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# On vector variational-like inequality and vector optimization problem with $(G, \alpha)$-univexity

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**Abstract**

In this paper, we introduce and study $(G, \alpha)$-univex functions by generalizing the $\alpha$-univex functions and establish the relationships between vector variational-like inequality problems and vector optimization problems. Furthermore, we formulate equivalence among the vector critical points, weak efficient points of vector optimization problems and the solution of weak vector variational-like inequality problems under pseudo $(G, \alpha)$-univexity assumptions. An example is also constructed to validate the main result.

Keywords: Pseudo-univexity, vector optimization, vector critical points, vector efficient points
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## 1 Introduction

In 1980, Giannessi [5] introduced the notion of vector variational inequality in finite dimensional Euclidean spaces, which have been generalized by numerous authors in various ways, see for example [1, 7, 9, 10, 12, 13, 16, 18, 20] and the references therein. The concept of pre-univex functions, univex functions, and pseudo-univex functions was introduced by Bector et al. [4], as a generalization of invex functions [6]. More information on the applications of univex and generalized univex functions can be found in [14, 15, 17, 19] and the references therein.

A new class of functions called $\alpha-$ invex functions was introduced by Noor [21], as a generalization of invex functions and studied some properties of $\alpha$-preinvex ($\alpha$-invex) functions. Ruiz-Garzon et al. [23] established relationship between vector variational-like inequality and optimization problems based on the concept of pseudo-invexity. Mishra et al. [19] proposed a new notion of $\alpha$-pseudo-univex function, which is a generalized convex function that combines the concepts of $\alpha$-invex functions and pseudo-univex. Furthermore, various relationships are established between vector variational-like inequalities and vector optimization problem under the assumptions of $\alpha$-pseudo-univex functions. Under the assumption of smooth (G,$\alpha$)-invex functions, the solution properties of the vector optimization problem, the Minty vector variational-like inequality problem, and the Stampacchia vector variational-like inequality problem were examined by Jayswal and Choudhury [8]. Li and Yu [24] introduced a class of generalized invex functions called $(\alpha, \rho, \eta)$-invex functions and established the connection between two types of vector variational-like inequalities and multiobjective programming problem. Pooja et al. [11], found some links between approximate convexity and generalized approximate convexity and established relationships between vector variational inequalities and nonsmooth vector optimization problems.

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Motivated and inspired by the work of Mishra et al. [19], and Jayswal and Choudhury [8], we introduce a new type of functions namely $(G, \alpha)$-univex functions and establish the relationship between vector variational-like inequality problems and vector optimization problems. We also establish relationship between the vector critical points and weak efficient solutions.

This paper is organized as follows: In section 2, we review several definitions and results that will be used in latter sections. In section 3, we establish relationship between vector variational-like inequality and vector optimization problems by using $(G, \alpha)$-univex function.

## 2 Preliminaries

The following stipulation for equalities and inequalities will be used throughout this paper. If $x, y \in \mathbb{R}^n$, then
$x \leq y \;\Leftrightarrow\; x_i \leq y_i,\; i = 1, 2, 3, ..., n$ with strict inequality holding for at least one i;
$x \leqq y \;\Leftrightarrow\; x_i \leq y_i,\; i = 1, 2, 3, ..., n$;
$x = y \;\Leftrightarrow\; x_i = y_i,\; i = 1, 2, 3, ..., n$;
$x < y \;\Leftrightarrow\; x_i < y_i,\; i = 1, 2, 3, ..., n$.

Suppose $X \subseteq \mathbb{R}^n$ be a nonempty set, $\eta : X \times X \to \mathbb{R}^n$ be a continuous map, $\alpha : X \times X \to \mathbb{R}_+ \setminus \{0\}$ be a bifunction. Let $b : X \times X \to \mathbb{R}_+$ and $\phi : \mathbb{R} \to \mathbb{R}$ be two functions.

First of all, we recall some definitions, known results and concepts which will be used in the sequel.

**Definition 2.1.** [21] A subset $X$ of $\mathbb{R}^n$ is said to be $\alpha$-*invex* set with respect to $\eta$ and $\alpha$, if for all $x, u \in X$, $t \in [0, 1]$, we have $x + t\alpha(u, x)\eta(u, x) \in X$.

It is well-known that $\alpha$-invex set may not be convex set [21]. From now onward, we suppose that $X \subseteq \mathbb{R}^n$ be nonempty, open and $\alpha$-invex set, unless otherwise specified.

**Definition 2.2.** [2] A function $G : \mathbb{R} \to \mathbb{R}$ is said to be increasing, if

$$x < y \;\Leftrightarrow\; G(x) < G(y),\; \forall\, x, y \in \mathbb{R}.$$

**Lemma 2.3.** [2] $G^{-1}$ is an increasing function iff $G$ is an increasing function.

Now, we introduce the concept of $(G, \alpha)$-univex function by generalizing $\alpha$-univex function introduced and studied by Mishra et al. [19] as follows:

**Definition 2.4.** Suppose $X \subseteq \mathbb{R}^n$ be a nonempty set and $G : \mathbb{R} \to \mathbb{R}$ be a continuous differentiable real valued increasing function. The differentiable function $f : X \to \mathbb{R}^p$ with $p \times n$ matrix as its Jacobian is said to be :

(i) $(G, \alpha)$-*univex with respect to* $\alpha,\ \eta,\ \phi$ *and* $b$, if

$$b(x, u)\ \phi[G(f_i(x)) - G(f_i(u))] \geq \langle G'(f_i(u))\alpha(x, u)\nabla f(u), \eta(x, u)\rangle\,,$$

$\forall\, x, u \in X,\; i = 1, 2, ..., p,$

(ii) *strictly* $(G, \alpha)$-*univex with respect to* $\alpha,\ \eta,\ \phi$ *and* $b$, if

$$b(x, u)\ \phi[G(f_i(x)) - G(f_i(u))] > \langle G'(f_i(u))\alpha(x, u)\nabla f(u), \eta(x, u)\rangle\,,$$

$\forall\, x, u \in X,\; x \neq y,\; i = 1, 2, ..., p,$

(iii) *pseudo* $(G, \alpha)$-*univex with respect to* $\alpha,\ \eta,\ \phi$ *and* $b$, if

$$b(x, u)\ \phi[G(f_i(x)) - G(f_i(u))] < 0 \;\Rightarrow\; \langle G'(f_i(u))\alpha(x, u)\nabla f(u), \eta(x, u)\rangle < 0,$$

$\forall\, x, u \in X,\; i = 1, 2, ..., p.$

**Remark 2.5.** If $G(x) = x$, then $(G, \alpha)$-univex function reduces to $\alpha$-univex function.

Given $X \subseteq \mathbb{R}^n$ and a function $f : X \to \mathbb{R}^p$, the vector optimization problem (for short, $VOP$) is given as follows:

$$\text{Min } \{f(x)\} \text{ such that } x \in X.$$

In vector optimization problems, objectives often conflict with each other. Consequently, the concept of optimality for single-objective optimization problems cannot be applied directly to vector optimization. In this regard, the concept of efficient solutions is more useful for vector optimization problems.

**Definition 2.6.** [20] Given an open set $X \subseteq \mathbb{R}^n$ and a function $f : X \to \mathbb{R}^p$. A point $z \in X$ is said to be *efficient (Pareto)*, iff there exists no $y \in X$ such that $f(y) \leq f(z)$.

**Definition 2.7.** [20] Given an open set $X \subseteq \mathbb{R}^n$ and a function $f : X \to \mathbb{R}^p$. A point $z \in X$ is said to be *weakly efficient (Pareto)*, iff there exists no $y \in X$ such that $f(y) < f(z)$.

The vector variational-like inequality problem is a generalized form of the vector variational inequality problem, which was introduced and studied by Siddiqi et al. [25] and Yang [26].

Suppose $X \subseteq \mathbb{R}^n$ be a nonempty set and $F : X \to \mathbb{R}^p$ be a function. The *variational-like inequality problem* (for short, $VLIP$), is to find a point $y \in X$ such that

$$\langle F(y), \eta(x, y)\rangle \geq 0, \ \forall \ x \in X.$$

A *vector variational-like inequality problem* (for short, $VVLIP$), is to find a point $y \in X$, there exists no $x \in X$ such that $\langle F(y), \eta(x, y)\rangle \leq 0$.

A *weak vector variational-like inequality problem* (for short, $WVVLIP$), is to find a point $y \in X$, there exists no $x \in X$ such that $\langle F(y), \eta(x, y)\rangle < 0$.

In general, for each $x, y \in X, \ \alpha(x, y) > 0$,

$$\langle \alpha(x, y) \ F(y), \eta(x, y)\rangle \leq 0 \Leftrightarrow \langle F(y), \eta(x, y)\rangle \leq 0,$$

and

$$\langle \alpha(x, y) \ F(y), \eta(x, y)\rangle < 0 \Leftrightarrow \langle F(y), \eta(x, y)\rangle < 0.$$

## 3 Relationship between $VVLIP$ and $VOP$

In this section, we shall extend the results in [19] from $\alpha$-univex functions to $(G, \alpha)$-univex functions.

**Theorem 3.1.** Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^p$ be a differentiable function. If $F = \nabla f$, $f$ is $(G, \alpha)$-univex with respect to $\alpha, \ \eta, \ \phi$ and $b$ with $\phi(t) \leq 0$, whenever $t \leq 0$.
If $y \in X$ solves the $VVLIP$ with respect to same $\alpha, \ \eta, \ \phi$ and $b$, and $G'(f_i(y)) > 0, \ i = 1, 2, ..., p$, then $y$ is an efficient solution to the $VOP$.

**Proof .** Suppose that $y$ is not an efficient solution to the $VOP$. Then there exist $x \in X$ such that $f(x) - f(y) \leq 0$. As $G$ is an increasing function, we have

$$G(f_i(x)) - G(f_i(y)) \leq 0, \ i = 1, 2, ..., p.$$

Since $\phi(t) \leq 0$, whenever $t \leq 0$ and $b : X \times X \to \mathbb{R}_+$, we have

$$b(x, y) \ \phi[G(f_i(x)) - G(f_i(y))] \leq 0, \ i = 1, 2, ..., p.$$

Using $(G, \alpha)$-univexity of $f$ with respect to $\alpha, \ \eta, \ \phi$ and $b$. It follows that there exist $x \in X$ such that

$$\langle G'(f_i(y))\alpha(x, y)\nabla f(y), \eta(x, y)\rangle \leq 0, \ i = 1, 2, ..., p.$$

Since $G'(f_i(y)) > 0, \ i = 1, 2, ..., p$ and $\alpha(x, y) > 0$, there exists $x \in X$ such that $\langle \nabla f(y), \eta(x, y)\rangle \leq 0$, which means that $y$ is not a solution of $VVLIP$, which is a contradiction. Hence $y$ must be an efficient solution to the $VOP$. □

The validity of the above theorem has been shown in the following example.

**Example 3.2.** Let $X = \{x = (x_1, y_1) \in \mathbb{R}^2 : x_1 \geq 1, y_1 \geq 1 \text{ and } x_1 \geq y_1\}$. Consider the following $VOP$ :

$$\text{Minimize } \; f(x = (x_1, y_1)) = log\left(\frac{x_1^2}{y_1^2}\right), \quad \text{subject to } x \in X,$$

where $f : X \to \mathbb{R}$ be a differentiable function. Suppose $x = (x_1, y_1), y = (x_2, y_2) \in X$, we define $G(t) = \sqrt{e^t},\ \alpha(x, y) = \frac{6y_2}{y_1},\ \eta(x, y) = (x_1 - x_2, y_1 - y_2),\ b(x, y) = 2$ and $\phi(t) = 3t$. Then $f$ is $(G, \alpha)$ univex function with respect to $\alpha,\ \eta,\ \phi$ and $b$. Here, $G'(t) = \frac{\sqrt{e^t}}{2},\ G'(f(y)) = \left(\frac{x_2}{2y_2}\right)$ and $\nabla f(y) = 2\left(\frac{1}{x_2}, -\frac{1}{y_2}\right)$. Now,

$$\begin{aligned}
& b(x,y)\ \phi[G(f(x)) - G(f(y))] - \langle G'(f(y))\alpha(x,y)\nabla f(y), \eta(x,y)\rangle \\
= \;& 6\left[\left(\frac{x_1}{y_1}\right) - \left(\frac{x_2}{y_2}\right)\right] - \left\langle \left(\frac{x_2}{2y_2}\right)\left(\frac{6y_2}{y_1}\right) 2\left(\frac{1}{x_2}, -\frac{1}{y_2}\right), (x_1 - x_2, y_1 - y_2)\right\rangle \\
= \;& 6\left[\frac{x_1y_2 - y_1x_2}{y_1y_2} - \left(\frac{x_1y_2 - y_1x_2}{y_1y_2}\right)\right] \\
= 0.&
\end{aligned}$$

We observe that $y = (1, 1)$ solves the $VVLIP$,

$$\begin{aligned}
\langle \nabla f(y), \eta(x,y)\rangle &= \left\langle 2\left(\frac{1}{x_2}, -\frac{1}{y_2}\right), (x_1 - x_2, y_1 - y_2)\right\rangle \\
&= 2(x_1 - y_1) \geq 0,\ \forall\ x = (x_1, y_1) \in X.
\end{aligned}$$

Hence by Theorem 3.1, $y = (1, 1)$ is an efficient to the $VOP$.

**Theorem 3.3.** Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^p$ be a differentiable function. If $F = \nabla f,\ -f$ is strictly $(G, \alpha)$-univex function with respect to $\alpha,\ \eta,\ \phi$ and $b$ with $\phi(t) \leq 0$, whenever $t \leq 0$. If $y \in X$ is a weak efficient solution to the $VOP$ and $G'(f_i(y)) > 0,\ i = 1, 2, ..., p$ , then $y$ solves the $VVLIP$ with respect to same $\alpha,\ \eta,\ \phi$ and $b$.

**Proof .** Suppose that $y$ does not solve $VVLIP$. Then there exist $x \in X$ such that

$$\langle \alpha(x,y)\nabla f(y), \eta(x,y)\rangle \leq 0.$$

Using the strict $(G, \alpha)$-univexity of the function $-f$ with respect to $\alpha,\ \eta,\ \phi$ and $b$ and $G'(f_i(y)) > 0,\ i = 1, 2, ..., p$, we have

$$b(x,u)\ \phi[G(f_i(x)) - G(f_i(y))] < \langle G'(f_i(y))\alpha(x,y)\nabla f(y), \eta(x,y)\rangle \leq 0,\ i = 1, 2, ..., p.$$

Therefore, there exists $x \in X$ such that

$$b(x,y)\ \phi[G(f_i(x)) - G(f_i(y))] < 0,\ i = 1, 2, ..., p.$$

Since $\phi(t) \leq 0$, whenever $t \leq 0$ and $b : X \times X \to \mathbb{R}_+$, we have

$$G(f_i(x)) - G(f_i(y)) < 0,\ i = 1, 2, ..., p.$$

As $G^{-1}$ is an increasing function, we have $f(x) - f(y) < 0$, which contradicts $y \in X$ being a weakly efficient solution of $VOP$. Hence $y$ must be a solution of $VVLIP$. □

As every efficient solution is also a weakly efficient solution to the $VOP$, so the following result is trivial to prove:

**Corollary 3.4.** Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^p$ be a differentiable function. If $F = \nabla f,\ -f$ is strictly $(G, \alpha)$-univex function with respect to $\alpha,\ \eta,\ \phi$ and $b$ with $\phi(t) \leq 0$, whenever $t \leq 0$. If $y \in X$ is an efficient solution to the $VOP$ and $G'(f_i(y)) > 0,\ i = 1, 2, ..., p$, then $y$ also solves the $VVLIP$ with respect to same $\alpha,\ \eta,\ \phi$ and $b$.

**Theorem 3.5.** Suppose $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^p$ be a differentiable function. If $F = \nabla f,\ f$ is pseudo $(G, \alpha)$-univex function with respect to $\alpha,\ \eta,\ \phi$ and $b$ with $\phi(t) \leq 0$, whenever $t \leq 0$.
If $y \in X$ solves the $WVVLIP$ with respect to same $\alpha,\ \eta,\ \phi$ and $b$ and $G'(f_i(y)) > 0,\ i = 1, 2, ..., p$, then $y$ is a weakly efficient solution to the $VOP$.

**Proof .** Suppose that $y$ is not a weakly efficient solution to the $VOP$. Then there exist $x \in X$ such that $f(x) - f(y) < 0$. Since $G$ is an increasing function, we have

$$G(f_i(x)) - G(f_i(y)) < 0, \ i = 1, 2, ..., p.$$

Also, since $\phi(t) \leq 0$, whenever $t \leq 0$ and $b : X \times X \to \mathbb{R}_+$, we have

$$b(x, y) \ \phi[G(f_i(x)) - G(f_i(y))] < 0, \ i = 1, 2, ..., p.$$

By using pseudo $(G, \alpha)$-univexity of $f$ with respect to $\alpha, \ \eta, \ \phi$ and $b$, there exist $x \in X$ such that

$$\langle G'(f_i(y))\alpha(x, y)\nabla f(y), \eta(x, y)\rangle < 0, \ i = 1, 2, ..., p.$$

Given that $G'(f_i(y)) > 0, \ i = 1, 2, ..., p$ and $\alpha(x, y) > 0$, it follows that there exists $x \in X$ such that $\langle \nabla f(y), \eta(x, y)\rangle < 0$, which means that $y$ is not a solution to the $WVVLIP$, which is a contradiction. □

**Theorem 3.6.** Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^p$ be a differentiable function. If $F = \nabla f, \ f$ is strictly $(G, \alpha)$-univex function with respect to $\alpha, \ \eta, \ \phi$ and $b$ with $\phi(t) \leq 0$, whenever $t \leq 0$. If $y \in X$ is a weak efficient solution of $VOP$ and $G'(f_i(y)) > 0, \ i = 1, 2, ..., p$, then $y \in X$ is an efficient solution to the $VOP$.

**Proof .** Suppose that $y$ is a weakly efficient solution to the $VOP$, but not an efficient solution to $VOP$. Then there exist $x \in X$ such that $f(x) - f(y) \leq 0$. As $G$ is an increasing function, so we have

$$G(f_i(x)) - G(f_i(y)) \leq 0, \ i = 1, 2, ..., p.$$

Since $\phi(t) \leq 0$, whenever $t \leq 0$ and $b : X \times X \to \mathbb{R}_+$, we have

$$b(x, y) \ \phi[G(f_i(x)) - G(f_i(y))] \leq 0, \ i = 1, 2, ..., p.$$

By the strict $(G, \alpha)$-univexity of the function $f$ with respect to $\alpha, \ \eta, \ \phi$ and $b$, we have

$$0 \geq b(x, y) \ \phi[G(f_i(x)) - G(f_i(y))] > \langle G'(f_i(y))\nabla f(y), \eta(x, y)\rangle, \ i = 1, 2, ..., p.$$

Thus, there exists $x \in X$ such that $\langle G'(f_i(y))\alpha(x, y)\nabla f(y), \eta(x, y)\rangle < 0, \ i = 1, 2, ..., p$. Since $G'(f_i(y)) > 0, \ i = 1, 2, ..., p$ and $\alpha(x, y) > 0$, there exists $x \in X$ such that $\langle \nabla f(y), \eta(x, y)\rangle < 0$. Therefore, $y$ does not solve $WVVLIP$. Thus, by Theorem 3.5 , we get a contradiction. Hence $y$ must be an efficient solution to the $VOP$. □

Now, we need the following definition and Lemma:

**Definition 3.7.** [22] A feasible solution $y \in X$ is said to be a *vectorial critical point* to the $VOP$, if there exists a vector $\lambda \in \mathbb{R}^p$ with $\lambda \geq 0$ such that $\langle \lambda, \nabla f(y)\rangle = 0$.

**Lemma 3.8.** (Gordan's Theorem) [3] If $A$ is $n \times m$ matrix, then we have either

(i) $Ax < 0$, for some $x \in \mathbb{R}^m$, or

(ii) $\langle A, y\rangle = 0, \ y \geq 0$, for some nonzero solution $y \in \mathbb{R}^n$,

but not both.

**Theorem 3.9.** Suppose $y \in X$ be a vector critical point to the $VOP$ and $G'(f_i(y)) > 0, \ i = 1, 2, ..., p$. Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^p$ be a differentiable function. If $F = \nabla f, \ f$ is pseudo $(G, \alpha)$-univex on $X$ with respect to $\alpha, \ \eta, \ \phi$ and $b$ with $\phi(t) \leq 0$, whenever $t \leq 0$. Then $y \in X$ is a weakly efficient solution to the $VOP$.

**Proof .** Suppose $y \in X$ be a vector critical point to the $VOP$, then there exists $\lambda \in \mathbb{R}^p$ with $\lambda \geq 0$ such that

$$\langle \lambda, \nabla f(y)\rangle = 0.$$

Suppose on the contrary that $y$ is not an efficient solution to the $VOP$. Then there exist $x \in X$ such that $f(x) - f(y) \leq 0$. As $G$ is an increasing function, we have

$$G(f_i(x)) - G(f_i(y)) \leq 0, \;\; i = 1, 2, ..., p.$$

Since $\phi(t) \leq 0$, whenever $t \leq 0$ and $b : X \times X \to \mathbb{R}_+$, we have

$$b(x, y) \; \phi[G(f_i(x)) - G(f_i(y))] \leq 0, \;\; i = 1, 2, ..., p.$$

Using pseudo $(G, \alpha)$-univexity of $f$ with respect to $\alpha, \; \eta, \; \phi$ and $b$, there exist $x \in X$ such that

$$\langle G'(f_i(y))\alpha(x, y)\nabla f(y), \eta(x, y)\rangle < 0, \;\; i = 1, 2, ..., p.$$

Since $G'(f_i(y)) > 0, \;\; i = 1, 2, ..., p$ and $\alpha(x, y) > 0$, there exists $x \in X$ such that $\langle \nabla f(y), \eta(x, y)\rangle < 0$. Applying the Gordan's theorem, we deduce that there is no $\lambda \geq 0$ such that

$$\langle \lambda, \nabla f(y)\rangle = 0,$$

which is a contradiction to the fact that $y$ is a vector critical point.

□

**Theorem 3.10.** Suppose $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^p$ be a differentiable function and $F = \nabla f$. Every vector critical points is a weakly efficient solutions to the $VOP$ iff $f$ is pseudo $(G, \alpha)$-univex on $X$ with respect to $\alpha, \; \eta, \; \phi$ and $b$ with $\phi(t) \leq 0$, whenever $t \leq 0$ and $G'(f_i(w)) > 0, \;\; i = 1, 2, ..., p$, where $w$ be any vector critical point to the $VOP$.

**Proof .** The sufficient condition can be shown from Lemma 2.1 [22]. Next, we prove the necessary condition, that is, if every vector critical point is a weak efficient solution, then $f$ is pseudo $(G, \alpha)$-univex with respect to $\alpha, \; \eta, \; \phi$ and $b$ with $\phi(t) \leq 0$, whenever $t \leq 0$. Suppose $y \in X$ be a weak efficient solution to the $VOP$. Then there is no $x \in X$ such that $f(x) - f(y) < 0$. As $G$ is an increasing function, we have

$$G(f_i(x)) - G(f_i(y)) < 0, \;\; i = 1, 2, ..., p.$$

Since $\phi(t) \leq 0$, whenever $t \leq 0$ and $b : X \times X \to \mathbb{R}_+$, we have

$$b(x, y) \; \phi[G(f_i(x)) - G(f_i(y))] < 0, \;\; i = 1, 2, ..., p.$$

If $y \in X$ is a vector critical point to the $VOP$, then there exists a vector $\lambda \in \mathbb{R}^p$ with $\lambda \geq 0$ such that $\langle \lambda, \nabla f(y)\rangle = 0$. From the Gordan Theorem, there is no $\eta$ such that $\langle \nabla f(y), \eta(x, y)\rangle \leq 0$. Since $G'(f_i(w)) > 0, \;\; i = 1, 2, ..., p$ and $\alpha(x, y) > 0$, we have

$$\langle G'(f_i(y))\alpha(x, y)\nabla f(y), \eta(x, y)\rangle < 0, \;\; i = 1, 2, ..., p.$$

Since every vector critical point is a weak efficient solution, so there exists $x \in X$ such that $f(x) - f(y) < 0$, and there exists $\eta$ such that

$$\langle G'(f_i(y))\alpha(x, y)\nabla f(y), \eta(x, y)\rangle < 0, \;\; i = 1, 2, ..., p.$$

This is precisely the pseudo $(G, \alpha)$-univexity condition for $f$. □

We can relate the vector critical points to the solutions of the $WVVLIP$ by using Theorem 3.5 and Theorem 3.10.

**Corollary 3.11.** Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^p$ be a differentiable function. If $F = \nabla f$, $f$ is pseudo $(G, \alpha)-$ univex function with respect to $\alpha, \; \eta, \; \phi$ and $b$ with $\phi(t) \leq 0$, whenever $t \leq 0$ and $G'(f_i(w)) > 0, \;\; i = 1, 2, ..., p$, where $w \in X$. Then the vector critical points, the weakly efficient points and the solutions of the $WVVLIP$ are equivalent.

## 4 Conclusions

In this paper, the concept of $(G, \alpha)$-univexity as a generalization of $\alpha$-univexity has been introduced and discussed the relationship between vector variational-like inequality problems and vector optimization problems under $(G, \alpha)$-univexity. Our results in this paper are generalization and refinement of some well-known results in the literature. Further research is needed to approximate the solution of weak vector variational- like inequalities and its related models with $(G, \alpha)$-univexity assumptions.

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