Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 763–788
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.24639.2797

Existence and asymptotic behavior for a logarithmic viscoelastic plate equation with distributed delay

Erhan Pişkin[a], Jorge Ferreira[b], Hazal Yüksekkaya[a], Mohammad Shahrouzi[c,*]

[a] *Dicle University, Department of Mathematics, Diyarbakir, Turkey*

[b] *Federal Fluminense University, Department of Exact Sciences, Volta Redonda, RJ, Brazil*

[c] *Department of Mathematics, Jahrom University, P.O.Box: 74137-66171, Jahrom, Iran*

(Communicated by Madjid Eshaghi Gordji)

Abstract

In this article, we consider a logarithmic viscoelastic plate equation with distributed delay. Firstly, we study the local and global existence of solutions by using the energy method combined with Faedo-Galerkin method. Then, by introducing a suitable Lyapunov functional, we prove the asymptotic behavior of the solution. Our results are more general than the earlier results.

Keywords: Existence, Asymptotic behavior, Distributed delay, Logarithmic viscoelastic plate equation
2010 MSC: 35B35, 35B40, 35L05, 93D15

1 Introduction

In this paper, we study the logarithmic viscoelastic plate equation with distributed delay

$$\begin{cases} |u_t|^{\rho} u_{tt} + \Delta^2 u - \Delta u_{tt} - \int_0^t h(t-\sigma)\Delta^2 u(x,\sigma)\,d\sigma \\ +\mu_1 u_t(x,t) + \int_{\tau_1}^{\tau_2} \mu_2(s)\,u_t(x,t-s)\,ds \\ = bu\ln u & \text{in } \Omega\times(0,+\infty), \\ u(x,t) = \frac{\partial u(x,t)}{\partial \upsilon} = 0 & \text{on } \Gamma\times(0,+\infty), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x) & \text{in } \Omega, \\ u_t(x,-t) = f_0(x,t) & \text{in } \Omega\times(0,\tau_2), \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^n, $n\in N$, with a smooth boundary $\partial\Omega=\Gamma$, $\rho>0$, μ_1 is a positive constant, b is a positive real number and h is a positive nonincreasing function defined on R^+. (u_0,u_1,f_0) are the initial data belonging to a suitable function space. Moreover, $\mu_2:[\tau_1,\tau_2]\to R$ is a bounded function, where τ_1 and τ_2 are two real numbers satisfy $0\le\tau_1<\tau_2$. υ is the unit outward normal vector.

*Corresponding author
Email addresses: episkin@dicle.edu.tr (Erhan Pişkin), ferreirajorge2012@gmail.com (Jorge Ferreira), hazally.kaya@gmail.com (Hazal Yüksekkaya), mshahrouzi@jahromu.ac.ir (Mohammad Shahrouzi)

Received: September 2021 *Accepted:* January 2022

We consider the vibration of a viscoelastic beam, in one-dimensional space. The constitutive relationship between the stress N and strain u satisfies

$$N(x,t)=\alpha u_{xxx}-\int_0^t g(t-s)\,u_{xxx}(x,s)\,ds,$$

where the constant α represents the tension stiffness, and g is so-called relaxation function. We can get, if there exists the load $F(x,t,u,u_t)$ on the beam, the following model:

$$u_{tt}+\frac{\alpha}{\rho A}u_{xxxx}-\frac{\alpha}{\rho A}\int_0^t g(t-s)\,u_{xxxx}ds=\frac{F}{\rho A},$$

where ρ and A represent the density and the cross-sectional area of the beam, respectively.

We have the Euler-Bernoulli viscoelastic model (when $\frac{\alpha}{\rho A}=1$, $F=0$), in high-dimensional space, as follows:

$$u_{tt}+\Delta^2 u-\int_0^t g(t-s)\,\Delta^2 u(s)\,ds=0,$$

where Δ represents the Laplacian operator with respect to the spatial variables in R^n $(n\geq 2)$ and

$$\Delta^2 u=\Delta(\Delta u)=\sum_{j=1}^{n}\left(\sum_{i=1}^{n}u_{x_ix_i}\right)_{x_jx_j},$$

[35].

- **Problems with logarithmic nonlinearity:**

Logarithmic nonlinearity generally appears in super symmetric field theories and in cosmological inflation. From Quantum Field Theory, logaritmic source term seems in nuclear physics, inflation cosmology, geophysics and optics (see [4, 11]).

For the literature review, firstly, we begin with the studies of Birula and Mycielski [5, 6]. The authors investigated the equation with logarithmic term as follows

$$u_{tt}-u_{xx}+u-\varepsilon u\ln|u|^2=0. \tag{1.2}$$

This equation is a relativistic version of logarithmic quantum mechanics. They are the pioneer of these kind of problems.

In 1980, Cazenave and Haraux [7] introduced the following equation

$$u_{tt}-\Delta u=u\ln|u|^k, \tag{1.3}$$

and the authors proved the existence and uniqueness of the solution for the Cauchy problem. Gorka [11] obtained the global existence results of solutions for one-dimensional of the equation (1.3). Bartkowski and Gorka [4], considered the weak solutions and obtained the existence results.

In [14], Hiramatsu et al. studied the equation as follows

$$u_{tt}-\Delta u+u+u_t+\left|u^2\right|u=u\ln u. \tag{1.4}$$

In [13], Han established the global existence of solutions for the equation (1.4).

In [3], Al-Gharabli and Messaoudi concerned with the plate equation with logarithmic term as follows

$$u_{tt}+\Delta^2 u+u+h(u_t)=ku\ln|u|. \tag{1.5}$$

They established the existence results by the Galerkin method and obtained the explicit and decay of solutions utilizing the multiplier method for the equation (1.5).

In [20], Liu introduced the plate equation with logarithmic term as follows

$$u_{tt}+\Delta^2 u+|u_t|^{m-2}u_t=|u|^{p-2}u\log|u|^k. \tag{1.6}$$

The author proved the local existence by the contraction mapping principle. Also, he studied the global existence and decay results. Moreover, under suitable conditions, the author proved the blow up results with $E(0)<0$.

- **Problems with time delay:**

Time delays often appear in many practical problems such as thermal, economic phenomena, biological, chemical and physical [15].

In 1986, Datko et al. [10] indicated that delay is a source of instability. In [23], Nicaise and Pignotti studied the equation as follows

$$u_{tt} - \Delta u + a_0 u_t (x,t) + a u_t (x, t-\tau) = 0, \tag{1.7}$$

where a_0, $a > 0$. They proved that, under the condition $0 \le a \le a_0$, the system is exponentially stable. The authors obtained a sequence of delays that shows the solution is instable in the case $a \ge a_0$. In the absence of delay, some other authors [19, 37] looked into exponentially stability for the equation (1.7). In [36], Xu et al., by using the spectral analysis approach, established the same result similar to the [22] for the one space dimension.

In [24], Nicaise et al. studied the wave equation in one space dimension in the presence of time-varying delay. In this article, the authors showed that the exponential stability results with the condition

$$a \le \sqrt{1-d}a_0,$$

here d is a constant and

$$\tau' (t) \le d < 1, \forall t > 0.$$

In [16], Kafini and Messaoudi studied wave equation with delay and logarithmic terms as follows

$$u_{tt} - \Delta u + \mu_1 u_t (x,t) + \mu_2 u_t (x, t-\tau) = |u|^{p-2} u \log |u|^k . \tag{1.8}$$

The authors proved the local existence and blow up results for the equation (1.8).

In [25], Park considered the equation with delay and logarithmic terms as follows

$$u_{tt} - \Delta u + \alpha u_t (t) + \beta u_t (x, t-\tau) = u \ln |u|^\gamma . \tag{1.9}$$

The author showed the local and global existence results for the equation (1.9). Also, the author investigated the decay and nonexistence results for the equation (1.9).

In the absence of the logarithmic source term ($bu \ln u$), the problem (1.1) can be reduced as follows

$$\begin{cases} |u_t|^l u_{tt} + \Delta^2 u - \Delta u_{tt} - \int_0^t h(t-\sigma) \Delta^2 u(x,\sigma) d\sigma \\ +\mu_1 u_t (x,t) + \int_{\tau_1}^{\tau_2} \mu_2 (s) u_t (x, t-s) ds \\ = 0. \end{cases} \tag{1.10}$$

Zineb et al. [32], considered the existence and proved stability of the solutions for the equation (1.10). Recently, some other authors studied related problems (see [2, 9, 16, 17, 21, 26, 27, 28, 29, 30, 31, 33, 34, 35]).

There is no research, to our best knowledge, about the logarithmic ($bu \ln u$) viscoelastic plate equation with a varying material density ($|u_t|^\rho$) and distributed delay term ($\int_{\tau_1}^{\tau_2} \mu_2 (s) u_t (x, t-s) ds$), hence, our paper is generalization of the previous ones. Our aim is to establish the local existence, global existence and asymptotic behavior of the solutions.

The outline of this paper is as follows: Firstly, in sect. 2, we give some assumptions and lemmas needed in our proof. Then, in sect. 3, we get the existence result. Moreover, in sect. 4, the asymptotic behavior result is established.

2 Preliminaries

In this part, to prove our main result, we give needed materials. We will use the Sobolev space $H_0^2 (\Omega)$ and the Lebesgue space $L^2 (\Omega)$ with their usual scalar products and norms. We denote by $(\cdot, \cdot)$ the inner product in $L^2 (\Omega)$. The constant C is a generic positive constant, throughout this paper.

We have the following assumptions:

(A1) The relaxation function $h : R^+ \to R^+$ is a bounded function of C^1 so that

$$\int_0^\infty h(\sigma) d\sigma = \beta < 1 \text{ and } 1 - \int_0^\infty h(\sigma) d\sigma = l, \; h(0) > 0, \tag{2.1}$$

and we suppose that there exists a positive constant ζ satisfy

$$h'(t) \leq -\zeta h(t). \tag{2.2}$$

(A2) We suppose that

$$\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds < \mu_1. \tag{2.3}$$

Let ξ be a positive constant satisfies

$$\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds + \frac{\xi(\tau_2 - \tau_2)}{2} < \mu_1. \tag{2.4}$$

(A3) b is the constant in (1.1) such that

$$1 < b < 2\pi l e^3. \tag{2.5}$$

We assume λ is the first eigenvalue of the spectral Dirichlet problem

$$\Delta^2 u = \lambda u, \text{ in } \Omega,\ u = \frac{\partial u}{\partial \eta} = 0 \text{ in } \Gamma,$$

$$\|\nabla u\|_2 \leq \frac{1}{\sqrt{\lambda}} \|\Delta u\|_2. \tag{2.6}$$

Lemma 2.1. [1] Assume that q be a number with

$$2 \leq q < +\infty,\ (n = 1, 2) \text{ or } 2 \leq q \leq 2n/(n-2),\ (n \geq 3),$$

then, there exists a constant $C_s = C_s(\Omega, q)$ satisfying

$$\|u\|_q \leq C_s \|\nabla u\|_2 \text{ for } u \in H_0^1(\Omega). \tag{2.7}$$

Lemma 2.2. [32] For $h, \Psi \in C^1([0, +\infty[, R)$ we have

$$\begin{aligned}\int_\Omega h * \Psi \Psi_t dx &= -\frac{1}{2} h(t) \|\Psi(t)\|_2^2 + \frac{1}{2}(h' \circ \Psi)(t) \\ &\quad -\frac{1}{2}\frac{d}{dt}\left[(h \circ \Psi)(t) - \left(\int_0^t h(s)\, ds\right) \|\Psi(t)\|_2^2\right].\end{aligned} \tag{2.8}$$

Lemma 2.3. [8, 12] (Logarithmic Sobolev inequality) Suppose that u is a function in $H_0^1(\Omega)$ and $a > 0$ is a number. Then,

$$\int_\Omega u^2 \ln|u|\, dx \leq \frac{1}{2}\|u\|_2^2 \ln \|u\|_2^2 + \frac{a^2}{2\pi}\|\nabla u\|_2^2 - (1 + \ln a)\|u\|_2^2. \tag{2.9}$$

Corollary 2.4. [2] Assume that u is a function in $H_0^2(\Omega)$ and $a > 0$ is a number. Then,

$$\int_\Omega u^2 \ln|u|\, dx \leq \frac{1}{2}\|u\|_2^2 \ln \|u\|_2^2 + \frac{a^2}{2\pi}\|\Delta u\|_2^2 - (1 + \ln a)\|u\|_2^2. \tag{2.10}$$

Lemma 2.5. [7] (Logarithmic Gronwall inequality) Suppose that $C > 0$, $\gamma \in L^1(0, T; R^+)$ and suppose that the function $w : [0, T] \to [1, \infty)$ satisfies

$$w(t) \leq C\left(1 + \int_0^t \gamma(s) w(s) \ln(w(s))\, ds\right),\ \forall t \in [0, T]. \tag{2.11}$$

Then

$$w(t) \leq C \exp\left(C \int_0^t \gamma(s)\, ds\right),\ \forall t \in [0, T]. \tag{2.12}$$

Lemma 2.6. [2] Let $\epsilon_0 \in (0,1)$. Then, there exists $d_{\varepsilon_0} > 0$ satisfying

$$s\left|\ln s\right| \leq s^2 + d_{\epsilon_0} s^{1-\epsilon_0},\ \forall s > 0. \tag{2.13}$$

Lemma 2.7. [2] Suppose that h satisfies (A1). Then, for $u \in H_0^2(\Omega)$, we obtain

$$\int_\Omega \left(\int_0^t h(t-s)(u(t)-u(s))\,ds\right)^2 dx \leq c(h \circ \Delta u)(t) \tag{2.14}$$

and

$$\int_\Omega \left(\int_0^t h'(t-s)(u(t)-u(s))\,ds\right)^2 dx \leq -c(h' \circ \Delta u)(t). \tag{2.15}$$

Formulation of the results:

Firstly, we introduce, similar to [22], the new variable

$$z(x,\kappa,s,t) = u_t(x, t-\kappa s), \quad (x,\kappa,s,t) \in \Omega \times (0,1) \times (\tau_1,\tau_2) \times (0,\infty),$$

which implies that

$$s z_t(x,\kappa,s,t) + z_\kappa(x,\kappa,s,t) = 0 \quad \text{in } \Omega \times (0,1) \times (\tau_1,\tau_2) \times (0,\infty).$$

Hence, problem (1.1) can be transformed as follows:

$$\begin{cases} |u_t|^\rho u_{tt} + \Delta^2 u - \Delta u_{tt} - \int_0^t h(t-\sigma)\Delta^2 u(x,\sigma)\,d\sigma & \\ +\mu_1 u_t(x,t) + \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t)\,ds & \\ = bu\ln u & \text{in } \Omega \times (0,+\infty), \\ s z_t(x,\kappa,s,t) + z_\kappa(x,\kappa,s,t) = 0 & \text{in } \Omega \times (0,1) \times (\tau_1,\tau_2) \times (0,\infty), \\ u(x,t) = \frac{\partial u(x,t)}{\partial \upsilon} = 0 & \text{on } \Gamma \times (0,+\infty), \\ z(x,0,s,t) = u_t(x,t) & \text{in } \Omega \times (0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{in } \Omega, \\ z(x,\kappa,s,0) = f_0(x,\kappa s) & \text{in } \Omega \times (0,1) \times (\tau_1,\tau_2). \end{cases} \tag{2.16}$$

We define the energy functional of (2.16) by

$$\begin{aligned} E(t) &= \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2}\left(1 - \int_0^t h(\sigma)\,d\sigma\right)\|\Delta u\|^2 + \frac{1}{2}\|\nabla u_t\|^2 \\ &\quad + \frac{1}{2}(h \circ \Delta u)(t) - \frac{b}{2}\int_\Omega |u|^2 \ln|u|\,dx + \frac{b}{4}\|u\|^2 \\ &\quad + \frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s(|\mu_2(s)| + \xi) z^2(x,\kappa,s,t)\,ds\,d\kappa\,dx, \end{aligned} \tag{2.17}$$

where

$$(h \circ \Delta \nu)(t) = \int_0^t h(t-\sigma)\|\Delta \upsilon(t) - \Delta \upsilon(\sigma)\|^2\,d\sigma.$$

Lemma 2.8. Assume that (u,z) is a solution of problem (2.16) and suppose that (A1)-(A3) satisfy. Then, $E(t)$ defined by (2.17) satisfies

$$\begin{aligned} E'(t) &\leq -\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\,ds - \frac{\xi(\tau_2 - \tau_2)}{2}\right)\|u_t\|^2 \\ &\quad - \frac{\xi}{2}\int_\Omega \int_{\tau_1}^{\tau_2} z^2(x,1,s,t)\,ds\,dx - \frac{1}{2}h(t)\|\Delta u\|^2 + \frac{1}{2}(h' \circ \Delta u)(t) \\ &\leq 0,\ \forall t \geq 0. \end{aligned} \tag{2.18}$$

Proof . We multiply the first equation in (2.16) by u_t and integrate over Ω and use integration by parts, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2}+\frac{1}{2}\|\nabla u_t\|^2+\frac{1}{2}\|\Delta u\|^2\right.\\
&\left.-\frac{b}{2}\int_\Omega |u|^2\ln|u|\,dx+\frac{b}{4}\|u\|^2\right]\\
&+\mu_1\|u_t\|^2+\int_{\tau_1}^{\tau_2}\mu_2(s)\int_\Omega z(x,1,s,t)\,u_t dxds\\
=&\int_\Omega\int_0^t h(t-\sigma)\,\Delta u(\sigma)\,\Delta u_t(\sigma)\,d\sigma dx. \qquad (2.19)
\end{aligned}$$

For the term on the right-hand side of (2.19), from Lemma 2.2, we have

$$\begin{aligned}
&\int_\Omega\int_0^t h(t-\sigma)\,\Delta u(\sigma)\,\Delta u_t(\sigma)\,d\sigma dx\\
=&\frac{1}{2}\frac{d}{dt}\left[\int_0^t h(\sigma)\,d\sigma\|\Delta u\|^2-(h\circ\Delta u)(t)\right]\\
&+\frac{1}{2}(h'\circ\Delta u)(t)-\frac{1}{2}h(t)\|\Delta u\|^2. \qquad (2.20)
\end{aligned}$$

Utilizing Young's inequality, we obtain

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2}\mu_2(s)\int_\Omega z(x,1,s,t)\,u_t dsdx\\
\le&\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)\int_\Omega u_t^2dx+\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|\left|z^2(x,1,s,t)\right|dsdx. \qquad (2.21)
\end{aligned}$$

We multiply the second equation in (2.16) by $(|\mu_2(s)|+\xi)\,z$ and integrate over $\Omega\times(0,1)\times(\tau_1,\tau_2)$ with respect to κ, x and s, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}s(|\mu_2(s)|+\xi)\,z^2(x,\kappa,s,t)\,dsd\kappa dx\\
=&-\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}(|\mu_2(s)|+\xi)\frac{\partial}{\partial\rho}z^2(x,\kappa,s,t)\,dsd\kappa dx\\
=&\frac{1}{2}\int_{\tau_1}^{\tau_2}(|\mu_2(s)|+\xi)\int_\Omega u_t^2dsdx\\
&-\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}(|\mu_2(s)|+\xi)\,z^2(x,1,s,t)\,dsdx\\
=&\frac{1}{2}\left[\xi(\tau_2-\tau_2)+\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right]\int_\Omega u_t^2dx\\
&-\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}(|\mu_2(s)|+\xi)\,z^2(x,1,s,t)\,dsdx. \qquad (2.22)
\end{aligned}$$

Consequently, by combining (2.19)-(2.22) and using (2.1)-(2.5) give (2.18), hence, we conclude the proof. □

Theorem 2.9. (Existence) Let $u_0\in H_0^2$, $u_1\in H_0^2(\Omega)$ and $f_0\in H_0^2\left(\Omega,\,H^2(0,1)\right)$ satisfies the compatibility condition

$$f_0(\cdot,0)=u_1.$$

Suppose that (A1)-(A3) hold. Hence, problem (1.1) has a weak solution:

$$\begin{aligned}
u &\in L^\infty\left([0,\infty);H_0^2\right),\ u_t\in L^\infty\left([0,\infty);H_0^2(\Omega)\right)\\
u_{tt} &\in L^2\left([0,\infty);H_0^1(\Omega)\right). \qquad (2.23)
\end{aligned}$$

Theorem 2.10. (Asymptotic behavior) Suppose that (A1)-(A3) hold. Then, $E(t)$ energy functional (2.17) satisfies,

$$E(t) \leq k_0 e^{-k_1 t} \ \forall t > 0, \tag{2.24}$$

where k_0 and k_1 are positive constants.

3 Existence

3.1 Local existence

Proof of Theorem 2.9:

In this part, we get the local existence result of (2.16) by using the Faedo-Galerkin method. We suppose $u_0 \in H_0^2(\Omega)$, $u_1 \in H_0^2(\Omega)$ and $f_0 \in H_0^2(\Omega) \cap H^2(0,1)$. Assume that $T > 0$ is fixed and suppose $\{w^k\}$, $k \in N$ be a basis of $H_0^2(\Omega)$ the space generated by w^1, w^2,$\cdots$,w^k. Next, we define, for $1 \leq j \leq k$, the sequence $\phi^j(x,\rho)$ as follows:

$$\phi^j(x,0) = w^j. \tag{3.1}$$

Next, we extend $\phi^j(x,0)$ by $\phi^j(x,\rho)$ over $L^2(\Omega \times (0,1))$ such that $(\phi^j)_j$ forms a basis of $L^2(\Omega) \times H^2(0,1)$ and show Z_k the space generated by $\{\phi^k\}$. We construct approximate solutions $(u^k$,$z^k)$, $k = 1, 2,\cdots$, in the form

$$u^k(t) = \sum_{j=1}^{k} c^{jk}(t) w^j(x), \ z^k(t) = \sum_{j=1}^{k} d^{jk}(t) \phi^j(x), \tag{3.2}$$

where c^{jk} and d^{jk} are determined by ordinary differential equations as follows

$$\begin{aligned} & \left(\left|u_t^k\right|^\rho u_{tt}^k, w^j\right) + \left(\Delta u^k, \Delta w^j\right) + \left(\nabla u_{tt}^k, \nabla w^j\right) + \mu_1\left(u_t^k, w^j\right) \\ & - \int_0^t h(t-\sigma)\left(\Delta u^k(\sigma), \Delta w^j\right) d\sigma + \int_0^t \mu_2(s)\left(z^k(x,1,s,t), w^j\right) ds \\ = \ & b \int_\Omega u^k \ln|u|^k w^j dx, \end{aligned} \tag{3.3}$$

$$z^k(x,0,s,t) = u_t^k(x,t), \tag{3.4}$$

$$u^k(0) = u_0^k = \sum_{j=1}^{k}\left(u_0, w^j\right) w^j \to u_0, \text{ in } H_0^2(\Omega) \text{ as } k \to +\infty, \tag{3.5}$$

$$u_t^k(0) = u_1^k = \sum_{j=1}^{k}\left(u_1, w^j\right) w^j \to u_1, \text{ in } H_0^2(\Omega) \text{ as } k \to +\infty, \tag{3.6}$$

and

$$\left(s z_t^k + z_\kappa^k, \phi^j\right) = 0, \ 1 \leq j \leq k, \tag{3.7}$$

$$z^k(0,\kappa,s,0) = z_0^k = \sum_{j=1}^{k}\left(f_0, \phi^j\right) \phi^j \to f_0 \ \text{ in } H_0^2\left(\Omega, H_0^2(0,1)\right) \text{ as } k \to +\infty. \tag{3.8}$$

As $0 < \rho \leq \frac{2}{n-2}$ if $n \geq 3$, by using the Sobolev embedding, we get

$$H_0^2(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega),$$

and the same occurs for $n = 1, 2$ where $\rho > 0$. By the generalized Hölder inequality, we note that $\frac{\rho}{2(\rho+1)} + \frac{1}{2(\rho+1)} + \frac{1}{2} = 1$, the nonlinear term $(\left|u_t^k\right|^\rho u_{tt}^k, w^j)$ in (3.3) makes sense.

The standard theory of ODE guarantees that the system (3.3)-(3.8) has an unique solution in $[0, t_k)$, with $0 < t_k < T$, utilizing Zorn lemma since the nonlinear terms in (3.3) are locally Lipschitz continuous. Noting that $u^k(t)$ is of class C^2.

Now, we get a priori estimate for the solution of (3.3)-(3.8), thus, it can be extended to $[0,T)$ and that the local solution is uniformly bounded independently of k and t.

First estimate:

By Lemma 2.8, since the sequences u_0^k, u_1^k converge, we find a positive constant C_1 independent of k satisfying

$$\begin{aligned}
& E^k(t) - E^k(0) \\
\leq & -\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds - \frac{\xi(\tau_2-\tau_2)}{2}\right)\int_0^t \left\|u_t^k(\sigma)\right\|^2 d\sigma \\
& -\frac{\xi}{2}\int_\Omega \int_{\tau_1}^{\tau_2}\int_0^t \left|z^k(x,1,s,\sigma)\right|^2 dsdxd\sigma \\
& -\frac{1}{2}\int_0^t h(\sigma)\left\|\Delta u^k(\sigma)\right\|^2 d\sigma + \frac{1}{2}\int_0^t \left(h' \circ \Delta u^k\right)(\sigma)\, d\sigma \\
\leq & -\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds - \frac{\xi(\tau_2-\tau_2)}{2}\right)\int_0^t \left\|u_t^k(\sigma)\right\|^2 d\sigma \\
& -\frac{\xi}{2}\int_\Omega \int_{\tau_1}^{\tau_2}\int_0^t \left|z^k(x,1,s,\sigma)\right|^2 dsdxd\sigma. \qquad (3.9)
\end{aligned}$$

Since h is a positive nonincreasing function, we obtain

$$\begin{aligned}
& E^k(t) + \left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds - \frac{\xi(\tau_2-\tau_2)}{2}\right)\int_0^t \left\|u_t(\sigma)\right\|^2 d\sigma \\
& +\frac{\xi}{2}\int_\Omega \int_{\tau_1}^{\tau_2}\int_0^t \left|z^k(x,1,s,\sigma)\right|^2 dsdxd\sigma \\
\leq & E^k(0) \leq C_1, \qquad (3.10)
\end{aligned}$$

where

$$\begin{aligned}
E^k(t) = & \frac{1}{\rho+2}\left\|u_t^k\right\|_{\rho+2}^{\rho+2} + \frac{1}{2}\left(1-\int_0^t h(\sigma)\, d\sigma\right)\left\|\Delta u^k\right\|^2 \\
& +\frac{1}{2}\left\|\nabla u_t^k\right\|^2 + \frac{1}{2}\left(h\circ \Delta u^k\right)(t) - \frac{b}{2}\int_\Omega \left|u^k\right|^2 \ln\left|u^k\right| dx \\
& +\frac{b}{4}\left\|u^k\right\|^2 + \frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s\left(|\mu_2(s)|+\xi\right)\left|z^k(x,\kappa,s,t)\right|^2 dsd\kappa dx.
\end{aligned}$$

By applying the Logarithmic Sobolev inequality, (3.10) yields

$$\begin{aligned}
& \left\|u_t^k\right\|_{\rho+2}^{\rho+2} + \left(l - \frac{ba^2}{2\pi}\right)\left\|\Delta u^k\right\|^2 + \left[\frac{b}{2} + b\left(1+\ln a\right)\right]\left\|u^k\right\|^2 \\
& + \left\|\nabla u_t^k\right\|^2 + \left(h\circ\Delta u^k\right)(t) \\
& + \int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s\left(|\mu_2(s)|+\xi\right)\left|z^k(x,\kappa,s,t)\right|^2 dsd\kappa dx \\
& + \int_\Omega\int_{\tau_1}^{\tau_2}\int_0^1 \left|z^k(x,1,s,\sigma)\right|^2 dsdxd\sigma \\
\leq & C_2 + \left\|u^k\right\|^2 \ln\left\|u^k\right\|^2,
\end{aligned}$$

where C_2 is a positive constant.

Choosing

$$e^{(-3/2)} < a < \sqrt{\frac{2\pi l}{b}}, \qquad (3.11)$$

will make

$$l - \frac{ba^2}{2\pi} > 0 \qquad (3.12)$$

and

$$\frac{b}{2}+b\left(1+\ln a\right)>0. \tag{3.13}$$

Thanks to (A3), this selection is possible. Hence, we obtain

$$\begin{aligned}
&\left\|u_t^k\right\|_{\rho+2}^{\rho+2}+\left\|\Delta u^k\right\|^2+\left\|\nabla u_t^k\right\|^2+\left\|u^k\right\|^2+\left(h\circ\Delta u^k\right)(t)\\
&+\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s\left(\left|\mu_2(s)\right|+\xi\right)\left|z^k(x,\kappa,s,t)\right|^2 ds d\kappa dx\\
&+\int_\Omega\int_{\tau_1}^{\tau_2}\int_0^t\left|z^k(x,1,s,\sigma)\right|^2 ds dx d\sigma\\
\le\ & c\left(1+\left\|u^k\right\|^2\ln\left\|u^k\right\|^2\right).
\end{aligned} \tag{3.14}$$

Let us note that

$$u^k(t)=u^k(0)+\int_0^t u_s^k(s)\,ds.$$

Then, utilizing Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned}
\left\|u^k\right\|^2 &\le \left\|u^k(0)\right\|^2+2\left\|\int_0^t u_s^k(s)\,ds\right\|^2\\
&\le \left\|u^k(0)\right\|^2+2T\int_0^t\left\|u_s^k(s)\right\|^2 ds.
\end{aligned} \tag{3.15}$$

Therefore, (3.14) gives

$$\left\|u^k\right\|^2\le C\left(1+\int_0^t\left\|u^k\right\|^2\ln\left\|u^k\right\|^2 ds\right), \tag{3.16}$$

where $C=\max\left\{2Tc,2\left\|u^k(0)\right\|^2\right\}$. Applying the Logarithmic Gronwall inequality to (3.16), we get

$$\left\|u^k\right\|^2\le Ce^{CT}. \tag{3.17}$$

Therefore, from (3.14), we obtain the first estimate:

$$\begin{aligned}
&\left\|u_t^k\right\|_{\rho+2}^{\rho+2}+\left\|\Delta u^k\right\|^2+\left\|\nabla u_t^k\right\|^2+\left\|u^k\right\|^2+\left(h\circ\Delta u^k\right)(t)\\
&\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s\left(\left|\mu_2(s)\right|+\xi\right)\left|z^k(x,\kappa,s,t)\right|^2 ds d\kappa dx\\
&+\int_\Omega\int_{\tau_1}^{\tau_2}\int_0^t\left|z^k(x,1,s,\sigma)\right|^2 ds dx d\sigma\\
\le\ & c\left(1+Ce^{CT}\ln\left(Ce^{CT}\right)\right)=A_1.
\end{aligned} \tag{3.18}$$

The estimate implies that the solution (u^k,z^k) exists in $[0,T)$ and it yields

$$u^k \text{ is bounded in } L_{loc}^\infty\left(0,\infty,H_0^2(\Omega)\right), \tag{3.19}$$

$$u_t^k \text{ is bounded in } L_{loc}^\infty\left(0,\infty,H_0^1(\Omega)\right), \tag{3.20}$$

$$s\left(\left|\mu_2(s)\right|+\xi\right)z^k(x,\kappa,s,t) \text{ is bounded in } L_{loc}^\infty\left(0,\infty,L^2\left(\Omega\times(0,1)\times(\tau_1,\tau_2)\right)\right), \tag{3.21}$$

$$z^k(x,1,s,t) \text{ is bounded in } L^2\left(\Omega\times(\tau_1,\tau_2)\times(0,T)\right). \tag{3.22}$$

Second estimate:

We replace w^j by $-\Delta w^j$ in (3.3), multiply by c_t^{jk} and sum up over j from 1 to k, such that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\left\|\nabla\Delta u^k\right\|^2+\left\|\Delta u_t^k\right\|^2\right]-\int_\Omega\left|u_t^k\right|^\rho u_{tt}^k\Delta u_t^k dx\\
&-\int_0^t h(t-\sigma)\int_\Omega \nabla\Delta u^k(t)\nabla\Delta u_t^k(\sigma)\,dxd\sigma\\
&+\int_{\tau_1}^{\tau_2}\mu_2(s)\int_\Omega \nabla z^k(x,1,s,t)\nabla u_t^k dsdx+\mu_1\left\|\nabla u_t^k\right\|^2\\
=\;& -b\int_\Omega \Delta u_t^k u^k\ln\left|u^k\right|dx. \qquad (3.23)
\end{aligned}$$

Utilizing Green's formula, we get

$$\begin{aligned}
&-\int_\Omega\left|u_t^k\right|^\rho u_{tt}^k\Delta u_t^k dx\\
=\;&\frac{d}{dt}\int_\Omega\left|u_t^k\right|^\rho\left|\nabla u_t^k\right|^2dx-(\rho+1)\int_\Omega\left|u_t^k\right|^\rho\nabla u_{tt}^k\nabla u_t^k dx. \qquad (3.24)
\end{aligned}$$

Replacing ϕ^j by $-\Delta\phi^j$ in (3.7), we multiply by $(|\mu_2(s)|+\xi)\,d^{jk}$ and sum up over j from 1 to k, we obtain

$$s(|\mu_2(s)|+\xi)\int_\Omega\nabla z_t^k\nabla z^k dx+(|\mu_2(s)|+\xi)\int_\Omega\nabla z_\kappa^k\nabla z^k dx=0. \qquad (3.25)$$

Then, we get

$$\frac{s(|\mu_2(s)|+\xi)}{2}\frac{d}{dt}\left\|\nabla z^k\right\|^2+\frac{|\mu_2(s)|+\xi}{2}\frac{d}{d\kappa}\left\|\nabla z^k\right\|^2=0, \qquad (3.26)$$

integrating over $(0,1)\times(\tau_1,\tau_2)$ to find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1\int_{\tau_1}^{\tau_2}s(|\mu_2(s)|+\xi)\int_\Omega\left|\nabla z^k(x,\kappa,s,t)\right|^2dsd\kappa dx\\
&+\frac{1}{2}\int_{\tau_1}^{\tau_2}(|\mu_2(s)|+\xi)\int_\Omega\left|\nabla z^k(x,1,s,t)\right|^2dsdx\\
&-\frac{1}{2}\int_{\tau_1}^{\tau_2}(|\mu_2(s)|+\xi)\int_\Omega\left|\nabla u_t^k\right|^2dsdx\\
=\;&0. \qquad (3.27)
\end{aligned}$$

Combining (3.23) and (3.27), taking into consideration Lemma 2.2, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\left(1-\int_0^t h(\sigma)\,d\sigma\right)\left\|\nabla\Delta u^k\right\|^2+\left\|\Delta u_t^k\right\|^2+\left(h\circ\nabla\Delta u^k\right)\right.\\
&\left.\int_{\tau_1}^{\tau_2}\int_0^1 s(|\mu_2(s)|+\xi)\int_\Omega\left|\nabla z^k(x,\kappa,s,t)\right|^2dsd\kappa dx+2\int_\Omega\left|u_t^k(t)\right|^\rho\left|\nabla u_t^k\right|^2dx\right]\\
&+\frac{1}{2}\int_{\tau_1}^{\tau_2}(|\mu_2(s)|+\xi)\int_\Omega\left|\nabla z^k(x,1,s,t)\right|^2dsdx\\
=\;&(\rho+1)\int_\Omega\left|u_t^k\right|^\rho\nabla u_{tt}^k\nabla u_t^k dx-\int_{\tau_1}^{\tau_2}\mu_2(s)\int_\Omega\nabla z^k(x,1,s,t)\nabla u_t^k dsdx\\
&-\mu_1\left\|\nabla u_t^k\right\|^2+\frac{1}{2}\int_{\tau_1}^{\tau_2}(|\mu_2(s)|+\xi)\int_\Omega\left|\nabla u_t^k\right|^2dsdx\\
&-\frac{1}{2}h(t)\left\|\nabla\Delta u^k\right\|^2+\frac{1}{2}\left(h'\circ\nabla\Delta u^k\right)-b\int_\Omega\Delta u_t^k u^k\ln\left|u^k\right|dx. \qquad (3.28)
\end{aligned}$$

By the first estimate (3.18) and utilizing Young's inequality, we obtain for $\eta > 0$

$$\begin{aligned}(\rho+1)\int_\Omega \left|u_t^k\right|^\rho \nabla u_{tt}^k \nabla u_t^k dx &\le (\rho+1)\, C_2^{\rho/(\rho+2)+1/2} \left\|\nabla u_{tt}^k\right\|_2 \\ &\le \eta \left\|\nabla u_{tt}^k\right\|^2 + \frac{(\rho+1)^2 C_2^{2\rho/(\rho+2)+1}}{4\eta}. \end{aligned} \tag{3.29}$$

Utilizing Young's inequality, we have

$$\begin{aligned} &\int_{\tau_1}^{\tau_2} \mu_2(s) \int_\Omega \nabla z^k(x,1,s,t)\, \nabla u_t^k ds dx \\ \le\ & \frac{1}{4\eta}\int_{\tau_1}^{\tau_2} |\mu_2(s)| \int_\Omega |\nabla u_t|^2 ds dx \\ &+\eta \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \left|\nabla z^k(x,1,s,t)\right|^2 ds dx \\ \le\ & \frac{\mu_1}{4\eta}\|\nabla u_t\|^2 + \eta \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \left|\nabla z^k(x,1,s,t)\right|^2 ds dx \\ \le\ & \frac{\mu_1}{4\eta} C_2 + \eta \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \left|\nabla z^k(x,1,s,t)\right|^2 ds dx \\ \le\ & C(\eta) + \eta \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \left|\nabla z^k(x,1,s,t)\right|^2 ds dx. \end{aligned} \tag{3.30}$$

From Lemma 2.6, to estimate the last term on the right-hand side of (3.28), with $\epsilon_0 = (1/2)$ and utilizing Young's, Cauchy-Schwartz's and the embedding inequalities, we get:

$$\begin{aligned} \left| b\int_\Omega \Delta u_t^k u^k \ln\left|u^k\right| dx\right| &\le b\int_\Omega \left|\Delta u_t^k\right| \left(\left|u^k\right|^2 + d_{\epsilon_0}\sqrt{|u^k|}\right) dx \\ &\le b\left(\eta \int_\Omega \left|\Delta u_t^k\right|^2 dx + \frac{1}{4\eta}\int_\Omega \left(\left|u^k\right|^2 + d_{\epsilon_0}\sqrt{|u^k|}\right)^2 dx\right) \\ &\le b\eta \int_\Omega \left|\Delta u_t^k\right|^2 dx + \frac{c}{4\eta}\left(\int_\Omega \left|u^k\right|^4 dx + \int_\Omega \left|u^k\right| dx\right) \\ &\le b\eta \left\|\Delta u_t^k\right\|^2 + \frac{c}{4\eta}\left(\left\|\nabla u^k\right\|^4 + \left\|u^k\right\|\right),\ \eta > 0. \end{aligned} \tag{3.31}$$

Taking into account (3.29)-(3.31) into (3.28), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\left(1-\int_0^t h(\sigma)\,d\sigma\right)\left\|\nabla\Delta u^k\right\|^2 + \left\|\Delta u_t^k\right\|^2 + \left(h\circ \nabla\Delta u^k\right)\right. \\ &\left.\int_{\tau_1}^{\tau_2}\int_0^1 s\left(|\mu_2(s)|+\xi\right)\left\|\nabla z^k(x,\kappa,s,t)\right\|^2 ds d\kappa + 2\int_\Omega \left|u_t^k(t)\right|^\rho \left|\nabla u_t^k\right|^2 dx\right] \\ &+\frac{1}{2}\int_{\tau_1}^{\tau_2}\left(|\mu_2(s)|+\xi-2\eta\right)\left\|\nabla z^k(x,1,s,t)\right\|^2 ds \\ \le\ & \eta\|\nabla u_{tt}\|^2 - \frac{1}{2}h(t)\left\|\nabla\Delta u^k\right\|^2 + \frac{1}{2}\left(h'\circ\nabla\Delta u^k\right) + C(\eta) \\ &+b\eta\left\|\Delta u_t^k\right\|^2 + \frac{c}{4\eta}\left(\left\|\nabla u^k\right\|^4 + \left\|u^k\right\|\right). \end{aligned} \tag{3.32}$$

We multiply (3.3) by c_{tt}^{jk} and sum up over j from 1 to k, we obtain

$$\begin{aligned} & \int_\Omega \left|u_t^k\right|^\rho \left|\nabla u_{tt}^k\right|^2 dx + \left\|\nabla u_{tt}^k\right\|^2 \\ = & -\int_\Omega \Delta^2 u^k u_{tt}^k dx + \int_0^t h(t-\sigma) \int_\Omega \Delta u^k(\sigma) \Delta u_{tt}^k(t) \, dx d\sigma \\ & -\mu_1 \int_\Omega u_t^k u_{tt}^k dx - \int_\Omega \int_{\tau_1}^{\tau_2} \mu_2(s) z^k(x,1,s,t) u_{tt}^k ds dx \\ & +b \int_\Omega u_{tt}^k u^k \ln\left|u^k\right| dx. \end{aligned} \tag{3.33}$$

Differentiating (3.7) with respect to t, we obtain

$$\left(s z_{tt}^k + z_{t\kappa}^k, \phi^j\right) = 0. \tag{3.34}$$

We multiply by $(|\mu_2(s)| + \xi) d_t^{jk}$ and sum up over j from 1 to k, to have

$$\frac{s(|\mu_2(s)|+\xi)}{2} \frac{d}{dt} \left\|\nabla z_t^k\right\|^2 + \frac{|\mu_2(s)|+\xi}{2} \frac{d}{d\kappa} \left\|\nabla z_t^k\right\|^2 = 0, \tag{3.35}$$

we integrate over $(0,1) \times (\tau_1, \tau_2)$ with respect to κ and s, to get

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt} \int_{\tau_1}^{\tau_2} \int_0^1 s(|\mu_2(s)|+\xi) \int_\Omega \left|\nabla z_t^k(x,\kappa,s,t)\right|^2 ds d\kappa dx \\ & +\frac{1}{2} \int_{\tau_1}^{\tau_2} (|\mu_2(s)|+\xi) \int_\Omega \left|\nabla z_t^k(x,1,s,t)\right|^2 ds dx \\ & -\frac{1}{2} \int_{\tau_1}^{\tau_2} (|\mu_2(s)|+\xi) \int_\Omega \left|\nabla u_{tt}^k\right|^2 ds dx \\ = & \ 0. \end{aligned} \tag{3.36}$$

Summing (3.33) and (3.36), we have

$$\begin{aligned} & \int_\Omega \left|u_t^k\right|^\rho \left|\nabla u_{tt}^k\right|^2 dx + \left\|\nabla u_{tt}^k\right\|^2 \\ & +\frac{1}{2}\frac{d}{dt} \int_{\tau_1}^{\tau_2} \int_0^1 s(|\mu_2(s)|+\xi) \int_\Omega \left|\nabla z_t^k(x,\kappa,s,t)\right|^2 ds d\kappa dx \\ & +\frac{1}{2} \int_{\tau_1}^{\tau_2} (|\mu_2(s)|+\xi) \int_\Omega \left|\nabla z_t^k(x,1,s,t)\right|^2 ds dx \\ = & -\int_\Omega \Delta^2 u^k u_{tt}^k dx + \int_0^t h(t-\sigma) \int_\Omega \Delta u^k(\sigma) \Delta u_{tt}^k(t) \, dx d\sigma \\ & -\mu_1 \int_\Omega u_t^k u_{tt}^k dx - \int_\Omega \int_{\tau_1}^{\tau_2} \mu_2(s) z^k(x,1,s,t) u_{tt}^k ds dx \\ & +\frac{1}{2} \int_{\tau_1}^{\tau_2} (|\mu_2(s)|+\xi) \int_\Omega \left|\nabla u_{tt}^k\right|^2 ds dx + b \int_\Omega u_{tt}^k u^k \ln\left|u^k\right| dx. \end{aligned} \tag{3.37}$$

Utilizing Young's inequality, the right hand side of (3.37) can be written

$$\int_\Omega \Delta^2 u^k u_{tt}^k dx \leq \eta \left\|\nabla u_{tt}^k\right\|^2 + \frac{1}{4\eta} \left\|\nabla \Delta u^k\right\|^2, \ \eta > 0, \tag{3.38}$$

and

$$\begin{aligned}
&\int_0^t h(t-\sigma)\int_\Omega \Delta u^k(\sigma)\,\Delta u_{tt}^k(t)\,dxd\sigma \\
=\;& -\int_0^t h(t-\sigma)\int_\Omega \nabla\Delta u^k(\sigma)\,\nabla u_{tt}^k(t)\,dxd\sigma \\
\leq\;& \eta\left\|\nabla u_{tt}^k\right\|^2+\frac{\beta^2}{4\eta}(1+\eta)\left|\nabla\Delta u^k\right|^2 \\
&+\frac{\beta}{4\eta}\left(1+\frac{1}{\eta}\right)\left(h\circ\nabla\Delta u^k\right). \qquad (3.39)
\end{aligned}$$

By using Young's inequality, we get

$$\begin{aligned}
\mu_1\int_\Omega u_t^k u_{tt}^k dx &\leq \eta\left\|u_{tt}^k\right\|^2+\frac{\mu_1^2}{4\eta}\left\|u_t^k\right\|^2 \\
&\leq \eta C_s^2\left\|\nabla u_{tt}^k\right\|^2+\frac{C_s^2\mu_1^2}{4\eta}\left\|\nabla u_t^k\right\|^2 \\
&\leq \eta C_s^2\left\|\nabla u_{tt}^k\right\|^2+C(\eta), \qquad (3.40)
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega\int_{\tau_1}^{\tau_2}\mu_2(s)\,z^k(x,1,s,t)\,u_{tt}^k dsdx \\
\leq\;& \eta C_s^2\int_{\tau_1}^{\tau_2}|\mu_2(s)|\int_\Omega\left|\nabla u_{tt}^k\right|^2 dsdx \\
&+\frac{1}{4\eta}\int_{\tau_1}^{\tau_2}|\mu_2(s)|\int_\Omega\left|z^k(x,1,s,t)\right|^2 dsdx \\
\leq\;& \eta C_s^2\mu_1\int_\Omega\left|\nabla u_{tt}^k\right|^2+\frac{1}{4\eta}\int_{\tau_1}^{\tau_2}|\mu_2(s)|\int_\Omega\left|z^k(x,1,s,t)\right|^2 dsdx. \qquad (3.41)
\end{aligned}$$

For the last term in the right-hand side of (3.37), by applying (2.13) with $\varepsilon_0=\frac{1}{2}$ and utilizing Young's, Cauchy-Schwarz's and the embedding inequalities, we have

$$\begin{aligned}
b\int_\Omega u_{tt}^k u^k\ln\left|u^k\right|dx &\leq c\int_\Omega u_{tt}^k\left(\left|u^k\right|^2+d\sqrt{u^k}\right)dx \\
&\leq c\left(\delta\int_\Omega\left|u_{tt}^k\right|^2dx+\frac{1}{4\delta}\int_\Omega\left(\left|u^k\right|^2+d\sqrt{u^k}\right)^2dx\right) \\
&\leq c\delta\left\|\nabla u_{tt}^k\right\|^2+\frac{c}{4\delta}\left(\int_\Omega\left|u^k\right|^4dx+\int_\Omega\sqrt{u^k}dx\right) \\
&\leq c\delta\left\|\nabla u_{tt}^k\right\|^2+\frac{c}{4\delta}\left(\left\|\Delta u^k\right\|^4+\left\|u^k\right\|\right). \qquad (3.42)
\end{aligned}$$

Taking into account (3.38)-(3.42) into (3.37), satisfies

$$\begin{aligned}
& \int_\Omega \left|u_t^k\right|^\rho \left|\nabla u_{tt}^k\right|^2 dx \\
& +\frac{1}{2}\frac{d}{dt}\int_{\tau_1}^{\tau_2}\int_0^1 s\left(\left|\mu_2(s)\right|+\xi\right)\int_\Omega \left|\nabla z_t^k(x,\kappa,s,t)\right|^2 dsd\kappa dx \\
& +\frac{1}{2}\int_{\tau_1}^{\tau_2}\left(\left|\mu_2(s)\right|+\xi\right)\int_\Omega \left|\nabla z_t^k(x,1,s,t)\right|^2 dsdx \\
& +\left(1-\eta\left(2+C_s^2+C_s^2\mu_1\right)-c\delta\right)\left\|\nabla u_{tt}^k\right\|^2 \\
\leq & \frac{1}{4\eta}\left(1+\beta^2(1+\eta)\right)\left\|\nabla\Delta u^k\right\|^2+\frac{\beta}{4\eta}\left(1+\frac{1}{\eta}\right)\left(h\circ\nabla\Delta u^k\right) \\
& +\frac{1}{4\eta}\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right|\int_\Omega \left|z^k(x,1,s,t)\right|^2 dsdx+C(\eta) \\
& +\frac{c}{4\delta}\left(\left\|\Delta u^k\right\|^4+\left\|u^k\right\|^2\right).
\end{aligned} \tag{3.43}$$

Therefore, by (3.32) and (3.43), we get

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\left[\left(1-\int_0^t h(\sigma)\right)\left\|\nabla\Delta u^k\right\|^2+\left\|\Delta u_t^k\right\|^2+\left(h\circ\nabla\Delta u^k\right)\right. \\
& +\int_{\tau_1}^{\tau_2}\int_0^1 s\left(\left|\mu_2(s)\right|+\xi\right)\left\|\nabla z^k(x,\kappa,s,t)\right\|^2 dsd\kappa \\
& \left.+\int_{\tau_1}^{\tau_2}\int_0^1 s\left(\left|\mu_2(s)\right|+\xi\right)\left\|\nabla z_t^k(x,\kappa,s,t)\right\|^2 dsd\kappa+2\int_\Omega \left|u_t^k\right|^\rho \left|\nabla u_t^k\right|^2 dx\right] \\
& +\frac{1}{2}\int_{\tau_1}^{\tau_2}\left(\left|\mu_2(s)\right|+\xi\right)\int_\Omega \left|\nabla z_t^k(x,1,s,t)\right|^2 dsdx \\
& +\frac{1}{2}\int_{\tau_1}^{\tau_2}\left(\left|\mu_2(s)\right|+\xi-2\eta\right)\left\|\nabla z^k(x,1,s,t)\right\|^2 ds \\
& +\left(1-\eta\left(3+C_s^2+C_s^2\mu_1\right)-c\delta\right)\left\|\nabla u_{tt}^k\right\|^2 \\
\leq & -\frac{1}{2}h(t)\left\|\nabla\Delta u^k\right\|^2+\frac{1}{2}\left(h'\circ\nabla\Delta u^k\right) \\
& +\frac{1}{4\eta}\left(1+\beta^2(1+\eta)\right)\left\|\nabla\Delta u^k\right\|^2 \\
& +\frac{\beta}{4\eta}\left(1+\frac{1}{\eta}\right)\left(h\circ\nabla\Delta u^k\right) \\
& +\frac{1}{4\eta}\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right|\int_\Omega \left|z^k(x,1,s,t)\right|^2 dsdx+C(\eta) \\
& +\frac{c}{4\delta}\left(\left\|\Delta u^k\right\|^4+\left\|u^k\right\|^2\right).
\end{aligned} \tag{3.44}$$

Choosing $\delta > 0$ and η small enough such that $\left(1 - \eta\left(3 + C_s^2 + C_s^2\mu_1\right) - c\delta\right) > 0$ and integrating over $(0, t)$, we get

$$
\begin{aligned}
&\frac{1}{2}\left[\left(1 - \int_0^t h(\sigma)\right)\left\|\nabla\Delta u^k\right\|^2 + \left\|\Delta u_t^k\right\|^2 + \left(h \circ \nabla\Delta u^k\right)\right. \\
&+ \int_{\tau_1}^{\tau_2}\int_0^1 s\left(\left|\mu_2(s)\right| + \xi\right)\left\|\nabla z^k(x,\kappa,s,t)\right\|^2 dsd\kappa \\
&\left.+ \int_{\tau_1}^{\tau_2}\int_0^1 s\left(\left|\mu_2(s)\right| + \xi\right)\left\|\nabla z_t^k(x,\kappa,s,t)\right\|^2 dsd\kappa + 2\int_\Omega \left|u_t^k\right|^\rho \left|\nabla u_t^k\right|^2 dx\right] \\
&+ \frac{1}{2}\int_0^t\int_{\tau_1}^{\tau_2}\left(\left|\mu_2(s)\right| + \xi\right)\int_\Omega \left|\nabla z_t^k(x,1,s,\sigma)\right|^2 dsdxd\sigma \\
&+ \left(1 - \eta\left(3 + C_s^2 + C_s^2\mu_1\right) - c\delta\right)\int_0^t \left\|\nabla u_{tt}^k\right\|^2 d\sigma \\
&+ \frac{1}{2}\int_{\tau_1}^{\tau_2}\left(\left|\mu_2(s)\right| + \xi - 2\eta\right)\int_0^t \left\|\nabla z^k(x,1,s,\sigma)\right\|^2 dsd\sigma \\
\leq\ & -\frac{1}{2}\int_0^t h(\sigma)\left\|\nabla\Delta u^k\right\|^2 d\sigma + \frac{1}{2}\int_0^t \left(h' \circ \nabla\Delta u^k\right) d\sigma \\
&+ \frac{1}{4\eta}\left(1 + \beta^2(1+\eta)\right)\int_0^t \left\|\nabla\Delta u^k\right\|^2 d\sigma \\
&+ \frac{\beta}{4\eta}\left(1 + \frac{1}{\eta}\right)\int_0^t \left(h \circ \nabla\Delta u^k\right)(\sigma)\, d\sigma \\
&+ \frac{1}{4\eta}\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right|\int_0^t\int_\Omega \left|z^k(x,1,s,t)\right|^2 dsdxd\sigma + C(\eta)T \\
&+ \frac{c}{4\delta}\int_0^t \left(\left\|\Delta u^k\right\|^4 + \left\|u^k\right\|^2\right) dt. \qquad (3.45)
\end{aligned}
$$

Using Gronwall's Lemma and taking $h_1 = \min\{h(t) \mid \text{for all } t \geq t_0\}$, we have

$$
\begin{aligned}
&\left\|\nabla\Delta u^k\right\|^2 + \left\|\Delta u_t^k\right\|^2 + \left(h \circ \nabla\Delta u^k\right) + \int_0^t \left\|\nabla u_{tt}^k(\sigma)\right\|^2 d\sigma \\
&+ \int_{\tau_1}^{\tau_2}\int_0^1 s\left(\left|\mu_2(s)\right| + \xi\right)\left\|\nabla z^k(x,\kappa,s,t)\right\|^2 dsd\kappa \\
&+ \int_0^t\int_{\tau_1}^{\tau_2} s\left(\left|\mu_2(s)\right| + \xi\right)\int_\Omega \left|\nabla z_t^k(x,\kappa,s,\sigma)\right|^2 dsdxd\sigma \\
&+ \int_0^t\int_{\tau_1}^{\tau_2} s\left(\left|\mu_2(s)\right| + \xi\right)\int_\Omega \left|\nabla z_t^k(x,1,s,\sigma)\right|^2 dsdxd\sigma \\
&+ \frac{1}{2}\int_{\tau_1}^{\tau_2}\left(\left|\mu_2(s)\right| + \xi - 2\eta\right)\int_0^t \left\|\nabla z^k(x,1,s,t)\right\|^2 dsd\sigma \\
\leq\ & C_3. \qquad (3.46)
\end{aligned}
$$

The estimate (3.46) yields

$$\left(u^k\right) \text{ is uniformly bounded in } L^\infty\left(0,T;H_0^2(\Omega)\right), \qquad (3.47)$$

$$\left(u_t^k\right) \text{ is uniformly bounded in } L^\infty\left(0,T;L^{\rho+2}(\Omega)\right) \cap L^\infty\left(0,T;H_0^2(\Omega)\right), \qquad (3.48)$$

$$\left(u_{tt}^k\right) \text{ is uniformly bounded in } L^2\left(0,T;H_0^2(\Omega)\right). \qquad (3.49)$$

We see that by the estimates (3.18) and (3.46) that there exists a subsequence $\{u^m\}$ of $\{u^k\}$ and a function u such that

$$u^m \to u \text{ weakly star in } L^\infty\left(0,T;H^2(\Omega)\right), \qquad (3.50)$$

$$u_t^m \to u_t \text{ weakly star in } L^\infty\left(0,T;L^{\rho+2}(\Omega)\right) \cap L^\infty\left(0,T;H_0^2(\Omega)\right), \qquad (3.51)$$

$$u^m \to u \text{ weakly in } L^2\left(0,T;H_0^2(\Omega)\right), \qquad (3.52)$$

$$u_t^m \to u_t \text{ weakly in } L^2\left(0,T;L^{\rho+2}(\Omega)\right)\cap L^2\left(0,T;H_0^2(\Omega)\right), \tag{3.53}$$

$$u_{tt}^m \to u_{tt} \text{ weakly in } L^2\left(0,T;H_0^2(\Omega)\right). \tag{3.54}$$

Analysis of the nonlinear terms:

First term ($u^k\ln\left|u^k\right|$): By using (3.46), we have (u^m) is bounded in $L^\infty\left(0,T;H_0^2(\Omega)\right)$ which implies, utilizing the embedding of $H_0^2(\Omega)$ in $L^\infty(\Omega)\left(\Omega\subset R^2\right)$, the boundness of (u^m) in $L^2(\Omega\times(0,T))$. In a similar way, (u_t^m) is bounded in $L^2(\Omega\times(0,T))$. Next, by using Aubin-Lions theorem, we have a subsequence such that

$$u^m \to u \text{ strongly in } L^2(\Omega\times(0,T)) \tag{3.55}$$

which implies

$$u^m \to u \text{ a.e. in } \Omega\times(0,T). \tag{3.56}$$

Because of the maps $s\to bs\ln|s|$ is continuous, we find the following convergence:

$$bu^m\ln|u^m| \to bu\ln|u| \text{ a.e. in } \Omega\times(0,T). \tag{3.57}$$

From the embedding of $H_0^2(\Omega)$ in $L^\infty(\Omega)\left(\Omega\subset R^2\right)$, we see that $b\left(u^m\ln|u^m|\right)$ is bounded in $L^\infty(\Omega\times(0,T))$. Now, taking into consideration the Lebesgue bounded convergence theorem (Ω is bounded), we have

$$bu^m\ln|u^m| \to bu\ln|u| \text{ strongly in } L^2\left(0,T;L^2(\Omega)\right). \tag{3.58}$$

Second term ($\left|u_t^k\right|^\rho\left|u_t^k\right|$): From (3.46), we see that (u_t^m) is uniformly bounded in $L^\infty\left(0,T;H^2(\Omega)\right)$ which implies the boundedness of (u_t^m) in $L^\infty(\Omega\times(0,T))$, and so in $L^2(\Omega\times(0,T))$. Also, we know that (u_{tt}^m) is bounded in $L^2\left((0,T);H_0^2(\Omega)\right)$ which implies that (u_{tt}^m) is bounded in $L^2(\Omega\times(0,T))$.

From the first estimate in (3.18) and Lemma 2.1, we conclude

$$\begin{aligned}\left\|\left|u_t^k\right|^\rho u_t^k\right\|_{L^2(0,T,L^2(\Omega))} &= \int_0^T\left\|u_t^k\right\|_{2(\rho+1)}^{2(\rho+1)}dt\\ &\le \left(\frac{C_s}{\sqrt{\lambda}}\right)^{2(\rho+1)}\int_0^T\left\|\Delta u_t^k\right\|_2^{2(\rho+1)}dt\\ &\le \left(\frac{C_s}{\sqrt{\lambda}}\right)^{2(\rho+1)}C_3^{2(\rho+1)}T.\end{aligned} \tag{3.59}$$

Now, using Aubin-Lions theorem, (see Lions [18]), there exists a subsequence $\{u^m\}$ of $\left\{u^k\right\}$ such that

$$u_t^m \to u_t \text{ strongly in } L^2\left(0,T;L^2(\Omega)\right) \tag{3.60}$$

which implies

$$u_t^m \to u_t \text{ almost everywhere in } \Omega\times(0,T). \tag{3.61}$$

Therefore,

$$|u_t^m|^\rho u_t^m \to |u_t|^\rho u_t \text{ almost everywhere in } \Omega\times(0,T). \tag{3.62}$$

Hence, by using (3.60)-(3.62) and utilizing Lions Lemma, we get

$$|u_t^m|^\rho u_t^m \to |u_t|^\rho u_t \text{ weakly in } L^2\left(0,T;L^2(\Omega)\right). \tag{3.63}$$

We multiply (3.3) by $\Theta(t)\in D(0,T)$ and integrate over $(0,T)$, we have

$$\begin{aligned}&-\frac{1}{\rho+1}\int_0^T\left(\left|u_t^k(t)\right|^\rho u_t^k(t),w^j\right)\theta'(t)\,dt+\int_0^T\left(\Delta u^k(t),\Delta w^j\right)\theta(t)\,dt\\ &+\int_0^T\left(\nabla u_{tt}^k,\nabla w^j\right)\theta(t)\,dt-\int_0^T\int_0^t h(t-\sigma)\left(\Delta u^k(\sigma),\Delta w^j\right)\theta(t)\,d\sigma dt\\ &+\mu_1\int_0^T\left(u_t^k,w^j\right)\theta(t)\,dt+\int_0^T\int_{\tau_1}^{\tau_2}\mu_2(s)\left(z(x,1,s,t),w^j\right)\theta(t)\,dsdt\\ =\;& b\int_0^T\left(u^k(s)\ln\left|u^k(s)\right|,w^j\right)\theta(t)\,dxdt,\end{aligned} \tag{3.64}$$

we multiply (3.7) by $\theta(t) \in D(0,T)$ and integrate over $(0,T)\times(0,1)$, to obtain

$$\int_0^T \int_0^1 \left(sz_t^k + z_\kappa^k, \phi^j\right) \theta(t)\, dx d\kappa = 0. \tag{3.65}$$

The convergence of (3.50)-(3.54) and (3.63), are sufficient to pass to the limit in (3.64) and (3.65) to get

$$\begin{aligned}
& -\frac{1}{\rho+1}\int_0^T \left(|u_t|^\rho u_t, w\right)\theta'(t)\, dt + \int_0^T (\Delta u, \Delta w)\,\theta(t)\, dt \\
& + \int_0^T (\nabla u_{tt}, \nabla w)\,\theta(t)\, dt - \int_0^T\int_0^t h(t-\sigma)\left(\Delta u(\sigma), \Delta w\right)\theta(t)\, d\sigma dt \\
& + \mu_1 \int_0^T (u_t, w)\,\theta(t)\, dt + \int_0^T \int_{\tau_1}^{\tau_2} \mu_2(s)\left(z(x,1,s,t), w\right)\theta(t)\, ds dt \\
= \; & b\int_0^T \left(u(s)\ln|u(s)|, w\right)\theta(t)\, dx dt,
\end{aligned}$$

and

$$\int_0^T \int_0^1 (sz_t + z_\kappa, \phi)\,\theta(t)\, dt d\kappa = 0.$$

Integrating over $(0,T)$, we have

$$\begin{aligned}
& \int_0^T \Bigg(|u_t|^\rho u_{tt} + \Delta^2 u - \Delta u_{tt} - \int_0^t h(t-\sigma)\,\Delta^2 u(s)\, d\sigma \\
& + \mu_1 u_t + \int_{\tau_1}^{\tau_2} \mu_2(s)\left(z(x,1,s,t)\, ds, w\right)\Bigg)\theta(t)\, dt \\
= \; & b\int_0^T \left(u(t)\ln|u(t)|, w\right)\theta(t)\, dx dt.
\end{aligned}$$

Consequently, we get the local existence of the problem (2.16).

3.2 Global existence

In this part, we obtain the global existence result. Firstly, we give the following functionals for this aim:

$$I(t) = \left(1 - \int_0^t h(s)\, ds\right)\|\Delta u\|^2 + \|\nabla u_t\|^2 + (h\circ\Delta u)(t) - 3b\int_\Omega u^2 \ln|u|\, dx, \tag{3.66}$$

$$\begin{aligned}
J(t) &= \frac{1}{2}\left(1 - \int_0^t h(s)\, ds\right)\|\Delta u\|^2 + \frac{1}{2}\|\nabla u_t\|^2 \\
&\quad + \frac{1}{2}(h\circ\Delta u)(t) - \frac{b}{2}\int_\Omega u^2\ln|u|\, dx + \frac{b}{4}\|u\|^2. \\
&= \frac{1}{3}\left[\left(1 - \int_0^t h(s)\, ds\right)\|\Delta u\|^2 + \|\nabla u_t\|^2\right. \\
&\quad \left. + (h\circ\Delta u)(t)\right] + \frac{b}{4}\|u\|^2 + \frac{1}{6} I(t).
\end{aligned} \tag{3.67}$$

We note that

$$E(t) = \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + J(t) + \frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s\left(|\mu_2(s)| + \xi\right) z^2(x,\kappa,s,t)\, ds d\kappa dx. \tag{3.68}$$

Lemma 3.1. [2] The following inequalities hold:

$$-bd_0\sqrt{|\Omega|\, c_*^3}\,\|\Delta u\|_2^{3/2} \le b\int_\Omega u^2\ln|u|\, dx \le bc_*^3\|\Delta u\|_2^3,\ \forall u\in H_0^2(\Omega), \tag{3.69}$$

where $d_0 = \sup_{0<s<1} \sqrt{s}\,|\ln s| = \frac{2}{e}$, $|\Omega|$ is the Lebesgue measure of Ω and c_* is the smallest embedding constant

$$\left(\int_\Omega |u|^3\,dx\right)^{1/3} \le c_* \|\Delta u\|_2\,,\ \forall u \in H_0^2(\Omega)\,, \tag{3.70}$$

(c_* exists thanks to the embedding of $H_0^2(\Omega)$ in $L^\infty(\Omega)$ and $\Omega \subset R^2$).

Lemma 3.2. Suppose that (A1)-(A3). Assume that $(u_0, u_1) \in H_0^2(\Omega) \times H_0^2(\Omega)$ such that

$$I(0) > 0 \text{ and } \sqrt{54}bc_*^3\left(\frac{E(0)}{l}\right)^{1/2} < l. \tag{3.71}$$

Then

$$I(t) > 0,\ \forall t \in [0, T)\,. \tag{3.72}$$

Proof . We have the energy functional as follow

$$E(t) = \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + J(t) + \frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s\left(|\mu_2(s)| + \xi\right) z^2(x,\kappa,s,t)\,dsd\kappa dx.$$

Hence, by (3.66), we get

$$b\int_\Omega u^2 \ln|u|\,dx = \frac{1}{3}\left(1 - \int_0^t h(s)\,ds\right)\|\Delta u\|^2 + \frac{1}{3}\|\nabla u_t\|^2 + \frac{1}{3}(h \circ \Delta u)(t) - \frac{1}{3}I(t)\,. \tag{3.73}$$

Substituting (3.73) in (3.67), we have

$$J(t) = \frac{1}{3}\left[\left(1 - \int_0^t h(s)\,ds\right)\|\Delta u\|^2 + \|\nabla u_t\|^2 + (h \circ \Delta u)(t)\right] + \frac{b}{4}\|u\|^2 + \frac{1}{6}I(t)\,. \tag{3.74}$$

Since I is continuous on $[0, T]$ and $I(0) > 0$, there exists $t_0 \in (0, T]$ such that $I(t) > 0$, for all $t \in [0, t_0)$. We show by t_0 the largest real number in $(0, T]$ such that $I > 0$ on $[0, t_0)$. If $t_0 = T$, then (3.72) is satisfied.

From contradiction, we suppose that $t_0 \in (0, T)$. Therefore, $I(t_0) = 0$ and

$$\|\Delta u\|^2 \le \frac{6}{l}J(t) \le \frac{6}{l}E(t) \le \frac{6}{l}E(0)\,,\ \ \forall t \in [0, t_0)\,. \tag{3.75}$$

If $\|\Delta u(t_0)\|^2 = 0$, then (3.69) and (3.70) give

$$\begin{aligned} 0 &= I(t_0) = \left(1 - \int_0^{t_0} h(s)\,ds\right)\|\Delta u(t_0)\|^2 + \|\nabla u_t(t_0)\|^2 + (h \circ \Delta u)(t_0) \\ &\quad -3b\int_\Omega u^2(t_0)\ln|u(t_0)|\,dx \\ &\le c\|\Delta u(t_0)\|^2 + (h \circ \Delta u)(t_0) = \int_0^{t_0} h(s)\,ds\,\|\Delta u(s)\|^2\,ds. \end{aligned} \tag{3.76}$$

As a result, if $g > 0$ on $[0, t_0)$, we obtain

$$\|\Delta u(s)\|_2 = 0,\ \ \forall s \in [0, t_0)\,.$$

Hence,

$$I(t) = 0,\ \ \forall t \in [0, t_0)\,,$$

which is not hold since $I > 0$ on $[0, t_0)$. If $h \neq 0$ on $[0, t_0)$, then assume that $t_1 \in [0, t_0)$ the smallest real number such that $h(t_1) = 0$. Since $h(0) > 0$ and h is continuous, nonincreasing and positive on R^+, hence, $t_1 > 0$ and $h = 0$ on $[t_1, \infty)$. Hence, by (3.76), we conclude that

$$0 = \int_0^{t_0} h(s)\|\Delta u(s)\|^2\,ds = \int_0^{t_1} h(s)\|\Delta u(s)\|^2\,ds,$$

then $\|\Delta u(s)\|_2 = 0$, for any $s \in [0, t_1)$, which specified that $I(t) = 0$, for any $t \in [0, t_1)$. Similar to above, this is a contradiction with the fact that $I > 0$ on $[0, t_0)$. Therefore, we infer that $\|\Delta u(t_0)\|^2 > 0$. Moreover, we get

$$I(t_0) \geq l \|\Delta u(t_0)\|^2 - 3b \int_\Omega u^2(t_0) \ln |u(t_0)| \, dx.$$

From (3.75) and by using Lemma 3.1, we obtain

$$I(t_0) \geq \left[l - 3bc_*^3 \left(\frac{6E(0)}{l} \right)^{1/2} \right] \|\Delta u(t_0)\|^2 .$$

Recalling (3.71), we conclude that $I(t_0) > 0$, which contradicts the assumption $I(t_0) = 0$. Thus, $t_0 = T$ and then

$$I(t) > 0, \quad \forall t \in [0, T).$$

Consequently, we completed the proof of Theorem 2.9. □

4 Asymptotic behavior

Proof of Theorem 2.10:

In this part, by constructing a suitable Lyapunov functional, we get asymptotic behavior result for our problem. Firstly, we define the functional as follows

$$L(t) = NE(t) + N_1 F_1(t) + F_2(t) + N_2 F_3(t), \tag{4.1}$$

where N, N_1 and N_2 are positive real numbers.

Next, we define the following functionals:

$$F_1(t) = \frac{1}{\rho+1} \int_\Omega |u_t|^\rho u_t u dx + \int_\Omega \nabla u_t \nabla u dx, \tag{4.2}$$

$$F_2(t) = \int_\Omega \left(\Delta u_t - \frac{1}{\rho+1} |u_t|^\rho u_t \right) \int_0^t h(t-\sigma)(u(t) - u(\sigma)) \, d\sigma dx, \tag{4.3}$$

$$F_3(t) = \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\kappa} (|\mu_2(s)| + \xi) z^2(x, \kappa, s, t) \, ds d\kappa dx. \tag{4.4}$$

To get our main result, we need the following lemmas:

Lemma 4.1. Suppose that (A1)-(A3) and (3.71) hold and let $\varepsilon_0 \in (0, 1)$. Let

$$0 < E(0) < \frac{el\pi}{4}. \tag{4.5}$$

Then, the functional $L(t)$, for N sufficiently large, satisfies

$$\lambda_0 E(t) \leq L(t) \leq \lambda_1 E(t), \forall t \geq 0, \tag{4.6}$$

where λ_0 and λ_1 are positive constants depending on N_1, N_2 and N. Hence, $L \sim E$ and for any $t_0 > 0$, there exists a positive constant m, such that

$$L'(t) \leq -mE(t) + c^*(h \circ \Delta u)(t) + c_{\epsilon_0} (h \circ \Delta u)^{1/(1+\epsilon_0)}(t), \quad \forall t \geq t_0. \tag{4.7}$$

Proof . The proof is similar to [2, 32], hence, we omit it. □

Lemma 4.2. Assume that (u, z) be a solution of problem (2.16). Then, the functional $F_1(t)$ satisfies

$$\begin{aligned} F_1'(t) \leq & \frac{1}{\rho+1} \|u_t\|_{\rho+2}^{\rho+2} - \left(1 - \beta - \eta - \frac{2\eta C_s^2}{\lambda} \mu_1 \right) \|\Delta u\|^2 + \left(1 + \frac{C_s^2 \mu_1}{4\eta} \right) \|\nabla u_t\|^2 \\ & + \frac{\beta}{4\eta} (h \circ \Delta u)(t) + \frac{1}{4\eta} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) \, ds dx + b \int_\Omega u^2 \ln |u| \, dx, \end{aligned} \tag{4.8}$$

for $\eta > 0$.

Proof . Taking the derivative of $F_1(t)$, from (2.16) and integrating by parts, we get

$$\begin{aligned} F_1'(t) &= \frac{1}{\rho+1}\int_\Omega \left(|u_t|^\rho u_t\right)' u dx + \frac{1}{\rho+1}\int_\Omega |u_t|^{\rho+2} dx + \int_\Omega \nabla u_{tt}\nabla u dx + \|\nabla u_t\|^2 \\ &= \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|^2 + \int_\Omega \left(|u_t|^\rho u_{tt} - \Delta u_{tt}\right) u dx \\ &= \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|^2 - \int_\Omega \left(\Delta^2 u - \int_0^t h(t-\sigma)\Delta^2 u(\sigma) d\sigma\right) u dx \\ &\quad - \int_\Omega \left(\mu_1 u_t + \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t) ds - bu\ln|u|\right) u dx \\ &= \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|^2 - \|\Delta u\|^2 + \int_\Omega \Delta u \int_0^t h(t-\sigma)\Delta u(\sigma) d\sigma dx \\ &\quad -\mu_1 \int_\Omega u u_t dx - \int_\Omega \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t) u ds dx + b\int_\Omega u^2 \ln|u| dx. \end{aligned} \tag{4.9}$$

Next, utilizing Sobolev embedding and Young's inequality, estimating the terms in the right hand side of (4.9), we have

$$\begin{aligned} &\left|\int_\Omega \Delta u \int_0^t h(t-\sigma)\Delta u(\sigma) d\sigma dx\right| \\ \leq\ & (\beta+\eta)\|\Delta u\|^2 + \frac{\beta}{4\eta}(h\circ\Delta u). \end{aligned} \tag{4.10}$$

Since

$$\left|\int_\Omega u u_t dx\right| \leq \eta \frac{C_s^2}{\lambda}\|\Delta u\|^2 + \frac{C_s^2}{4\eta}\|\nabla u_t\|^2, \tag{4.11}$$

and

$$\begin{aligned} &\left|\int_\Omega \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t) u ds dx\right| \\ \leq\ & \eta\frac{C_s^2}{\lambda}\int_{\tau_1}^{\tau_2} |\mu_2(s)| \int_\Omega |\Delta u|^2 ds dx \\ & + \frac{1}{4\eta}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \left|z^2(x,1,s,t) ds dx\right| \\ \leq\ & \eta\frac{\mu_1 C_s^2}{\lambda}\|\Delta u\|^2 + \frac{1}{4\eta}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \left|z^2(x,1,s,t)\right| ds dx. \end{aligned} \tag{4.12}$$

The estimate (4.8) followed by substituting (4.10)-(4.12) into (4.9). Hence, we completed the proof. □

Lemma 4.3. Suppose that (u,z) be the solution of problem (2.16). Then, $F_2(t)$ satisfies

$$\begin{aligned} F_2'(t) \leq\ & \delta\left(2\beta^2 + 1 + \frac{1}{4}\right)\|\Delta u\|^2 + \left(\delta + \frac{\delta a_0}{\rho+1} - h_0\right)\|\nabla u_t\|^2 \\ & - \frac{1}{\rho+1} h_0 \|u_t\|_{\rho+2}^{\rho+2} + \beta\left(2\delta + \frac{1}{4\delta} + \frac{\mu_1 C_s^2}{2\delta\lambda} + \frac{c}{\delta\beta}\right)(h\circ\Delta u)(t) \\ & - \frac{h(0)}{4\delta\lambda}\left(1 + \frac{C_s^2}{l+1}\right)(h'\circ\Delta u) + \mu_1\delta\int_\Omega \int_{\tau_1}^{\tau_2} \mu_2(s) z^2(x,1,s,t) ds dx \\ & + \mu_1\delta\|u_t\|^2 + c_{\epsilon_0,\delta}(h\circ\Delta u)^{1/(1+\epsilon_0)}(t). \end{aligned} \tag{4.13}$$

for any $\delta > 0$, where $\int_0^t h(\sigma) d\sigma \geq \int_0^{t_0} h(\sigma) d\sigma = h_0$, $\forall t \geq t_0$.

Proof . Utilizing the Leibnitz formula, and the first equation of (2.16), we get

$$\begin{aligned}
F_2'(t) &= -\int_\Omega \left(\int_0^t h(t-\sigma)\Delta u(\sigma)\,d\sigma\right)\left(\int_0^t h(t-\sigma)(\Delta u(t)-\Delta u(\sigma))\,d\sigma\right)dx \\
&\quad +\int_\Omega \Delta u(t)\left(\int_0^t h(t-\sigma)(\Delta u(t)-\Delta u(\sigma))\,d\sigma\right)dx \\
&\quad +\mu_1\int_\Omega u_t(t)\left(\int_0^t h(t-\sigma)(u(t)-u(\sigma))\,d\sigma\right)dx \\
&\quad +\int_\Omega\int_{\tau_1}^{\tau_2}\mu_2(s)\,z(x,1,s,t)\int_0^t h(t-\sigma)(u(t)-u(\sigma))\,d\sigma dx \\
&\quad -\int_\Omega \nabla u_t(t)\int_0^t h'(t-\sigma)(\nabla u(t)-\nabla u(\sigma))\,d\sigma dx \\
&\quad -\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t\int_0^t h'(t-\sigma)(u(t)-u(\sigma))\,d\sigma dx \\
&\quad -b\int_\Omega u\ln|u|\int_0^t h(t-\sigma)(u(t)-u(\sigma))\,d\sigma dx \\
&\quad -\int_0^t h(s)\,ds\,\|\nabla u_t\|^2-\frac{1}{\rho+1}\int_0^t h(s)\,ds\,\|u_t\|_{\rho+2}^{\rho+2} \\
&= I_1+\cdots+I_7-\int_0^t h(\sigma)\,d\sigma\,\|\nabla u_t\|^2-\frac{1}{\rho+1}\int_0^t h(\sigma)\,d\sigma\,\|u_t\|_{\rho+2}^{\rho+2}.
\end{aligned}$$

Now, we will estimate $I_1,\cdots,I_7$. Hence, for $\delta>0$, we obtain

$$\begin{aligned}
|I_1| &\le \delta\int_\Omega\left(\int_0^t h(t-\sigma)\Delta u(\sigma)\,d\sigma\right)^2dx \\
&\quad +\frac{1}{4\delta}\int_\Omega\left(\int_0^t h(t-\sigma)|\Delta u(t)-\Delta u(\sigma)|\,d\sigma\right)^2dx \\
&\le 2\delta\left(\int_0^t h(\sigma)\,d\sigma\right)^2\|\Delta u\|^2 \\
&\quad +\left(2\delta+\frac{1}{4\delta}\right)\int_0^t h(\sigma)\,d\sigma\,(h\circ\Delta u)(t) \\
&\le 2\delta\beta^2\|\Delta u\|^2+\beta\left(2\delta+\frac{1}{4\delta}\right)(h\circ\Delta u)(t).
\end{aligned}$$

In a similar attitude,

$$|I_2|\le\delta\|\Delta u\|^2+\frac{\beta}{4\delta}(h\circ\Delta u)(t),$$

$$|I_3|\le\delta C_s^2\mu_1\|\nabla u_t\|^2+\frac{\beta C_s^2\mu_1}{4\delta\lambda}(h\circ\Delta u)(t),$$

$$|I_4|\le\delta\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,z^2(x,1,s,t)\,dsdx+\frac{\beta\mu_1C_s^2}{4\delta\lambda}(h\circ\Delta u)(t),$$

$$\begin{aligned}
|I_5| &\le \delta\|\nabla u_t\|^2+\frac{1}{4\delta}\int_\Omega\int_0^t -h'(s)\,ds\int_0^t -h'(t-s)|\nabla u(t)-\nabla u(s)|^2\,dsdx \\
&\le \delta\|\nabla u_t\|^2-\frac{h(0)}{4\delta\lambda}(h'\circ\Delta u)(t),
\end{aligned}$$

$$\begin{aligned} |I_6| &\leq \frac{\delta}{\rho+1}\int_\Omega \left||u_t|^\rho u_t\right|^2 \\ &\quad + \frac{1}{4\delta(\rho+1)}\int_\Omega \left(\int_0^t h'(t-\sigma)(u(t)-u(\sigma))\right)^2 dx \\ &\leq \frac{\delta}{\rho+1}\int_\Omega \|u_t\|_{2\rho+2}^{2\rho+2} - \frac{h(0)C_s^2}{4\delta\lambda(\rho+1)}(h'\circ\Delta u)(t) \\ &\leq \frac{\delta a_0}{\rho+1}\|\nabla u_t\|_2^2 - \frac{h(0)C_s^2}{4\delta\lambda(\rho+1)}(h'\circ\Delta u)(t), \end{aligned}$$

where $a_0 = C_s^{2(\rho+1)}(2E(0))^\rho$.

To estimate I_7, we apply (2.13) for $s = |u|$, use the embedding of $H_0^2(\Omega)$ in $L^\infty(\Omega)$, for any $\delta_* > 0$ and any $\varepsilon_0 \in (0,1)$, we obtain

$$\begin{aligned} |I_7| &\leq -b\int_\Omega u\ln|u|\int_0^t h(t-\sigma)(u(t)-u(\sigma))\,d\sigma dx \\ &\leq b\int_\Omega \left(u^2 + d_{\epsilon_0}|u|^{1-\epsilon_0}\right)\left|\int_0^t h(t-\sigma)(u(t)-u(\sigma))\,d\sigma dx\right| \\ &\leq c\int_\Omega |u^2|\left|\int_0^t h(t-\sigma)(u(t)-u(\sigma))\,d\sigma\right| dx + \delta_*\int_\Omega u^2 dx \\ &\quad + c_{\epsilon_0,\delta_*}\int_\Omega \left|\int_0^t h(t-\sigma)(u(t)-u(\sigma))\,d\sigma\right|^{2/(1+\epsilon_0)} dx \\ &\leq c\delta_*\|\Delta u\|^2 + \frac{c}{\delta_*}\int_\Omega \left|\int_0^t h(t-\sigma)(u(t)-u(\sigma))\,d\sigma\right|^2 dx \\ &\quad + c_{\epsilon_0,\delta_*}\int_\Omega \left|\int_0^t h(t-\sigma)(u(t)-u(\sigma))\,d\sigma\right|^{2/(1+\epsilon_0)} dx, \end{aligned}$$

then, putting $\delta/4 = c\delta_*$ and utilizing Hölder's inequality and from Lemma 2.7, we have

$$\begin{aligned} & -b\int_\Omega u\ln|u|\int_0^t h(t-\sigma)(u(t)-u(\sigma))\,d\sigma dx \\ \leq\ & \frac{\delta}{4}\|\Delta u\|^2 + \frac{c}{\delta}(h\circ\Delta u)(t) + c_{\epsilon_0,\delta}(h\circ\Delta u)^{1/(1+\epsilon_0)}(t). \end{aligned}$$

□

Lemma 4.4. [32] The functional $F_3(t)$ defined by (4.4), satisfies

$$\begin{aligned} F_3'(t) &\leq -m\int_\Omega\int_{\tau_1}^{\tau_2}(|\mu_2(s)|+\xi)z^2(x,1,s,t)\,dsdx \\ &\quad -m\int_\Omega\int_{\tau_1}^{\tau_2}\int_0^1 s(|\mu_2(s)|+\xi)z^2(x,\kappa,s,t)\,d\kappa dsdx \\ &\quad +\mu_1\int_\Omega u_t^2 dx, \end{aligned} \tag{4.14}$$

where $m = e^{-\tau_2}$.

Now, we give the proof of the Theorem 2.10:

Proof . Hence, from (2.18), (4.1), (4.8), (4.13) and (4.14), we arrive at

$$\begin{aligned} L'(t) \leq & -N\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds - \frac{\xi(\tau_2-\tau_2)}{2}\right)\|u_t\|^2 \\ & -\frac{1}{\rho+1}(h_0 - N_1)\|u_t\|_{\rho+2}^{\rho+2} - \xi\left(\frac{N}{2} + mN_2\right)\int_\Omega \int_{\tau_1}^{\tau_2} z^2(x,1,s,t)\, dsdx \\ & -\left(mN_2 - \frac{N_1}{4\eta} - \mu_1\delta\right)\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,s,t)\, dsdx \\ & -\left(\frac{N}{2}h(t) + N_1\left(1-\beta-\eta-\frac{2\eta C_s^2}{\lambda}\mu_1\right) - \delta\left(2\beta^2+1+\frac{1}{4}\right)\right)\|\Delta u\|^2 \\ & -\left(h_0 - \delta - \frac{\delta a_0}{\rho+1} - N_1 - \frac{N_1\mu_1}{4\eta} - \mu_1 C_s^2(1+\delta)\right)\|\nabla u_t\|^2 \\ & +\left(\frac{N_1\beta}{4\eta} + \beta\left(\left(2\delta + \frac{1}{4\delta} + \frac{\mu_1 C_s^2}{2\delta\lambda} + \frac{c}{\delta\beta}\right)\right)\right)(h\circ\Delta u)(t) \\ & +\left(\frac{N}{2} - \frac{h(0)}{4\delta\lambda}\left(1 + \frac{C_s^2}{\rho+1}\right)\right)(h'\circ\Delta u)(t) \\ & +bN_1\int_\Omega u^2 \ln|u|\, dx + c_{\epsilon_0,\delta}(h\circ\Delta u)^{1/(1+\epsilon_0)}(t) \\ & -mN_2\int_\Omega \int_{\tau_1}^{\tau_2}\int_0^1 s(|\mu_2(s)| + \xi)\, z^2(x,\kappa,s,t)\, d\kappa ds dx. \end{aligned}$$

By taking $N_1 < h_0$, $\delta > 0$ sufficiently small and N_2 large enough so that $mN_2 - \frac{N_1}{4\eta} - \mu_1\delta > 0$

$$\gamma_0 = \frac{1}{\rho+1}(h_0 - N_1) > 0,\ \gamma_1 = h_0 - N_1 - \frac{N_1\mu_1}{4\eta} - \delta\left(1 + \frac{a_0}{\rho+1}\right) - \mu_1 C_s^2(1+\delta) > 0.$$

$$\gamma_2 = \frac{N}{2}h(t) + N_1\left(1-\beta-\eta-\frac{2\eta C_s^2}{\lambda}\mu_1\right) - \delta\left(2\beta^2+1+\frac{1}{4}\right) > 0,$$

$$\gamma_3 = \zeta\left[\frac{N}{2} - \frac{h(0)}{4\delta\lambda}\left(1+\frac{C_s^2}{\rho+1}\right)\right] - \left[\frac{N_1\beta}{4\eta} + \beta\left(2\delta + \frac{1}{4\delta} + \frac{\mu_1 C_s^2}{2\delta\lambda} + \frac{c}{\delta\beta}\right)\right] > 0,$$

$$N\left[\mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(s)|\, ds - \frac{\xi(\tau_2-\tau_2)}{2}\right] > 0,\ mN_2 - \frac{N_1}{4\eta} - \mu_1\delta > 0.$$

Therefore, by letting $\gamma_4 = mN_2$, we obtain

$$\begin{aligned} L'(t) \leq & -\gamma_0\|u_t\|_{\rho+2}^{\rho+2} - \gamma_1\|\nabla u_t\|^2 - \gamma_2\|\Delta u\|^2 - \gamma_3(h\circ\Delta u)(t) \\ & -\gamma_4\int_\Omega\int_{\tau_1}^{\tau_2}\int_0^1 s(|\mu_2(s)|+\xi)\, z^2(x,\kappa,s,t)\, d\kappa ds dx \\ & +bN_1\int_\Omega u^2\ln|u|\, dx + c_{\epsilon_0,\delta}(h\circ\Delta u)^{1/(1+\epsilon_0)}(t). \end{aligned}$$

Thus, we have

$$\begin{aligned} L'(t) \leq & -m_1E(t) + \left(N_1 - \frac{m_1}{2}\right)b\int_\Omega u^2\ln|u|\, dx \\ & +\left(c + \frac{m_1}{2}\right)(h\circ\Delta u)(t) + \frac{m_1 b}{4}\|u\|^2 + c_{\epsilon_0,\delta}(h\circ\Delta u)^{1/(1+\epsilon_0)}(t). \end{aligned}$$

Utilizing the Logarithmic Sobolev inequality (2.10), we obtain

$$\begin{aligned} L'(t) \leq & -m_1E(t) - \left(N_1 - \frac{m_1}{2}\right)\frac{b}{2}\left(2(1+\ln a) - \ln\|u\|^2\right)\|u\|^2 + \left(c + \frac{m_1}{2}\right)(h\circ\Delta u)(t) \\ & +\left(N_1 - \frac{m_1}{2}\right)\frac{ba^2}{2\pi}\|\Delta u\|^2 + \frac{m_1 b}{4}\|u\|^2 + c_{\epsilon_0,\delta}(h\circ\Delta u)^{1/(1+\epsilon_0)}(t). \end{aligned}$$

Choosing m_1 and b small enough and from (3.11), for

$$m_1 \leq N_1 \leq \frac{m_1}{2}(b+1),$$

we get

$$\frac{m_1 b}{4} \leq \left(N_1 - \frac{m_1}{2}\right)\frac{b}{2}.$$

Thanks to (A3), this selection is possible. Hence, we obtain

$$\begin{aligned} L'(t) &\leq -m_1 E(t) - \left(N_1 - \frac{m_1}{2}\right)\frac{b}{2}\left(1 + 2\ln a - \ln\|u\|^2\right)\|u\|^2 \\ &\quad + c^*(h \circ \Delta u)(t) + c_{\epsilon_0}(h \circ \Delta u)^{1/(1+\epsilon_0)}(t). \end{aligned} \tag{4.15}$$

Recalling that $E'(t) \leq 0$ and $I(t) > 0$ and by (2.17), (3.67) and (4.5), we get

$$\begin{aligned} \ln\|u\|^2 &\leq \ln\left(\frac{4}{b}J(t)\right) \leq \ln\left(\frac{4}{b}E(t)\right) \\ &\leq \ln\left(\frac{4}{b}E(0)\right) \leq \ln\left(\frac{el\pi}{b}\right). \end{aligned}$$

By taking a satisfying

$$\max\left\{e^{-3/2}, \sqrt{\frac{l\pi}{b}}\right\} < a < \sqrt{\frac{2l\pi}{b}}.$$

(So (3.11) is satisfied), and we guarantee

$$1 + 2\ln a - \ln\|u\|^2 \geq 0.$$

From (2.17), (4.7) and (4.15), we get

$$L'(t) \leq -\alpha E(t),\ \forall t \geq 0, \tag{4.16}$$

for some $\alpha > 0$. By combining (4.6) and (4.16) satisfies

$$L'(t) \leq -k_1 L(t),\ \forall t \geq 0, \tag{4.17}$$

where $k_1 = \alpha/\alpha_1$. Thus, a simple integration of (4.17) over $(0,t)$ yields

$$L'(t) \leq L(0)\, e^{-k_1 t},\ \forall t \geq 0. \tag{4.18}$$

As a result, a combination of (4.6) and (4.18), we get (2.24) with $k_0 = \frac{\alpha_1 E(0)}{\alpha_0}$, hence, the proof is completed. □

5 Conclusions and open problems

In recent years, there has been published much work concerning the wave equation with constant delay or time-varying delay. However, to the best of our knowledge, there were no local existence, global existence and asymptotic behavior results for the logarithmic viscoelastic plate equation with distributed delay. In this work, we used the energy method combined with Faedo-Galerkin method to establish the local and global existence, moreover, by introducing a suitable Lyapunov functional we proved the asymptotic behavior of the solution with the logarithmic source term $f(u) = bu\ln|u|$ type. We like to point out that the local existence, global existence and asymptotic behavior of solutions to problem (1.1) with the logarithmic source term $f(u) = u|u|^{p-2}\ln|u|^k$ type is still open problem.
The problem (1.1) may be studied with variable exponents and logarithmic source term as follows:

$$\begin{cases} |u_t|^\rho u_{tt} + \Delta^2 u - \Delta u_{tt} - \int_0^t h(t-\sigma)\Delta^2 u(x,\sigma)\,d\sigma + \mu_1 u_t(x,t)|u_t|^{m(x)-2}(x,t) \\ +\mu_2 u_t(x,t-\tau)|u_t|^{m(x)-2}(x,t-\tau) = bu\ln|u|. \end{cases}$$

Also, this equation could be studied with $m \equiv m(x,t)$ instead of $m \equiv m(x)$ under different initial and boundary conditions. Furthermore, different mathematical behavior such as blow up, attractor... etc. may be established for the equation (1.1).

Acknowledgments

The authors are grateful to DUBAP (ZGEF.20.009) for research funds.

References

[1] R.A. Adams, *Sobolev spaces, pure and applied mathematics*, vol. 65 Academic Press, Cambridge, 1978.

[2] M.M. Al-Gharabli, A. Guesmia and S.A. Messaoudi, *Well-posedness and asymptotic stability results for a viscoelastic plate equation with a logarithmic nonlinearity*, Appl. Anal. **99** (2020), no. 1, 50–74.

[3] M.M. Al-Gharabli and S.A. Messaoudi, *Existence and a general decay result for a plate equation with nonlinear damping and a logarithmic source term*, J. Evol. Equ. **18** (2018), 105–125.

[4] K. Bartkowski and P. Gorka, *One dimensional Klein-Gordon equation with logarithmic nonlinearities*, J. Phys. A: Math. Theor. **41** (2008), no. 35, 355201.

[5] I. Bialynicki-Birula and J. Mycielski, *Wave equations with logarithmic nonlinearities*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom Phys. **23** (1975), no. 4, 461–466.

[6] I. Bialynicki-Birula and J. Mycielski, *Nonlinear wave mechanics*, Ann. Phys.. **100** (1976), no. 1-2, 62–93.

[7] T. Cazenave and A. Haraux, *Equations d'evolution avec non-linearite logarithmique*, Ann Fac. Sci. Touluse Math. **2** (1980), no. 1, 21–51.

[8] H. Chen, P. Luo and G.W. Liu, *Global solution and blow-up of a semilinear heat equation with logarithmic non linearity*, J. Math. Anal. Appl. **422** (2015), 84–98.

[9] A. Choucha, D. Ouchenane and K. Zennir, *Exponential growth of solution with L_p-norm for class of non-linear viscoelastic wave equation with distributed delay term for large initial data*, Open J. Math. Anal. **3** (2020), no. 1, 76–83.

[10] R. Datko, J. Lagnese and M.P. Polis, *An example on the effect of time delays in boundary feedback stabilization of wave equations*, SIAM J. Control Optim. **24** (1986), no. 1, 152–156.

[11] P. Gorka, *Logarithmic Klein-Gordon equation*, Acta Phys. Polon. B **40** (2009), no. 1, 59–66.

[12] L. Gross, *Logarithmic Sobolev inequalities*, Amer. J. Math. **97** (1975), no. 4, 1061–1083.

[13] X. Han, *Global existence of weak solutions for a logarithmic wave equation arising from q-ball dynamics*, Bull. Korean Math. Soc. **50** (2013), no. 1, 275–283.

[14] T. Hiramatsu, M. Kawasaki and F. Takahashi, *Numerical study of q-ball formation in gravity mediation*, J. Cosmol. Astropart. Phys. **2010** (2010), no. 6, 008.

[15] M. Kafini and S.A. Messaoudi, *A blow-up result in a nonlinear wave equation with delay*, Mediterr. J. Math. **13** (2016), 237–247.

[16] M. Kafini and S.A. Messaoudi, *Local existence and blow up of solutions to a logarithmic nonlinear wave equation with delay*, Appl. Anal. **99** (2020), no. 3, 530–547.

[17] M. Kirane and B.S. Houari, *Existence and asymptotic stability of a viscoelastic wave equation with a delay*, Z. Angew. Math. Phys. **62** (2011), 1065–1082.

[18] J.L. Lions, *Quelques methodes de resolution des problemes aux limites non lineaires*, Dunod, Paris, 1969.

[19] K. Liu, *Locally distributed control and damping for the conservative systems*, SIAM J. Control Optim. **35** (1997), 1574–1590.

[20] G. Liu, *The existence, general decay and blow-up for a plate equation with nonlinear damping and a logarithmic source term*, ERA **28** (2020), no. 1, 263–289.

[21] N. Mezouar, S. Boulaaras and A. Allahem, *Global existence of solutions for the viscoelastic Kirchhoff equation with logarithmic source terms*, Complexity **2020** (2020), 1–25.

[22] S. Nicaise and C. Pignotti, *Stabilization of the wave equation with boundary or internal distributed delay*, Differ. Integral Equ. **21** (2008), no. 9-10, 935–958.

[23] S. Nicaise and C. Pignotti, *Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks*, SIAM J. Control Optim. **45** (2006), no. 5, 1561–1585.

[24] S. Nicaise, J. Valein and E. Fridman, *Stabilization of the heat and the wave equations with boundary time-varying delays*, DCDIS-S, **2** (2009), no. 3, 559–581.

[25] S.H. Park, *Global existence, energy decay and blow-up of solutions for wave equations with time delay and logarithmic source*, Adv. Differ. Equ. **2020** (2020), 631.

[26] E. Pişkin and H. Yüksekkaya, *Local existence and blow up of solutions for a logarithmic nonlinear viscoelastic wave equation with delay*, Comput. Methods Differ. Equ. **9** (2021), no. 2, 623–636.

[27] E. Pişkin and H. Yüksekkaya, *Nonexistence of global solutions of a delayed wave equation with variable-exponents*, Miskolc Math. Notes 22 (2021), no. 2, 841-859.

[28] E. Pişkin, H. Yüksekkaya, *Decay of solutions for a nonlinear Petrovsky equation with delay term and variable exponents*, Aligarh Bull. Math. **39** (2020), no. 2, 63–78.

[29] E. Pişkin and H. Yüksekkaya, *Blow-up of solutions for a logarithmic quasilinear hyperbolic equation with delay term*, J. Math. Anal. **12** (2021), no. 1, 56–64.

[30] E. Pişkin and H. Yüksekkaya, *Blow up of solution for a viscoelastic wave equation with m-Laplacian and delay terms*, Tbil. Math. J. **SI** (2021), no. 7, 21–32.

[31] E. Pişkin and H. Yüksekkaya, *Non-existence of solutions for a Timoshenko equations with weak dissipation*, Math. Morav. **22** (2018), no. 2, 1–9.

[32] Z. Sabbagh, A. Khemmoudj, M. Ferhat and M. Abdelli, *Existence of global solutions and decay estimates for a viscoelastic Petrovsky equation with internal distributed delay*, Rend. Circ. Mat. Palermo Ser. (2) **68** (2019), 477–498.

[33] S.T. Wu, *Asymptotic behavior for a viscoelastic wave equation with a delay term*, Taiwanese J. Math. **17** (2013, no. 3,) 765–784.

[34] S.T. Wu, *Blow-up of solution for a viscoelastic wave equation with delay*, Acta Math. Sci. **39B** (2019), no. 1, 329–338.

[35] Z. Yang, *Existence and energy decay of solutions for the Euler-Bernoulli viscoelastic equation with a delay*, Z. Angew. Math. Phys. **66** (2015), 727–745.

[36] C.Q. Xu, S.P. Yung and L.K. Li, *Stabilization of the wave system with input delay in the boundary control*, ESAIM. Control Optim. Calc. Var. **12** (2006), 770–785.

[37] E. Zuazua, *Exponential decay for the semi-linear wave equation with locally distributed damping*, Commun. Partial Differ. Equ. **15** (1990), 205–235.