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Different types of dominating sets of the discrete topological graph

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Abstract

The dominating problem is one of the topics in graph theory because of its applications in different areas. In this paper, some dominating sets are applied in the discrete topological graph. Such as independent domination, connected domination, doubly connected domination, total domination, restrained domination, strongly domination and co-neighborhood domination with their inverses.

Keywords: Independent domination, doubly connected domination, total domination, restrained domination.
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1 Introduction

The study of the mathematical properties in the topological graphs appeared in [24, 25, 26, 27, 28, 29]. We work here on certain types of domination on a topological graph. Let $G = (V, E)$ be a graph has vertex set V and edge set E. For any vertex $v \in V$, the degree of v means the number of edges that incident on v and is denoted by $\deg(v)$, where $\delta(G)$ is the minimum degree and $\Delta(G)$ is the maximum degree of vertices in the graph G. For more detailed see [20]. A set $D \subseteq V$ is a dominating set if every vertex out of it is adjacent to a vertex or more in it. A dominating set D is said to be minimal if there is no proper dominating subset in it. The domination number $\gamma(G_\tau)$ is the cardinality of the minimum dominating set in G. For more detailed see [21, 22, 23]. There are several types of domination because of their importance and various applications, see [1]-[16, 32, 33]. The dominating set D is an independent dominating set if all vertices of $G[D]$ are isolated vertices [19]. D is the total dominating set if $G[D]$ has no isolated vertex [17]. Also, it is called a connected dominating set if $G[D]$ is connected [34]. And D is doubly connected if both $G[D]$ and $G[V - D]$ are connected subgraphs [18]. D is a restrained dominating set if every vertex in $V - D$ is adjacent to another vertex in $V - D$ [31]. Let $uv \in E(G)$ and u, v dominate each other, if $\deg(u) \geq \deg(v)$, then the vertex u is strongly dominates v. Where D is called a strong dominating set if every vertex in $V - D$ is strongly dominated by at least one vertex in D [35]. Also, it is equality co-neighborhood if every vertex in D is adjacent to an equal number of vertices in $V - D$ [30]. The aim of this paper is to study all the above dominating sets in the discrete topological graph G_τ.

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2 Some General Properties of Topological Graph

In this section, we recall some basic definitions and results of the discrete topological graph G_τ.

Definition 2.1. [24] Let X be a non-empty set and τ be a discrete topological graph denoted by $G_\tau = (V, E)$ is a graph of the vertex set $V(G_\tau) = \{A; A \in \tau \text{ and } A \neq \emptyset, X\}$ and the edge set defined by $E(G_\tau) = \{AB; A \subseteq B$ and $A \neq B\}$.

Theorem 2.2. [24] Let G_τ be a discrete topological graph on X, where $|X| = n$ $(n \geq 3)$, then G_τ is $(n-1)$-partite graph.

Theorem 2.3. [24] Let G_τ be a discrete topological graph on a non-empty set X for $|X| = n(n \geq 3)$, then G_τ is a connected graph.

Theorem 2.4. [24] Let G_τ be a discrete topological graph on a non-empty set X with $|X| = n$. Then:

i. $\delta(G_\tau) = \Delta(G_\tau) = 2$, if and only if $n = 3$.

ii. $\delta(G_\tau) = \sum_{i=1}^{\lceil \frac{n}{2} \rceil - 1} \binom{\lceil \frac{n}{2} \rceil}{i} + \sum_{i=\lceil \frac{n}{2} \rceil + 1}^{n-1} \binom{n - \lceil \frac{n}{2} \rceil}{i - \lceil \frac{n}{2} \rceil}$, for $n \geq 4$.

iii. $\Delta(G_\tau) = \sum_{i=1}^{n-2} \binom{n-1}{i}$, for $n \geq 4$.

Theorem 2.5. [25] Let G_τ be a discrete topological graph on X, where $|X| = n$. Then, $\gamma(G_\tau) = 2$.

Proposition 2.6. [25] Let G_τ be a discrete topological graph on X, where $|X| =$ n. Then, $\gamma^{-1}(G_\tau) = 2$.

3 New results of domination in the topological graph

In this section, many types of domination are applied on the discrete topological graph G_τ. The inverse domination for all these types are studied also.

Proposition 3.1. Let G_τ be a discrete topological graph on X, where $|X| = n$. Then, G_τ has an independent dominating set such that $\gamma_i(G_\tau) = 2$.

Proof . Let $D \subseteq V(G_\tau)$. Then, by the similar technique of Theorem 2.5, the minimum dominating set D has only two vertices say u and u^c and there is no edge between them. This implies that $D = \{u, u^c\}$ is an independent dominating set, that is $\gamma_i(G_\tau) = 2$, see Fig 1 . □

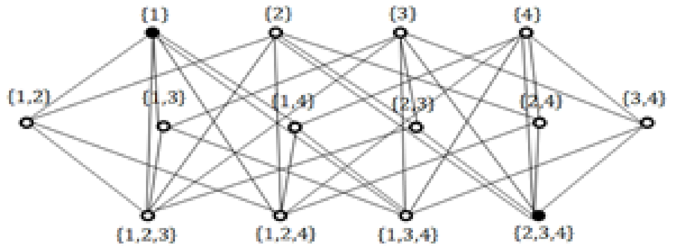
Proposition 3.2. Let G_τ be a discrete topological graph on X, where $|X| = n$. Then, G_τ has an inverse independent dominating set such that $\gamma_i^{-1}(G_\tau) = 2$.

Proof . Let $D^{-1} = \{w, w^c\}$ such that w has a singleton element and w^c has $n-1$ elements. The vertices w, w^c each of them dominates a certain part of the vertices in a graph G_τ. By similar technique of Proposition 2.6. This means $D^{-1} = \{w, w^c\}$ be the inverse independent dominating set. Therefore, $\gamma_i^{-1}(G_\tau) = 2$. See Fig 2. □

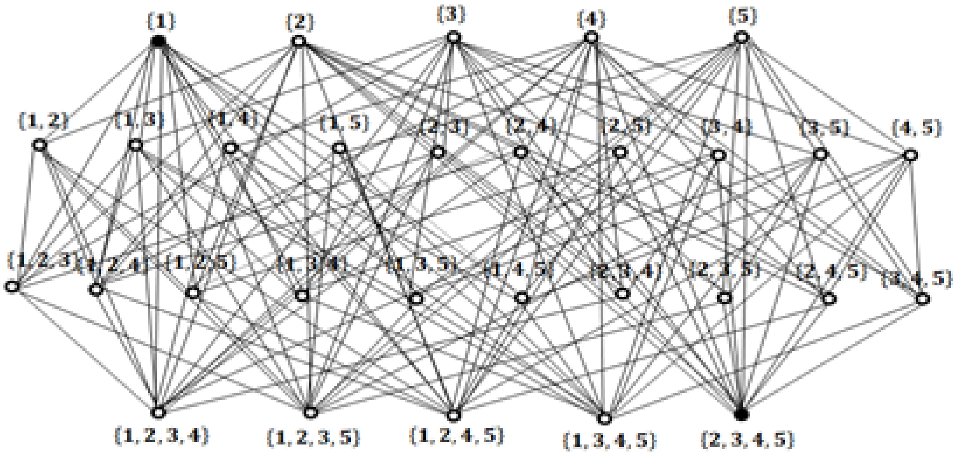
Proposition 3.3. Let G_τ be a discrete topological graph on $X, |X| = n$. If $n = 2$, then G_τ has no connected dominating set.

Proof . According to Theorem 2.5, $\gamma_c(G_t) = 2$. The dominating set has an isolated vertex, this contradicts the definition of a connected dominating set. Therefore, G_τ no connected dominating set. □

Theorem 3.4. Let G_τ be a discrete topological graph on X, where $|X| = n$ $(n \geq 3)$. Then, G_τ has a connected dominating set such that $\gamma_c(G_\tau) = 4$.

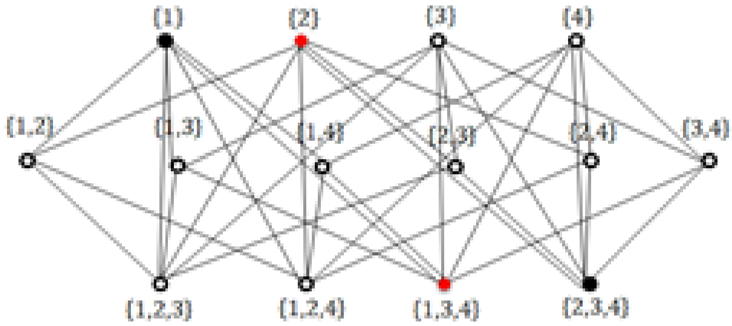


(a) $\gamma_i\,(K_{4,6,4}) = 2$



(b) $\gamma_i\,(K_{5,10,10,5}) = 2$

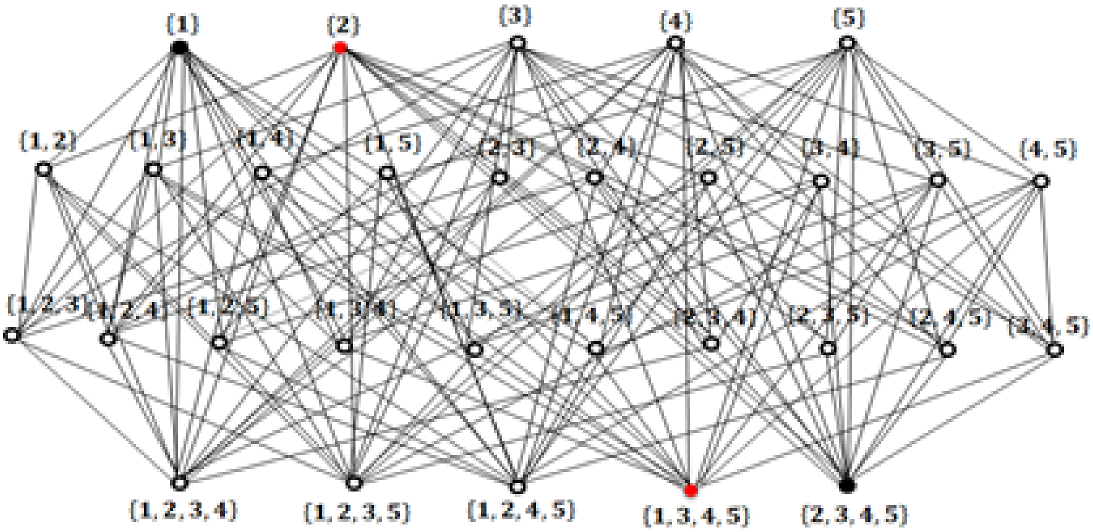
Figure 1: A minimum independent dominating set in G_τ



(a) $\gamma_i^{-1}\,(K_{4,6,4}) = 2$

Proof . Let $D = \{u, v, w, u^c\}$ such that u and w have singleton elements, so v and u^c have $n-1$ elements and $u \subseteq v, w \subseteq v, w \subseteq u^c$. Since $u \subseteq v$, there is an edge between them. Also, $w \subseteq v$, then there is an edge between them. Also, $w \subseteq u^c$, then there exist an edge between them. Hence, $D = \{u, v, w, u^c\}$ be a connected dominating set. Now, to prove that D is the minimum connected dominating set. Suppose that $\grave{D}$ is a connected dominating set in G_τ such that $|\grave{D}| < |D|$. Since $|D| = 4$, $|\grave{D}| = 2$ or $|\grave{D}| = 3$. If $|\grave{D}| = 2$, then $G[\grave{D}]$ is not connected graph by Theorem 2.5. Now, if $|\grave{D}| = 3$, then $\grave{D} = \{u, u^c, t\}$, where $e = ut \in E\,(G_t)$ such that there is no $\grave{e} = tu^c$. Since $t \subseteq u$, $t \nsubseteq u^c$ and the vertex u is adjacent to all vertices which u is a subset of them and not adjacent which any vertex that subset of u^c. Thus, $G[\bar{D}]$ is not a connected graph, then D is a minimum connected dominating set in G_t. Therefore, $\gamma_c\,(G_\tau) = 4$. □

Example 3.5. Consider a discrete topological graph G_τ on X.
i. Let $X = \{1, 2, 34\}$. Then, the minimum connected dominating set is $D = \{\{1\}, \{1,2,4\}, \{4\}, \{2,3,4\}\}$, so that



(b) $\gamma_i^{-1}(K_{5,10,10,5}) = 2$

Figure 2: A minimum inverse independent dominating set in G_τ

$\gamma_c(G_\tau) = 4$, see Fig 3.

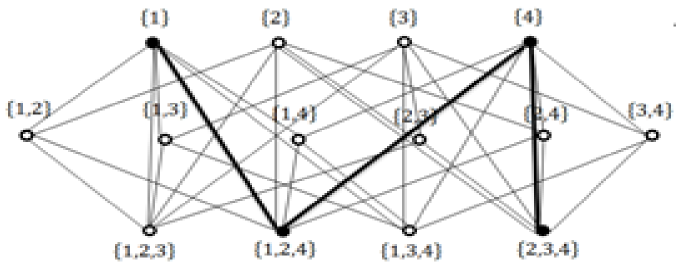


Figure 3: A minimum connected dominating set G_τ, for $|X| = 4$.

ii. Let $X = \{1,2,3,4,5\}$, then the minimum connected dominating set $D = \{\{1\},\{1,2,3,5\},\{5\},\{2,3,4,5\}\}$, so that $\gamma_c(G_\tau) = 4$. see Fig 4.

Proposition 3.6. Let G_τ be a discrete topological graph on X, where $|X| = n$ $(n \geq 3)$. Then, G_τ has an inverse connected dominating set and $\gamma_c^{-1}(G_\tau) = 4$.

Proof . Let $D^{-1} = \{x, y, z, x^c\}$ such that x and y have singleton element also z and x^c have $n-1$ elements. Such that the vertices x, y, z, x^c each of them dominate a certain part of the vertices in a graph G_τ. By the similar technique of Theorem 3.4. Hence, $D^{-1} = \{x, y, z, x^c\}$ be the minimum inverse connected dominating set. Therefore, $\gamma_c^{-1}(G_\tau) = 4$, see Fig 5 and Fig 6 . □

Proposition 3.7. Let G_τ be a discrete topological graph on X, where $|X| = 2$. Then, G_τ has no doubly connected dominating set.

Proof . By similar technique of Theorem 3.3. □

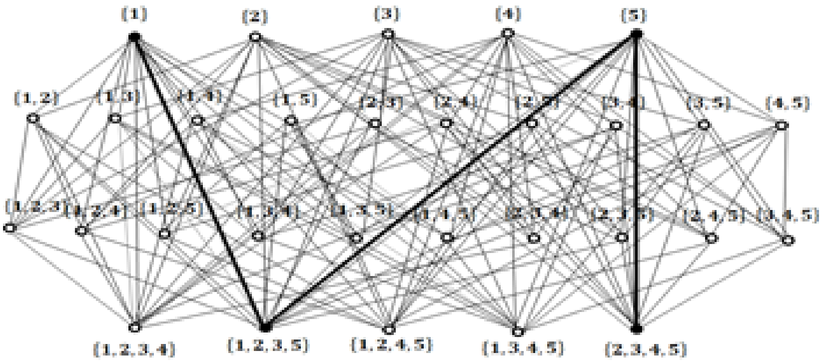


Figure 4: A minimum connected dominating set in G_τ, for $|X| = 4$.

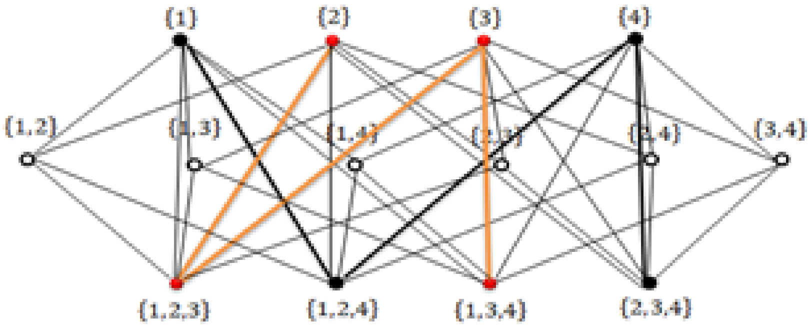


Figure 5: A minimum inverse connected dominating set in G_τ, for $|X| = 4$.

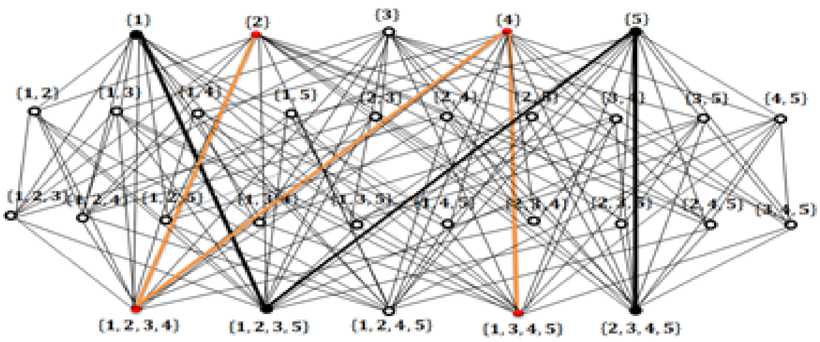


Figure 6: A minimum inverse connected dominating set , for $|X| = 5$.

Proposition 3.8. Let G_τ be a discrete topological graph on X, where $|X| = n$ $(n \geq 3)$. Then, G_τ has a doubly connected dominating set such that $\gamma_{cc}(G_\tau) = 4$.

Proof . Let $D = \{u, u^c\}$. By similar technique of Theorem 3.4, implies that G_τ has a connected dominating set D, and $G[D]$ is connected graph. Now, by similar technique of Theorem 2.3, we get $G[V - D]$ is connected. Since $G[D]$ and $G[V - D]$ are connected subgraphs, G_τ has a doubly connected dominating set and $\gamma_{cc}(G_\tau) = 4$ (see Fig 3 and Fig 4). □

Proposition 3.9. Let G_τ be a discrete topological graph on X, where $|X| = n$ $(n \geq 3)$. Then, G_τ has an inverse doubly connected dominating set such that $\gamma_{cc}^{-1}(G_\tau) = 4$.

Proof . According to the choosing of D in Proposition 3.8. Let $D^{-1} = \{x, y, z, x^c\}$. Thus, x, y, z, and x^c dominate all the vertices in G_τ. By similar technique of Proposition 3.6 and Proposition 3.8, deduce that D^{-1} is the inverse doubly connected dominating set. Therefore, $\gamma_{cc}^{-1}(G_\tau) = 4$, see Fig 5 and Fig 6 . □

Proposition 3.10. Let G_τ be a discrete topological graph on X, where $|X| = n$ $(n \geq 3)$. Then, G_τ has total dominating set such that $\gamma_t(G_\tau) = 4$.

Proof . By similar technique of Theorem 3.4. □

Proposition 3.11. Let G_τ be a discrete topological graph on X, where $|X| = n$ $(n \geq 3)$. Then, G_τ has a restrained dominating set such that $\gamma_r(G_\tau) = 2$.

Proof . Let $D = \{u, u^c\}$ such that u has a singleton element and u^c has $n-1$ elements, by the similar technique of Theorem 2.5. Now, according to Theorem 2.2, we divided the graph into $(n-1)$ disjoint sets $V_1, V_2, \ldots, V_{n-1}$, such that V_1 contains vertices that have singleton element, V_2 contain vertices that have two elements, and so on V_{n-1} contain vertices have $n-1$ elements. Such that there is no an edge between any two vertices in V_i for all $i = 1, 2, \ldots, n-1$. Then, for every edge in $G[V-D]$ from the remaining vertices of V_i and remaining vertices of V_j which belong to $V-D$ for all $i, j = 1, 2, \ldots, n-1$. So, $G[V-D]$ has no isolated vertices. That is, $D = \{u, u^c\}$ restrained dominating set with $\gamma_r(G_\tau) = 2$, see Fig 1. □

Proposition 3.12. Let G_τ be a discrete topological graph on X, where $|X| = n$ $(n \geq 3)$. Then, G_τ has an inverse restrained dominating set such that $\gamma_r^{-1}(G_\tau) = 2$.

Proof . By similar technique of Propositions 2.6 and Proposition 3.11. □

Proposition 3.13. Let G_τ be a discrete topological graph on X, where $|X| = n$. Then, G_τ has strong dominating set such that $\gamma_{st}(G_\tau) = 2$.

Proof . By similar technique of Theorem 2.5. If $n = 2$, let $D = \{u_1, u_2\}$ such that u_1, u_2 having degree zero. These vertices are strong dominated by themselves. Therefore, $\gamma_{st}(G_\tau) = 2$.
If $n > 2$, let $D = \{u, u^c\}$ such that u has a singleton element and u^c has $n-1$ elements. By using Theorem 2.4 these vertices have the maximum degrees. If the vertex u dominates any vertex v that has two elements, then $\deg(u) > \deg(v)$. So if the vertex u dominates any vertex w that has three elements, then $\deg(u) > \deg(w)$. And so on if the vertex u dominates any vertex r that has $n-1$ elements, then $\deg(u) = \deg(r)$. Similarly, if the vertex u^c dominates any vertex x that has a singleton element, then $\deg(u^c) = \deg(x)$. So if the vertex u^c dominates any vertex y that has two elements, then $\deg(u^c) > \deg(y)$. And so on if the vertex u^c dominates any vertex z that have $n-2$ elements, then $\deg(u^c) > \deg(y)$. That is, the vertices of D have a maximal degree, and hence, $D = \{u, u^c\}$ is strong dominating set. Therefore, $\gamma_{st}(G_\tau) = 2$, see Fig 1. □

Proposition 3.14. Let G_τ be a discrete topological graph on X, where $|X| = n$. Then, G_τ has an inverse strong dominating set such that $\gamma_{st}^{-1}(G_\tau) = 2$.

Proof . By similar technique to Proposition 2.6 and Proposition 3.13, let $D^{-1} = \{w, w^c\}$ such that w and w^c dominate all vertices in a graph G_τ. Therefore, $\gamma_{st}^{-1}(G_\tau) = 2$. See Fig 2 . □

Proposition 3.15. Let G_τ be a discrete topological graph on X, where $|X| = n$ $(n \geq 3)$. Then, G_τ has an equality co-neighborhood dominating set such that $\gamma_{en}(G_\tau) = 2$.

Proof . By a similar technique in Theorem 2.5, let $D = \{u, u^c\}$, where u has a singleton element and u^c has $n-1$ elements. Each of them dominates a certain part of the vertices in a graph G_τ. Then these vertices have the same degree, since $|N(u) \cap (V-D)| = |N(u^c) \cap (V-D)|$. Then, D is equality co-neighborhood dominating set and $\gamma_{en}(G_\tau) = 2$ (see Fig 1). □

Proposition 3.16. Let G_τ be a discrete topological graph on X, where $|X| = n$. Then, G_τ has an inverse equality co-neighborhood dominating set such that $\gamma_{en}^{-1}(G_\tau) = 2$.

Proof . By similar technique of Proposition 2.6 and Proposition 3.15, let $D^{-1} = \{w, w^c\}$ such that w and w^c dominate all vertices in a graph G_τ. Therefore, $\gamma_{st}^{-1}(G_\tau) = 2$ (see Fig 2). □

Conclusions

Several types of domination parameters are applied in this paper on the discrete topological graph. The inverse dominating sets are also studied here.

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