau\\ &\quad+\int_0^t\|\mathit{\Gamma}_1(t-\tau)\|_{\mathcal{B}}\lambda_1\widehat{m}\big(C(\tau)\big)d\tau\\ &\le N_1\big(N_1\alpha_1+aN_1\lambda_1+a\lambda_1\big)\|l\|_\infty\\ &\le \delta_1\|l\|_\infty\end{aligned}$$

and

$$\begin{aligned}m\left(\Lambda_2(C)(t)\right) &\le N_2\big(N_2\alpha_2+aN_2\lambda_2+a\lambda_2\big)\|l\|_\infty\\ &\le \delta_1\|l\|_\infty\end{aligned}$$

Therefore,

$$l(t)\le\delta\|l\|_\infty.$$

Case 2: For $t\in(s_i,t_{i+1}]$; $i=1,\cdots,m$, We have

$$\begin{aligned}m\left(\Lambda_1(C)(t)\right) &\le \|\mathit{\Gamma}_1(t)\|_{\mathcal{B}}\alpha_1\sup_{t\in[0,a]}\widehat{m}\left(\big(C(t)\big)\right)+\int_{s_i}^t\|\mathit{\Gamma}_1(t-\tau)\|_{\mathcal{B}}\lambda_1\widehat{m}\big(C(\tau)\big)d\tau\\ &\le N_1\big(\alpha_1+a\lambda_1\big)\|l\|_\infty\\ &\le \delta_1\|l\|_\infty\end{aligned}$$

and

$$\begin{aligned}m\left(\Lambda_2(C)(t)\right) &\le N_2\big(\alpha_2+a\lambda_2\big)\|l\|_\infty\\ &\le \delta_1\|l\|_\infty\end{aligned}$$

Then,

$$l(t)\le\delta\|l\|_\infty.$$

Case 3: For $t \in (t_i, s_i]$; $i = 1, \cdots, m$, We have

$$\begin{aligned} m\left(\Lambda_1(C)(t)\right) &\leq \|T(t-t_i)\|\sigma_1 \sup_{t\in[0,a]} \widehat{m}\big(C(t)\big) \\ &\leq \sigma_1 M_T e^{\omega_T a}\|l\|_\infty \\ &\leq \delta_2\|l\|_\infty \end{aligned}$$

and

$$\begin{aligned} m\left(\Lambda_2(C)(t)\right) &\leq \sigma_2 M_S e^{\omega_S a}\|l\|_\infty \\ &\leq \delta_2\|l\|_\infty \end{aligned}$$

Then,

$$l(t) \leq \delta\|l\|_\infty.$$

Hence, from above cases we can deduce that

$$\|l\|_\infty \leq \delta\|l\|_\infty.$$

Since $\delta < 1$, so obviously we have $\|l\|_\infty = 0$ which is equivalent to saying that $\widehat{m}\big(C(t)\big) = 0$. So, according to the first property of Definition 2.1, $C(t)$ is relatively compact in $X \times X$. Then, by the Ascoli-Arzelà theorem, it is relatively compact in B_r.

Thus, all conditions of Theorem 3.3 are satisfied, and consequently our problem has a solution. □

4 Examples

In this section we present two examples to illustrate our existence results.

Example 4.1. We consider the following problem:

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = \dfrac{\partial^2}{\partial x^2}u(t,x) + \int_0^t L_1(t-\tau)\dfrac{\partial^2}{\partial x^2}u(\tau,x)d\tau + \dfrac{1}{18N_1^2}\big(\cos(u(t,x)) + \sin(v(t,x))\big),\ t \in (0,1]\cup(2,3],\ x\in[0,1] \\ \dfrac{\partial}{\partial t}v(t,x) = \dfrac{\partial^2}{\partial x^2}v(t,x) + \int_0^t L_2(t-\tau)\dfrac{\partial^2}{\partial x^2}v(\tau,x)d\tau + \dfrac{1}{18N_2^2}\big(\cos(u(t,x)) + v(t,x)\big),\ t \in (0,1]\cup(2,3],\ x\in[0,1] \\ u(t,x) = T(t-1)\dfrac{1}{14}\big(\sin(u(t,x)) + \sin(v(t,x))\big),\ t\in(1,2],\ x\in[0,1] \\ v(t,x) = T(t-1)\dfrac{1}{14}\big(\cos(u(t,x)) + \sin(u(t,x))\big),\ t\in(1,2],\ x\in[0,1] \\ u(t,0) = v(t,0) = u(t,1) = v(t,1) = 0,\ t\in(0,1]\cup(2,3] \\ u(0,x) = u(3,x),\ x\in[0,1] \\ v(0,x) = v(3,x),\ x\in[0,1] \\ u(0,x) + \dfrac{1}{8N_1^2}\left(1+\sin(u)+v\right) = 1 + e^x,\ x\in[0,1] \\ u(2,x) + \dfrac{1}{8N_1^2}\left(1+\sin(u)+v\right) = 2 + e^x,\ x\in[0,1] \\ v(0,x) + \dfrac{1}{8N_2^2}\left(1+\cos(u)+v\right) = 1 + e^x,\ x\in[0,1] \\ v(2,x) + \dfrac{1}{8N_2^2}\left(1+\cos(u)+v\right) = 2 + e^x,\ x\in[0,1] \end{cases} \tag{4.1}$$

where $L_1,\ L_2 \in \mathcal{C}^1([0,3],\mathbb{R})$.

The previous problem can be abstracted into problem 1.1, where $X = L^2([0,1])$ endowed with the norm $\|u\| = \left(\int_0^1 |u(x)|^2 dx\right)^{\frac{1}{2}}$ which is a Banach space, and $Au = Bu = \dfrac{\partial^2}{\partial x^2}u$, for $u \in D(A) = \left\{u \in X : \dfrac{\partial}{\partial x}u,\ \dfrac{\partial^2}{\partial x^2}u \in X,\ u(0) = u(1) = 0\right\}$.

A is the generator of a strongly continuous and compact semigroup $\{T(t),\ t \geq 0\}$ on X and $\|T(t)\| \leq 1,$ for all $t \geq 0$.

$$
\begin{aligned}
&B_1(t) = L_1(t)A,\ B_2(t) = L_2(t)A,\ \varphi_1(t,u,v) = \frac{1}{18N_1^2}\big(\cos(u(t,x)) + \sin(v(t,x))\big),\\
&\varphi_2(t,u,v) = \frac{1}{18N_2^2}\big(\cos(u(t,x)) + v(t,x)\big),\ \psi_{11}(t,u,v) = \frac{1}{14}\big(\sin(u(t,x)) + \sin(v(t,x))\big),\\
&\psi_{21}(t,u,v) = \frac{1}{14}\big(\cos(u(t,x)) + \sin(v(t,x))\big)\\
&g_1(u,v) = \frac{1}{8N_1^2}\left(1+\sin(u)+v\right),\ g_2(u,v) = \frac{1}{8N_2^2}\left(1+\cos(u)+v\right).
\end{aligned}
$$

Clearly, we have $L_{\varphi_1} = \frac{1}{18N_1^2},\ L_{\varphi_2} = \frac{1}{18N_2^2},\ L_{\psi_{11}} = L_{\psi_{21}} = \frac{1}{14N},\ L_{g_1} = \frac{1}{8N_1^2},\ L_{g_2} = \frac{1}{8N_2^2},\ M_T = M_S = 1,\ a = 3$ and $\omega_T = \omega_S = 0$. Then $\kappa_{11} \leq \frac{17}{72}$, $\kappa_{12} \leq \frac{17}{72}$ and $\kappa_{21} = \kappa_{22} = \frac{1}{14}$, therefore $\kappa \leq \frac{17}{36} < 1$. Hence, according to theorem 3.2, problem (4.1) has a unique mild solution.

Example 4.2. To illustrate our second result of existence, we present the following problem:

$$
\begin{cases}
\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + \int_0^t L_1(t-\tau)\frac{\partial^2}{\partial x^2}u(\tau,x)d\tau + \left(\frac{1}{e^9} + \frac{1}{e^{t+x+9}}\right)\frac{t^2(1+u(t,x)+v(t,x))}{36N_1^2(1+\|u\|+\|v\|)},\ t\in(0,1]\cup(2,3],\ x\in[0,1]\\
\frac{\partial}{\partial t}v(t,x) = \frac{\partial^2}{\partial x^2}v(t,x) + \int_0^t L_2(t-\tau)\frac{\partial^2}{\partial x^2}u(\tau,x)d\tau + \left(\frac{1}{e^9} + \frac{1}{e^{t+x+9}}\right)\frac{t^2(1+u(t,x))}{36N_2^2(1+\|u\|+\|v\|)},\ t\in(0,1]\cup(2,3],\ x\in[0,1]\\
u(t,x) = T(t-1)\frac{u(t,x)}{24(1+\|u\|+\|v\|)},\ t\in(1,2],\ x\in[0,1]\\
v(t,x) = T(t-1)\frac{v(t,x)}{24(1+\|u\|+\|v\|)},\ t\in(1,2],\ x\in[0,1]\\
u(t,0) = u(t,1) = v(t,0) = v(t,1) = 0,\ t\in(0,1]\cup(2,3]\\
u(0,x) = u(3,x),\ x\in[0,1]\\
v(0,x) = v(3,x),\ x\in[0,1]\\
u(0,x) + \frac{1}{8N1^2}\left(1+\sin(u)+cos(v)\right) = 1+e^x,\ x\in[0,1]\\
u(2,x) + \frac{1}{8N_1^2}\left(1+\sin(u)+cos(v)\right) = 2+e^x,\ x\in[0,1]\\
v(0,x) + \frac{1}{8N_2^2}\left(1+\sin(u)+v\right) = 1+e^x,\ x\in[0,1]\\
v(2,x) + \frac{1}{8N_2^2}\left(1+\sin(u)+v\right) = 2+e^x,\ x\in[0,1]
\end{cases}
\tag{4.2}
$$

The previous problem can be written as problem (1.1), where

$$
\begin{aligned}
&\varphi_1(t,u,v) = \left(\frac{1}{e^9} + \frac{1}{e^{t+x+9}}\right)\frac{t^2(1+u(t,x)+v(t,x))}{36N_1^2(1+\|u\|+\|v\|)},\\
&\varphi_2(t,u,v) = \left(\frac{1}{e^9} + \frac{1}{e^{t+x+9}}\right)\frac{t^2(1+u(t,x))}{36N_2^2(1+\|u\|+\|v\|)},\\
&\psi_{11}(t,u,v) = \frac{u(t,x)}{24(1+\|u\|+\|v\|)},\\
&\psi_{21}(t,u,v) = \frac{v(t,x)}{24(1+\|u\|+\|v\|)},\\
&g_1(u,v) = \frac{1}{8N_1^2}\left(1+\sin(u)+cos(v)\right),\\
&g_2(u,v) = \frac{1}{8N_2^2}\left(1+\sin(u)+v\right).
\end{aligned}
$$

It's easy to verify that $\lambda_1 = \frac{1}{2N_1^2e^9},\ \lambda_2 = \frac{1}{2N_2^2e^9},\ \sigma_1 = \sigma_2 = \frac{1}{24},\ \alpha_1 = \frac{1}{8N_1^2}$ and $\alpha_2 = \frac{1}{8N_2^2}$. Thus, $\delta_1 \leq \frac{24+e^9}{4e^9} < 1$ and $\delta_2 = \frac{1}{12} < 1$. Therefore, by using Theorem (3.3), our problem (4.2) has a solution.

Acknowledgements

The authors express their sincere thanks to the anonymous referees for numerous helpful and constructive suggestions which have improved the manuscript.

References

[1] M. Alpha Diallo, K. Ezzinbi and A. Sene, *Impulsive integro-differential equations with nonlocal conditions in Banach spaces*, Trans. A. Razmadze Math. Instit. **171** (2017), 304–315.

[2] D.D. Bainov and P.S. Simeonov, *Impulsive differential equations: Periodic solutions and applications*, Longman Scientific and Technical, New York, 1993.

[3] M. Dieye, M.A. Diop, K. Ezzinbi and H. Hmoyed, *On the existence of mild solutions for nonlocal impulsive integrodifferential equations in Banach spaces*, Le Mat. **LXXIV** (2019), no. 1, 13-–34.

[4] Drumi Bainov, Zdzislaw Kamont, Emil Minchev, *Periodic boundary value problem for impulsive hyperbolic partial differential equations of first order*, Appl. Math. Comput. **68** (1995), no. 2-3, 95–104.

[5] A. El Allaoui, Y. Allaoui, S. Melliani and L.S. Chadli, *Coupled system of mixed hybrid differential equations: linear perturbation of first and second type*, J. Univer. Math., **1** (2018), no. 1, 24–31.

[6] M. Feckan and J.R. Wang, *A general class of impulsive evolution equations*, Topol. Methods Nonlinear Anal. **46** (2015), 915-934.

[7] E. Hernandez , O'Regan D., *On a new class of abstract impulsive differential equations*, Proc. Amer. Math. Soc. **141** (2013), 641–649.

[8] J.H. Liu, *Nonlinear impulsive evolution equations*, Dyn. Contin. Discrete Impuls. Syst. **6** (1999), no. 1, 77–85.

[9] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer, Berlin, 1983.

[10] M. Pierri, D. O'Regan and V. Rolnik, *Existence of solutions for semi-linear abstract differential equations with not instantaneous impulses*, Appl. Math. Comput. **219** (2013), 6743–6749.

[11] A.M. Samoilenko and N.A. Perestyuk, *Impulsive differential equations*, World Scientific, Singapore, 1995.

[12] V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, *Theory of impulsive differential equations*, World Scientific, Singapore, 1989.

[13] S. Melliani, L.S. Chadli and A. El Allaoui, *Periodic boundary value problems for controlled nonlinear impulsive evolution equations on Banach spaces*, Int. J. Nonlinear Anal. Appl. **8** (2017), no. 1, 301–314.

[14] S. Melliani, A. El Allaoui and L.S. Chadli, *A general class of periodic boundary value problems for controlled nonlinear impulsive evolution equations on Banach spaces*, Adv. Differ. Equ. **2016** (2016), no. 1, 1–13.

[15] H. Mönch, *Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces*, Nonlinear Anal. **4** (1980), no. 5, 985—999.

[16] J.R. Wang, M. Feckan and Y. Zhou, *Noninstantaneous impulsive models for studying periodic evolution processes in pharmacotherapy*, Mathematical Modeling and Applications in Nonlinear Dynamics. Springer, Cham, 2016, 87–107.

[17] W. Wei, X. Xiang and Y. Peng, *Nonlinear impulsive integro-differential equations of mixed type and optimal controls*, Optim. **55** (2006), 141–156.

[18] X. Yu and J.R. Wang, *Periodic boundary value problems for nonlinear impulsive evolution equations on Banach spaces*, Commun. Nonlinear Sci. Nume. Simul. **22** (2015), 980–989.

[19] L. Zhu and Q. Huang, *Nonlinear impulsive evolution equations with nonlocal conditions and optimal controls*, Adv. Differ. Equ. **2015** (2015), no. 1, 1–12.