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Bipolar $\mathcal{R}$-metric space and fixed point result

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Abstract

In this paper, we introduce the notion of Bipolar $\mathcal{R}$-metric space that in a way enriches the present literature. Here, on associating a binary relation $\mathcal{R}$ with bipolar metric space, we obtain a fixed point result which is well supported with an illustrative example.

Keywords: $\mathcal{R}$-metric space, Bipolar metric space, Bipolar $\mathcal{R}$-metric space, Fixed point.
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1 Introduction

One of the much acclaimed result of the fixed point theory is Banach Contraction Principle [2]. This contraction principle ensures the existence and uniqueness of the fixed point for a self-map on a complete metric space under certain contractive condition and also gives an insight to the iteration process that leads to the fixed point. The Banach Contraction Principle has wide applicability since in many of the cases the solution to a differential equation is comparable to its subsequent fixed point form and because of this, many have succeeded in generalizing and unifying its concept (see [3, 4, 5, 9, 10] and references cited therein).

One such attempt was made by Alam and Imdad in [1], where the authors discussed fixed point result for a self-map on a complete metric space associated with an arbitrary binary relation together with a contraction condition. For more results on relation-theoretic metric space one may see [6, 7] and references cited therein.

In 2016, Mutlu and Gürdal [8] coined the notion of bipolar metric space and proved certain fixed point results in this setting. The idea behind the notion of bipolar metric space was to generalize the distance between points belonging to two different sets.

Through this paper, we introduce the idea of bipolar $\mathcal{R}$-metric space. We also discuss the related topological terms for bipolar $\mathcal{R}$-metric space and obtain certain fixed point result that is supported by an illustration at the end.

2 Preliminaries

In this section, we discuss some definitions used in main section. Throughout the discussion X and Y are two non-empty sets, $\mathcal{R}$ denotes a binary relation on $X \times Y$, $\mathbb{R}^+ = [0, +\infty)$ and $\mathbb{N}$ denotes the set of natural numbers.

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Definition 2.1. [8] For two non-empty sets X and Y and a map $d : X \times Y \to \mathbb{R}^+$ the triplet (X, Y, d) is said to be a bipolar metric space if the following are satisfied:

(i) $d(\rho, \nu) = 0$ if and only if $\rho = \nu$ for all $(\rho, \nu) \in X \times Y$;

(ii) $d(\rho, \nu) = d(\nu, \rho)$ for all $\rho, \nu \in X \cap Y$;

(iii) $d(\rho_1, \nu_2) \leq d(\rho_1, \nu_1) + d(\rho_2, \nu_1) + d(\rho_2, \nu_2)$ for all $\rho_1, \rho_2 \in X$ and $\nu_1, \nu_2 \in Y$.

Definition 2.2. [8] In a bipolar metric space (X, Y, d), where X and Y are non-empty sets:

(i) a point is said to be left, right or central point depending if it belongs to X, Y or $X \cap Y$ respectively,

(ii) a sequence $(\{\rho_n\}, \{\nu_n\})_{n\in\mathbb{N}}$ on the set $X \times Y$ is said to be bisequence on (X, Y, d),

(iii) a bisequence $(\{\rho_n\}, \{\nu_n\})_{n\in\mathbb{N}}$ is said to be convergent if both $\{\rho_n\}_{n\in\mathbb{N}}$ and $\{\nu_n\}_{n\in\mathbb{N}}$ are convergent to respective right and left point. In addition, if both $\{\rho_n\}_{n\in\mathbb{N}}$ and $\{\nu_n\}_{n\in\mathbb{N}}$ converge to the same centre point, then the bisequence $(\{\rho_n\}, \{\nu_n\})_{n\in\mathbb{N}}$ is said to be biconvergent.

3 Main Results

To begin with, we first put forward some of the terminologies used throughout the proof of the main theorem and also of the supportive lemma. At the end of the section, an example is discussed that helps to validate the result proved.

Definition 3.1. Two non-empty sets X, Y together with metric $d : X \times Y \to \mathbb{R}^+$ and binary relation $\mathcal{R} \subseteq X \times Y$ is called Bipolar $\mathcal{R}$-metric space (denoted by $(X, Y, d, \mathcal{R})$) if:

(i) (X, Y, d) is a bipolar metric space, and,

(ii) $\mathcal{R}$ is a binary relation on $X \times Y$.

Definition 3.2. Let $(X, Y, d, \mathcal{R})$ be a bipolar $\mathcal{R}$-metric space. Then:

(i) a bisequence $(\{\rho_n\}, \{\nu_n\})_{n\in\mathbb{N}}$ in $X \times Y$ is said to be an $\mathcal{R}$-bisequence if $(\rho_n, \nu_{n+1}) \in \mathcal{R}$ or $(\rho_{n+1}, \nu_n) \in \mathcal{R}$ for all $n \in \mathbb{N}$,

(ii) an $\mathcal{R}$-bisequence $(\{\rho_n\}, \{\nu_n\})_{n\in\mathbb{N}}$ is said to be convergent $\mathcal{R}$-bisequence if both $\{\rho_n\}_{n\in\mathbb{N}}$ and $\{\nu_n\}_{n\in\mathbb{N}}$ are convergent to respective right and left point,

(iii) an $\mathcal{R}$-bisequence $(\{\rho_n\}, \{\nu_n\})_{n\in\mathbb{N}}$ is said to be biconvergent $\mathcal{R}$-bisequence if both $\{\rho_n\}_{n\in\mathbb{N}}$ and $\{\nu_n\}_{n\in\mathbb{N}}$ are convergent to the same central point,

(iv) a map $g : X \cup Y \to X \cup Y$ is said to be $\mathcal{R}$-continuous at $(\rho, \nu) \in X \times Y$ if for some $\mathcal{R}$-bisequence $(\{\rho_n\}, \{\nu_n\})_{n\in\mathbb{N}}$ in $X \times Y$ with $\rho_n \to \nu\ (\in Y)$ and $\nu_n \to \rho\ (\in X)$ as $n \to +\infty$ then $g\rho_n \to g\nu$ and $g\nu_n \to g\rho$ as $n \to +\infty$,

(v) a map $g : X \cup Y \to X \cup Y$ is said to be bipolar $\mathcal{R}$-continuous if for any convergent $\mathcal{R}$-bisequence $(\{\rho_n\}, \{\nu_n\})_{n\in\mathbb{N}}$ in $X \times Y$ such that

$$\begin{aligned} &\rho_n \to \nu \quad \text{and} \quad \nu_n \to \rho \quad \text{as} \quad n \to +\infty, \\ \text{implies,} \quad &g\rho_n \to g\nu \quad \text{and} \quad g\nu_n \to g\rho \quad \text{as} \quad n \to +\infty, \end{aligned}$$

(vi) $(X, Y, d, \mathcal{R})$ is said to be complete bipolar $\mathcal{R}$-metric space if every Cauchy $\mathcal{R}$-bisequence is convergent $\mathcal{R}$-bisequence.

Lemma 3.3. In a bipolar $\mathcal{R}$-metric space, every convergent Cauchy $\mathcal{R}$-bisequence is biconvergent $\mathcal{R}$-bisequence.

Proof: Let $(\{\rho_n\}, \{\nu_n\})_{n\in\mathbb{N}}$ be a convergent Cauchy $\mathcal{R}$-bisequence, that is, there exist some bisequence $(\{\rho_n\}, \{\nu_n\})_{n\in\mathbb{N}} \subset X\times Y$ that $\rho_n \to \nu$ (in Y) and $\nu_n \to \rho$ (in X) as $n \to +\infty$. Let $n, n_0, m \in \mathbb{N}$ where $n, m > n_0$, then

$$d(\rho, \nu) \le d(\rho, \nu_m) + d(\rho_n, \nu) + d(\rho_n, \nu_m).$$

Taking limit as $n, m \to +\infty$, we get $\rho = \nu$. Hence $(\{\rho_n\}, \{\nu_n\})_{n\in\mathbb{N}}$ is biconvergent $\mathcal{R}$-bisequence.

Theorem 3.4. Let $(X, Y, d, \mathcal{R})$ be a complete bipolar $\mathcal{R}$-metric space with $g : X \cup Y \to X \cup Y$ a map such that:

(I) $g(X) \subseteq X$ and $g(Y) \subseteq Y$;

(II) there exists $\lambda \in (0, 1)$ such that $d(g\rho, g\nu) \le \lambda d(\rho, \nu)$ for all $\rho, \nu \in \mathcal{R}$;

(III) there exists some $(\rho_0, \nu_0) \in X \times Y$ such that $(\rho_0, \nu_0) \in \mathcal{R}$ and $(\rho_0, g\nu_0) \in \mathcal{R}$;

(IV) g is bipolar $\mathcal{R}$-continuous;

(V) for each $(\rho, \nu) \in \mathcal{R}$, we have $(g\rho, g\nu) \in \mathcal{R}$.

Then, g possesses at least one fixed point.

Proof. Define the bisequence $(\{\rho_n\}, \{\nu_n\})_{n\in\mathbb{N}\cup\{0\}}$ in $X \times Y$ such that

$$g\rho_{n-1} = \rho_n \quad \text{and} \quad g\nu_{n-1} = \nu_n \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

By condition *(III)*, we obtain that there exist some $(\rho_0, \nu_0) \in X\times Y$ such that $(\rho_0, \nu_0) \in \mathcal{R}$ and $(\rho_0, \nu_1) = (\rho_0, g\nu_0) \in \mathcal{R}$. On using condition *(V)*, we have

$$(g\rho_0, g\nu_0) = (\rho_1, \nu_1) \in \mathcal{R} \quad \text{and} \quad (g\rho_0, g\nu_1) = (\rho_1, \nu_2) \in \mathcal{R},$$

continuing this process, we get $(\rho_n, \nu_n) \in \mathcal{R}$ and $(\rho_n, \nu_{n+1}) \in \mathcal{R}$, for all $n \in \mathbb{N} \cup \{0\}$. Thus $(\{\rho_n\}, \{\nu_n\})_{n\in\mathbb{N}\cup\{0\}}$ is an $\mathcal{R}$-bisequence. Now from condition *(II)*, we obtain

$$d(\rho_{n+1}, \nu_{n+1}) = d(g\rho_n, g\nu_n) \le \lambda d(\rho_n, \nu_n) \le ... \le \lambda^{n+1} d(\rho_0, \nu_0).$$

Furthermore,

$$d(\rho_n, \nu_{n+1}) = d(g\rho_{n-1}, g\nu_n) \le \lambda d(\rho_{n-1}, \nu_n) \le ... \le \lambda^n d(\rho_0, \nu_1).$$

Next, for some $n, m \in \mathbb{N} \cup \{0\}$ with $m > n$, we obtain

$$\begin{aligned} d(\rho_n, \nu_m) &\le d(\rho_n, \nu_{n+1}) + d(\rho_{n+1}, \nu_{n+1}) + d(\rho_{n+1}, \nu_m) \\ &\le d(\rho_n, \nu_{n+1}) + d(\rho_{n+1}, \nu_{n+1}) + d(\rho_{n+1}, \nu_{n+2}) + d(\rho_{n+2}, \nu_{n+2}) + d(\rho_{n+2}, \nu_m), \\ &\le (d(\rho_{n+1}, \nu_{n+1}) + d(\rho_{n+2}, \nu_{n+2}) + ... + d(\rho_{m-1}, \nu_{m-1})) \\ &\quad +(d(\rho_n, \nu_{n+1}) + d(\rho_{n+1}, \nu_{n+2}) + ... + d(\rho_{m-1}, \nu_m)), \\ &\le \sum_{k=n}^{m-2} d(\rho_{k+1}, \nu_{k+1}) + \sum_{k=n}^{m-1} d(\rho_k, \nu_{k+1}), \\ &\le \sum_{k=n}^{+\infty} \lambda^{k+1} d(\rho_0, \nu_0) + \sum_{k=n}^{+\infty} \lambda^k d(\rho_0, \nu_1), \\ &= \frac{\lambda^{n+2} d(\rho_0, \nu_0)}{1-\lambda} + \frac{\lambda^{n+1} d(\rho_0, \nu_1)}{1-\lambda} \to 0 \quad \text{as} \quad n \to +\infty. \end{aligned}$$

Thus $(\{\rho_n\}, \{\nu_n\})_{n\in\mathbb{N}\cup\{0\}}$ is a Cauchy $\mathcal{R}$-bisequence and since $(X, Y, d, \mathcal{R})$ is a complete bipolar $\mathcal{R}$-metric space, $(\{\rho_n\}, \{\nu_n\})_{n\in\mathbb{N}\cup\{0\}}$ is convergent $\mathcal{R}$-bisequence.
By Lemma 3.3, there exists $\eta \in X \cap Y$ such that

$$\rho_n \to \eta \quad \text{and} \quad \nu_n \to \eta \quad \text{as} \quad n \to +\infty.$$

As g is an $\mathcal{R}$-continuous, so we have

$$\begin{aligned} &\lim_{n\to+\infty} g\rho_n = g\eta \qquad \text{and} \qquad \lim_{n\to+\infty} g\nu_n = g\eta, \\ \text{that is,} \quad &\lim_{n\to+\infty} \rho_{n+1} = g\eta \qquad \text{and} \qquad \lim_{n\to+\infty} \nu_{n+1} = g\eta, \\ &\text{then,} \quad \eta = g\eta. \end{aligned}$$

Thus, g possesses at least one fixed point.

Example 3.5. Let $X = [0, 1/2]$ and $Y = [-1/2, 0]$ where for $(\rho, \nu) \in X \times Y$ we define $d(\rho, \nu) = |\rho - \nu|$. Define a binary relation $\mathcal{R}$ on $X \times Y$ as $(\rho, \nu) \in \mathcal{R}$ if and only if $\rho.\nu = 0$.
Define $g : X \cup Y \to X \cup Y$ as:

$$g(\rho) = \begin{cases} \dfrac{29\rho}{73} & for\ \rho \in [0, 1/2]; \\ \dfrac{-\rho^2}{6} & for\ \rho \in [-1/2, 0). \end{cases}$$

Clearly, $g(X) \subseteq X$ and $g(Y) \subseteq Y$. For $(\rho, \nu) \in \mathcal{R}$ we have $(g\rho, g\nu) \in \mathcal{R}$. Also, g is a bipolar $\mathcal{R}$-continuous, since for any convergent $\mathcal{R}$-bisequence $(\{\rho_n\}, \{\nu_n\})_{n\in\mathbb{N}} \in X \times Y$, we have $\rho_n \to 0$ and $\nu_n \to 0$ as $n \to +\infty$ then $g\rho_n \to g0 = 0$ and $g\nu_n \to g0 = 0$ as $n \to +\infty$.
Next, we verify condition (II) of Theorem 3.4. For $(\rho, \nu) \in \mathcal{R}$ either $\rho = 0$ and/or $\nu = 0$, therefore we have following cases:
Case I: If $\rho = 0$ and $\nu \in [-1/2, 0)$ and for $\lambda = 1/11$, we have

$$d(g\rho, g\nu) \leq \lambda d(\rho, \nu).$$

Case II: If $\rho \in (0, 1/2]$ and $\nu = 0$ and for $\lambda > 29/73$, we have

$$d(g\rho, g\nu) \leq \lambda d(\rho, \nu).$$

Since all the hypothesis of Theorem 3.4 holds, g possesses a fixed point viz. 0.

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