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# Double reduction of the Gibbons-Tsarev equation using admitted Lie point symmetries and associated conservation laws

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**Abstract**

In this article, the double reduction method is used to find solutions to a $(1+1)$ nonlinear partial differential equation that arises in the theory of dispersionless integrable systems. Four nontrivial conservation laws of the equation are constructed via the multiplier method, based on a particular form of admitted multipliers. Two of the constructed conservation laws are found to have associated Lie point symmetries and are utilised to construct invariant solutions.

Keywords: Double reduction, Gibbons-Tsarev Equation, Lie symmetry analysis, Conservation law, Invariant solution, Multiplier method.
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## 1 Introduction

The double reduction method [38, 39] is a well-known, but relatively recent, group-theoretical method for efficiently finding invariant solutions of a given partial differential equation (PDE). The method proposed by Sjöberg [38, 39], and recently improved by Anco and Gandarias [5], is applicable in every case where a given PDE admits a Lie point or Lie Bäcklund symmetry with an associated conservation law (in the sense defined later under preliminaries). In the case of a scalar PDE with two independent variables, the double reduction method reduces a PDE of order $q$ to an ordinary differential equation (ODE) of order $q-1$. Bokhari et al [11, 10] have extended the double reduction method to the case of a system of PDEs.

The method of double reduction, when applicable, is particularly attractive because determination of conservation laws of PDEs and admitted symmetries can be achieved easily using systematic routines. For Lagrangian PDE

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systems, the set of admitted Noether symmetries can be shown to lead to the set of all admitted local conservation laws [26, 8, 3, 4]. For PDEs that may not admit Noether symmetries, other approaches have been proposed for constructing conservation laws [1, 2, 24, 40, 25, 3, 4, 26, 20, 15, 16, 36]. Furthermore, establishing which conservation laws have associated symmetries is easily done through the definition of the association [24].

Application of the double reduction method has been reported in the literature (see, for example, [31, 17, 37, 21]). In this paper, we apply double reduction to a second-order nonlinear PDE known as the Gibbons-Tsarev (GT) equation

$$u_{tt} - u_x u_{tx} + u_t u_{xx} - 1 = 0, \tag{1.1}$$

which arises in the theory of dispersionless integrable systems (see, for example, [18]). The equation has attracted interest from many researchers. For example, Kaptsov et al [22, 23] applied the method of differential constraints to find solutions of the GT equation that are expressible in terms of solutions of Painlevé equations. Lelito and Morozov [29], on the other hand, used methods of group analysis of differential equations to find solutions of the GT equation that arise as invariant solutions (in the "classical" way [27, 28, 13, 34, 15, 30]) from Lie point symmetries admitted by the equation. Baran et al [7], using a known Lax pair, constructed an infinite series of conservation laws and the algebra of nonlocal symmetries associated with these conservation laws for the GT equation.

In this paper, we employ the double reduction method and find invariant solutions of the GT equation. The multiplier method is utilised to construct low-order multipliers of the equation of a particular form, namely, polynomial functions of degree at most three in the variables $x$, $t$, $u$ and the first-order derivatives of $u$. This leads to four non-trivial conservation laws of (1.1), two of which have associated Lie point symmetries. Solutions are then constructed systematically, from the two conservation laws and associated Lie point symmetries, according to the double reduction algorithm.

The rest of the paper is organized as follows. In Section 2, basic definitions and results that are relevant to the double reduction method are discussed. The main elements of the method are presented in Section 3. Construction of conservation laws of the GT equation is reported in Section 4. In Section 5, two exact solutions of the GT equation are constructed by the double reduction method. Concluding remarks are given in Section 6.

## 2 Preliminaries

Let us consider a PDE of order q ($q \geq 1$) with $n$ independent variables $x = (x^1, x^2, \dots, x^n)$ and one dependent variable $u = u(x)$,

$$F(x, u, \dots, u_{(q)}) = 0, \tag{2.1}$$

where $u_{(q)}$ denotes the collection $\{u_q\}$ of $q$th-order partial derivatives.

A one parameter Lie group of infinitesimal transformations in $(x, u)$ given by

$$\begin{aligned} \widetilde{x}^i &= x^i + \varepsilon\, \xi^i(x, u) + O(\varepsilon^2), \\ \widetilde{u} &= u + \varepsilon\, \eta(x, u) + O(\varepsilon^2), \end{aligned} \tag{2.2}$$

where $\varepsilon$ is the group parameter, with associated infinitesimal generator

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta(x, u)\frac{\partial}{\partial u}. \tag{2.3}$$

Eq. (2.3) is a symmetry of (2.1) if and only if

$$X^{(q)} F(x, u, \dots, u_{(q)})\big|_{(2.1)} = 0, \tag{2.4}$$

where $X^{(q)}$ is the $q$th extension of (2.3). An $n$-tuple

$$T = \left(T^1(x, u, \dots, u_{(q-1)}), \dots, T^n(x, u, \dots, u_{(q-1)})\right), \tag{2.5}$$

is a conservation vector of (2.1) if

$$D_i T^i = 0 \tag{2.6}$$

on solutions of (2.1). Here $D_i$ is the total derivative operator with respect to $x^i$ defined by

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + \cdots + u_{ii_1i_2\dots i_n}\frac{\partial}{\partial u_{i_1i_2\dots i_n}} + \cdots. \tag{2.7}$$

Equation (2.6) is called a conservation law of (2.1). A function $\Lambda(x, u, u_{(1)}, \ldots)$ for which the equation

$$\Lambda F = D_i T^i \tag{2.8}$$

holds identically for arbitrary functions $u(x)$ is called a multiplier of (2.1). It follows from (2.8) that the conservation law (2.6) holds for all solutions of (2.1). Determining equations for the admitted multipliers are obtained by taking the variational derivative of (2.8) (see, for example, [33]),

$$\frac{\delta}{\delta u}(\Lambda F) = 0, \tag{2.9}$$

which must hold for arbitrary functions $u(x)$, where $\delta/\delta u$ is the Euler operator defined by

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum (-1)^k D_{i_1}, \ldots, D_{i_k} \frac{\partial}{\partial u_{i_1,\ldots,i_k}}. \tag{2.10}$$

A Lie symmetry generator $X$ of the form (2.3) is associated with a conserved vector $T$ of (2.1) if $X$ and $T$ satisfy the relations [24]

$$X\left(T^i\right) + T^i D_k\left(\xi^k\right) - T^k D_k\left(\xi^i\right) = 0, \quad i = 1, \ldots, n. \tag{2.11}$$

## 3 The Double Reduction Method

Consider the PDE (2.1), with $n = 2$ and $x = (x^1, x^2) = (t, x)$, which admits a Lie point symmetry with operator $X$ that is associated with a conservation law

$$D_t T^t + D_x T^x = 0. \tag{3.1}$$

The aim is to find similarity variables $r, s, w$ such that in the new variables $X = \frac{\partial}{\partial s}$, so that (3.1) becomes [38, 39]

$$D_r T^r + D_s T^s = 0, \tag{3.2}$$

with

$$T^r = \frac{T^t D_t(r) + T^x D_x(r)}{D_t(r)D_x(s) - D_x(r)D_t(s)} \tag{3.3}$$

and

$$T^s = \frac{T^t D_t(s) + T^x D_x(s)}{D_t(r)D_x(s) - D_x(r)D_t(s)}. \tag{3.4}$$

The components $T^t$ and $T^x$ depend on $(t, x, u, u_{(1)}, u_{(2)}, \ldots, u_{(q-1)})$. This means $T^r$ and $T^s$ depend on $(r, s, w, w_r, w_{rr}, \ldots, w_{r^{(q-1)}})$ for solutions invariant with respect to $X$. Therefore, the conservation law in canonical variables (3.2) becomes

$$\frac{\partial T^s}{\partial s} + D_r T^r = 0. \tag{3.5}$$

It follows from the association of $X$ with $T$ that, in canonical variables,

$$XT^r \equiv \frac{\partial T^r}{\partial s} = 0 \quad \text{and} \quad XT^s \equiv \frac{\partial T^s}{\partial s} = 0. \tag{3.6}$$

This leads to further reduction of the conservation law (3.5) to

$$D_r T^r = 0, \tag{3.7}$$

or, equivalently, the reduced ODE of order $q - 1$, namely

$$T^r = k, \tag{3.8}$$

where $k$ is an arbitrary constant.

## 4 Conservation laws of the GT equation and associated Lie symmetries

We seek conservation laws of (1.1) via the multiplier approach [33, 3, 32]. Let us consider first-order multipliers of the form

$$\Lambda = \Lambda(t, x, u, u_x, u_t), \tag{4.1}$$

on the equation (1.1). Determining equations for the multipliers can be obtained by taking the variational derivative defined in (2.9),

$$\frac{\delta}{\delta u}\left[\Lambda\left(u_{tt} - u_x u_{tx} + u_t u_{xx} - 1\right)\right] = 0. \tag{4.2}$$

The determining equations that arise from (4.2) are intractable for an arbitrary $\Lambda$. For the purpose of double reduction, however, a particular form of $\Lambda$ would suffice provided that it leads to conservation laws that have symmetries of the GT equation associated with them. In light of this, we assume that $\Lambda$ is a sum of monomials of degree up to three involving the variables $t$, $x$, $u$, $u_t$, $u_x$,

$$\begin{aligned}\Lambda(t,x,u,u_x,u_t) = {} & \delta_0 + \delta_1 x + \delta_2 t + \delta_3 u + \delta_4 u_x + \delta_5 u_t + \delta_6 u^2 + \delta_7 u u_x \\ & + \delta_8 u_x^2 + \delta_9 u u_t + \delta_{10} u_x u_t + \delta_{11} u_t^2 + \delta_{12} x u + \delta_{13} x u_x \\ & + \delta_{14} x u_t + \delta_{15} x^2 + \delta_{16} t u + \delta_{17} t u_x + \delta_{18} t u_t + \delta_{19} t x \\ & + \delta_{20} t^2 + \delta_{21} u^3 + \delta_{22} u^2 u_x + \delta_{23} u u_x^2 + \delta_{24} u_x^3 + \delta_{25} u^2 u_t \\ & + \delta_{26} u u_x u_t + \delta_{27} u_x^2 u_t + \delta_{28} u u_t^2 + \delta_{29} u_x u_t^2 + \delta_{30} u_t^3 \\ & + \delta_{31} x u^2 + \delta_{32} x u u_x + \delta_{33} x u_x^2 + \delta_{34} x u u_t + \delta_{35} x u_x u_t \\ & + \delta_{36} x u_t^2 + \delta_{37} x^2 u + \delta_{38} x^2 u_x + \delta_{39} x^2 u_t + \delta_{40} x^3 + \delta_{41} t u^2 \\ & + \delta_{42} t u u_x + \delta_{43} t u_x^2 + \delta_{44} t u u_t + \delta_{45} t u_x u_t + \delta_{46} t u_t^2 \\ & + \delta_{47} t x u + \delta_{48} t x u_x + \delta_{49} t x u_t + \delta_{50} t x^2 + \delta_{51} t^2 u \\ & + \delta_{52} t^2 u_x + \delta_{53} t^2 u_t + \delta_{54} t^2 x + \delta_{55} t^3,\end{aligned} \tag{4.3}$$

where $\delta_i$ are arbitrary constants. This assumption on the form of admitted multipliers allows the determining equations from (4.2) to be solved easily, and we obtain

$$\Lambda = \delta_0 + \delta_4 u_x + \delta_5\left(\frac{2u_t - 3t - 3u_x^2}{2}\right) + \delta_{24}\left(\frac{4tu_x + 2u_x^3 - 3u_x u_t - x}{2}\right). \tag{4.4}$$

Using the multiplier in (4.4), we determine $T^t$ and $T^x$ such that

$$\Lambda\left(u_{tt} - u_x u_{tx} + u_t u_{xx} - 1\right) = D_t T^t + D_x T^x, \tag{4.5}$$

according to (2.8). We obtain

$$\begin{aligned} T^t \;=\; & \delta_{24}\left(t u_x^3 - 2t u_x u_t + \frac{u_x^5}{4} - u_x^3 u_t - \frac{x u_x^2}{2} + \frac{3u_x u_t^2}{4} + \frac{x u_t}{2}\right) \\ & + \delta_0\left(u_x^2 - u_t\right) + \delta_4\left(\frac{u_x^3}{2} - u_x u_t\right) + \delta_5\biggl(\frac{3}{2}u_t\left(t + u_x^2\right) \\ & - \frac{3t u_x^2}{2} - \frac{u}{2} - \frac{u_x^4}{2} - \frac{u_t^2}{2}\biggr) - \lambda u_x + \psi(x), \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} T^x \;=\; & \delta_{24}\left(2tu - t u_x^2 u_t + t u_t^2 - \frac{u_x^4 u_t}{4} + \frac{3u_x^2 u_t^2}{4} + \frac{x u_x u_t}{2} - \frac{u_t^3}{4} - \frac{x^2}{4}\right) \\ & + \delta_5\left(\frac{1}{2}u_x u_t(3t - 2u_t) - \frac{3tx}{2} + \frac{u_x^3 u_t}{2}\right) + \delta_0\left(x - u_x u_t\right) \\ & + \delta_4\left(u - \frac{u_x^2 u_t}{2} + \frac{u_t^2}{2}\right) - \lambda u_t + \phi(t), \end{aligned} \tag{4.7}$$

for arbitrary functions $\psi$ and $\phi$ and a constant $\lambda$. If $u(t,x)$ is a solution of (1.1), the left hand side of (4.5) vanishes. Therefore, the following non-trivial conserved vectors of (1.1) arise from multipliers of the form (4.3):

$$\begin{aligned} T_1^t &= u_x^2 - u_t, \\ T_1^x &= x - u_x u_t, \end{aligned} \tag{4.8}$$

$$\begin{aligned} T_2^t &= \frac{u_x^3}{2} - u_x u_t, \\ T_2^x &= u - \frac{u_x^2 u_t}{2} + \frac{u_t^2}{2}, \end{aligned} \tag{4.9}$$

$$\begin{aligned} T_3^t &= \frac{3}{2} u_t \left(t + u_x^2\right) - \frac{3tu_x^2}{2} - \frac{u}{2} - \frac{u_x^4}{2} - \frac{u_t^2}{2}, \\ T_3^x &= \frac{1}{2} u_x u_t (3t - 2u_t) - \frac{3tx}{2} + \frac{u_x^3 u_t}{2}, \end{aligned} \tag{4.10}$$

$$\begin{aligned} T_4^t &= tu_x^3 - 2tu_x u_t + \frac{u_x^5}{4} - u_x^3 u_t - \frac{xu_x^2}{2} + \frac{3u_x u_t^2}{4} + \frac{xu_t}{2}, \\ T_4^x &= 2tu - tu_x^2 u_t + tu_t^2 - \frac{u_x^4 u_t}{4} + \frac{3u_x^2 u_t^2}{4} + \frac{xu_x u_t}{2} - \frac{u_t^3}{4} - \frac{x^2}{4}. \end{aligned} \tag{4.11}$$

It is to be noted that a different form of $\Lambda$ in (4.3) might lead to other conservation laws of (1.1).

Next is to establish which of the constructed conservation laws have associated Lie point symmetries of (1.1). Lie point symmetries of (1.1) can be determined from Lie's algorithm [19, 33, 12, 9]. This has been implemented in many computer packages including MathLie [6] and SADE [35].

As did Lu and Zhang [30], we determined that (1.1) admits the following Lie point symmetries:

$$X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_3 = \frac{3}{2} x\partial_x + t\partial_t + 2u\partial_u, \quad X_4 = \partial_u, \quad X_5 = -\frac{1}{2} t\partial_x + x\partial_u. \tag{4.12}$$

A linear combination of these symmetries is

$$X = \Sigma_{i=1}^5 \kappa_i X_i \equiv \xi^t \partial_t + \xi^x \partial_x + \eta\partial_u, \tag{4.13}$$

where $\kappa_i$ are arbitrary constants. Extending $X$ once gives

$$\begin{aligned} X^{(1)} &= (\kappa_2 + \kappa_3 t)\,\partial_t + \left(\kappa_1 + \frac{3\kappa_3 x}{2} - \frac{\kappa_5 t}{2}\right)\partial_x + (2\kappa_3 u + \kappa_4 + \kappa_5 x)\,\partial_u \\ &\quad + \left(\kappa_3 u_t + \frac{\kappa_5 u_x}{2}\right)\partial_{u_t} + \left(\frac{\kappa_3 u_x}{2} + \kappa_5\right)\partial_{u_x}. \end{aligned} \tag{4.14}$$

For each of the constructed conservation vectors $(T_i^t, T_i^x)$, $i = 1, 2, \ldots, 4$, parameters in (4.13) can be determined for which (2.11) is satisfied. That is

$$\begin{aligned} X^{(1)} T^t + T^t D_x \xi^x - T^x D_x \xi^t = 0, \\ X^{(1)} T^x + T^x D_t \xi^t - T^t D_t \xi^x = 0. \end{aligned} \tag{4.15}$$

It turns out that two of the conserved vectors (4.8) and (4.9) have associated Lie point symmetries of the form

$$\Phi_1 = \kappa_1 \partial_x + \kappa_2 \partial_t = \kappa_1 X_1 + \kappa_2 X_2 \tag{4.16}$$

and

$$\Phi_2 = \kappa_2 \partial_t + \kappa_4 \partial_u = \kappa_2 X_2 + \kappa_4 X_4, \tag{4.17}$$

respectively.

## 5 Double reduction of the GT Equation

Canonical variables $r$, $s$ and $w$ are determined as a starting point. These variables transform the generator $\Phi_i$ to its canonical form $\widetilde{\Phi}_i = \partial/\partial s$. This means that they must satisfy the equations

$$\Phi_i r = \Phi_i w = 0, \Phi_i s = 1. \tag{5.1}$$

Clearly $r$ and $w$ are invariants of $\Phi_i$ and are therefore easily obtained from the characteristic equations

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta}. \tag{5.2}$$

The variable $s$ can be determined by inspection. More systematically, it can be obtained from an invariant $J = v - s(x,y)$ of the extended operator $\Phi_i + \partial_v$, where $v$ is an auxiliary variable [33].

### 5.1 Double reduction of (1.1) by $\Phi_1$

The following canonical variables are obtained from $\Phi_1$ :

$$r = \frac{\kappa_2 x - \kappa_1 t}{\kappa_2}, \quad s = \frac{t}{\kappa_2}, \quad w = u, \quad \kappa_2 \neq 0, \tag{5.3}$$

where $w = w(r)$. The inverse canonical coordinates follow from (5.3) and are given by

$$t = \kappa_2 s, \quad x = \kappa_1 s + r, \quad u = w. \tag{5.4}$$

From (5.4) partial derivatives of $u$ in terms of the canonical variables are derived following the routine outlined in [19]. We obtain

$$\begin{aligned} &u_x = w_r, \qquad u_t = -\frac{\kappa_1}{\kappa_2} w_r, \\ &u_{xx} = w_{rr}, \quad u_{tx} = -\frac{\kappa_1}{\kappa_2} w_{rr}, \quad u_{tt} = \frac{\kappa_1^2}{\kappa_2^2} w_{rr}, \quad \kappa_2 \neq 0. \end{aligned} \tag{5.5}$$

By substituting (5.4) and (5.5) into $T_1^t$ and $T_1^x$, and using (3.3) and (3.4), we obtain the following components of the reduced vector:

$$T_1^r = \kappa_2 w - \frac{\kappa_1^2 w_r^2}{2\kappa_2}, \quad T_1^s = -\frac{\kappa_1 w_r^2}{\kappa_2} - \frac{w_r^3}{2}. \tag{5.6}$$

The first component in (5.6), in accordance with (3.8), results in a known first-order nonlinear ODE called Chrystal's equation (see [14] and the references therein)

$$2\kappa_2^2 w - \kappa_1^2 (w')^2 = k, \tag{5.7}$$

where $k$ is an arbitrary constant. Equation (5.7) admits the translation symmetry $\partial_r$, and is thus tractable via Lie symmetry methods for ODEs (see, for example, [14, 9, 19, 33, 12]). The solution is

$$w = \frac{\kappa_2^2 r^2 - 2c_1 \kappa_2 r + c_1^2}{2\kappa_1^2}, \tag{5.8}$$

where $c_1$ is an arbitrary constant. Finally, from (5.4) and (5.8), a solution of the GT equation (1.1) obtained via the double reduction method and using $\Phi_1$ is

$$u(t,x) = \frac{(\kappa_2 x - \kappa_1 t)^2}{2\kappa_1^2} - \frac{c_1(\kappa_2 x - \kappa_1 t)}{\kappa_1^2} + \frac{c_1^2}{2\kappa_1^2}. \tag{5.9}$$

### 5.2 Double deduction of (1.1) by $\Phi_2$

The canonical coordinates in this case are

$$r = x, \quad s = \frac{t}{\kappa_2}, \quad w = \frac{\kappa_2 u - \kappa_4 t}{\kappa_2}, \tag{5.10}$$

where $w = w(r)$. From (5.10) the inverse canonical coordinates are given by

$$t = \kappa_2 s, \quad x = r, \quad u = \kappa_4 s + w. \tag{5.11}$$

It follows from (5.11) that the partial derivatives of $u$ in terms of the canonical variables are given by

$$u_x = w_r, \quad u_t = \frac{\kappa_4}{\kappa_2}, \quad u_{xx} = w_{rr}, \quad u_{tx} = u_{tt} = 0. \tag{5.12}$$

By substituting (5.11) and (5.12) into $T_2^t$ and $T_2^x$, and using (3.3) and (3.4), we obtain components of the reduced vector

$$T^r = \kappa_2 r - \kappa_4 w_r, \quad T^s = \frac{\kappa_4}{\kappa_2} - w_r^2. \tag{5.13}$$

From the first component in (5.13), we deduce, in accordance with (3.8), that the reduced conservation law in this case satisfies the simple ODE

$$\kappa_2 r - \kappa_4 w_r = k, \tag{5.14}$$

where $k$ is an arbitrary constant. The solution of (5.14) is

$$w(r) = \frac{\kappa_2 r^2}{2\kappa_4} + c_1, \tag{5.15}$$

where $c_1$ is an arbitrary constant. The solution of the GT equation (1.1) obtained via the double reduction method and using $\Phi_2$ follows from (5.11) and (5.15):

$$u(t,x) = \frac{\kappa_2^2 x^2 + 2\kappa_4^2 t}{2\kappa_2\kappa_4} + c_1. \tag{5.16}$$

## 6 Concluding remarks

In this paper the double reduction method was used to find exact solutions of the GT equation, an equation that arises in the theory of dispersionless integrable systems. The multiplier method was employed to construct conservation laws of the equation. It is noteworthy that we did not insist on determining all the admitted multipliers for the equation, in which case the determining equations are intractable. Instead, only multipliers (and hence conservation laws) corresponding to a particular form of the multiplier were determined. We determined multipliers for the GT equation that are polynomial functions in the variables $t$, $x$, $u$ and the first derivatives of $u$, and with degree of at most three. Four non-trivial conservation laws were determined, two of which were found to have associated Lie point symmetries. The double reduction method was then invoked in the two cases, leading to first-order ODEs that were easily solved and ultimately yielded two general solutions of the GT equation.

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