On automorphisms of strong semilattice of $\pi$-groups

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Abstract

In this paper, we make a start by considering the automorphisms of strong semilattice of $\pi$-groups, relating them to the automorphisms of underlying $\pi$-groups. We also provide a condition under which an automorphism of strong semilattice of $\pi$-groups can be constructed.

Keywords: Automorphisms, Linking homomorphisms, $\pi$-groups, $\pi$-regular

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1 Introduction

Let $\Lambda$ be a semilattice and for each $\alpha \in \Lambda$, let $S_\alpha$ be a semigroup and suppose $S_\alpha \cap S_\beta = \emptyset$ for $\alpha \neq \beta$. For every $\alpha, \beta \in \Lambda$ with $\alpha \geq \beta$, there exists a homomorphism $f_{\alpha,\beta}: S_\alpha \to S_\beta$ satisfying the following conditions:

(i) $f_{\alpha,\alpha} = \text{Id}_{S_\alpha}$ for any $\alpha \in \Lambda$.
(ii) For any $\alpha, \beta, \gamma \in \Lambda$ with $\alpha \geq \beta \geq \gamma$, $f_{\beta,\gamma} f_{\alpha,\beta} = f_{\alpha,\gamma}$.

The semigroup operation on $S = \cup_{\alpha \in \Lambda} S_\alpha$ is defined in terms of the multiplication in the components $S_\alpha$ and the homomorphism $f_{\alpha,\beta}$ (called linking homomorphism) by $st = f_{\alpha,\beta}(s)f_{\beta,\gamma}(t)$ for $s \in S_\alpha$ and $t \in S_\beta$, where $\gamma = \alpha \wedge \beta$. Then $S$ with multiplication defined above is a strong semilattice $\Lambda$ of semigroup $S_\alpha$, and is denoted by $S = (\Lambda, \{S_\alpha\}_{\alpha \in \Lambda}, \{f_{\alpha,\beta}\}_{\alpha \geq \beta})$.

A semigroup $S$ is said to be a $\pi$-group if there exists a subgroup $G^S$ of $S$ which is an ideal, and for any $s \in S$, there exists a natural number $n \in \mathbb{N}$ such that $s^n \in G^S$. An element $s \in S$ is said to be regular if there exists an element $a \in S$ such that $sas = s$ and $S$ is said to be regular if every element of $S$ is regular. An element $s$ of $S$ is said to be $\pi$-regular if there exists a positive integer $n \in \mathbb{N}$ such that $s^n \in s^n S s^n$ and $S$ is said to be $\pi$-regular if every element of $S$ is $\pi$-regular. Infact, $\pi$-regular semigroups is one of the important classes of non-regular semigroups. Let $R^S$ denote the set of all regular elements of $S$. We write, $S = R^S \cup N^S$, where $N^S = S \setminus R^S$ is the set of non-regular elements of $S$.

The set of idempotents in $S$ will be denoted by $E_S$. Thus $E_S = \{e_\alpha; \alpha \in \Lambda\}$. If $S$ is a $\pi$-group and $s \in R^S$, then $s = se$ for the (unique) idempotent $e$, and so $s \in G^S$. Since obviously $G^S \subseteq R^S$, so we have $G^S = R^S$ in a $\pi$-group. In this paper, we are looking for automorphisms of strong semilattice of $\pi$-groups.

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2 Automorphisms

In this section, first, we fix some notations without further mention. Let $S$ be a strong semilattice of $\pi$-groups. We write $S_\alpha = R_\alpha \cup N_\alpha$, where $N_\alpha = S_\alpha \setminus R_\alpha$ is the set of non-regular elements of $S_\alpha$ and it is the partial semigroup by definition of $\pi$-group.

Lemma 2.1. Let $S$ be a strong semilattice of $\pi$-groups. Let $\phi \in \text{Aut}(S)$, then the following hold:

(i) $\phi|E_S$ is an automorphism of semilattices.

(ii) If $G \subseteq S$ is a group, then there exists $\alpha \in \Lambda$ such that $G \subseteq S_\alpha$.

(iii) For each $\alpha \in \Lambda$, $\phi|_{S_\alpha}$ is an isomorphism of $\pi$-groups from $S_\alpha$ to $S_\tau$, where $\phi(e_\alpha) = e_\tau$.

Proof. Let $\phi \in \text{Aut}(S)$.

(i) Suppose $e_\alpha \in E_S$, we have $\phi(e_\alpha) = \phi(e_\alpha e_\alpha) = \phi(e_\alpha)\phi(e_\alpha)$, that is, $\phi(e_\alpha)$ is idempotent, hence $\phi(E_S) \subseteq E_S$.

Now for any $e_\gamma \in E_S$, since $\phi$ is onto, therefore there exists some $s \in S$ such that $\phi(s) = e_\gamma$. Now we show that $s \in E_S$. For this we have

$$\phi(s) = e_\gamma$$

$$= e_\gamma e_\gamma \quad (\text{as } e_\gamma \in E_S)$$

$$= \phi(s)\phi(s)$$

$$= \phi(s^2) \quad (\text{as } \phi \text{ is homomorphism}).$$

That is, $\phi(s) = \phi(s^2)$. Since $\phi$ is injective, therefore we have $s = s^2$, implies, $s$ is idempotent. Hence we have $e_\gamma = \phi(s) \in \phi(E_S)$, that is, $E_S \subseteq \phi(E_S)$. Thus we have $\phi(E_S) = E_S$. Since each $S_\alpha$ contains a unique idempotent $e_\alpha$, and $\phi \in \text{Aut}(S)$ permutes the idempotents, $\phi$ induces a bijection on $\Lambda$. Since $e_\alpha e_\beta = e_\beta$ if and only if $\alpha \geq \beta$, then $\phi$ preserves the order on $\Lambda$.

(ii) Suppose $G$ is a subgroup of $S$. Let $e$ be the identity element of $G$. Then $e = e_\alpha \in S_\alpha$ for some $\alpha \in \Lambda$. We show that $G$ is a subgroup of $S_\alpha$. Let $g \in G$, then $g \in S_\beta$ for some $\beta \in \Lambda$. Since $e$ is the identity element of $G$, so $ge = g$. Also, $g = ge = f_{\beta,\alpha}(g)f_{\alpha,\beta}(e) \in S_{\alpha\beta}$. So $\beta = \alpha\beta$, as $S_{\beta \cap S_{\alpha\beta}} = \emptyset$, this implies, $\beta \leq \alpha$.

Let $g^{-1}$ be the inverse of $g$, then $g^{-1} \in S_\eta$ for some $\eta \in \Lambda$. Thus $gg^{-1} = e \in S_\alpha$. Also, $e = gg^{-1} = f_{\beta,\eta}(g)f_{\eta,\beta}(g^{-1}) \in S_{\beta\eta}$. So $e \in S_{\beta\eta}$, implies, $\alpha = \beta\eta$ and so $\alpha \leq \beta$. Hence $\alpha = \beta$ and $g \in S_{\alpha}$, that is, $G \subseteq S_{\alpha}$.

(iii) Let $g \in S_{\alpha}$, since $S_{\alpha}$ is a $\pi$-group, so there exists a subgroup $G^{S_{\alpha}}$ of $S_\alpha$ which is an ideal of $S_\alpha$ and there exists $n \in \mathbb{N}$ such that $g^n \in G^{S_{\alpha}}$. Since $G^{S_{\alpha}}$ is a group, it implies the inverse of $g^n$ exists in $G^{S_{\alpha}}$. That is, $g^{-n} \in G^{S_{\alpha}} \subseteq S_{\alpha}$ such that $g^n g^{-n} = e_\alpha \in S_{\alpha}$. Let $\phi(g^n) \in S_\gamma$ for some $\gamma \in \Lambda$. Also, by part (i), $\phi(e_\alpha) = e_\tau \in S_\tau$ for some $\tau \in \Lambda$ and $\phi(g^{-n}) \in S_\gamma$.

Now we have

$$\phi(e_\alpha) = \phi(g^n g^{-n})$$

$$= \phi(g^n)\phi(g^{-n}) \in S_\gamma.$$ 

That is, $S_\tau = S_\gamma$. Hence $\phi(S_\alpha) \subseteq S_\tau$. Since $\phi$ is an automorphism and so $\phi^{-1}$ exists and will do same and hence $\phi^{-1}(S_\tau) \subseteq S_\alpha$, that is, $S_\tau \subseteq \phi(S_\alpha)$ and from part (i), we are done. □

By the above lemma we know that every automorphism of $S$ induces an automorphism of $\Lambda$. We will denote this automorphism of semilattices by $\phi_\Lambda$. Hence, we can write $\phi_\Lambda(\alpha) = \tau$, where $\phi(e_\alpha) = e_\tau$. Let $\phi \in \text{Aut}(S)$. Then we write $\phi_\alpha$ for $\phi|_{S_\alpha}$, where $\alpha \in \Lambda$. By Lemma 2.1, we know $\phi_\alpha$ is an isomorphism. So given an automorphism $\phi$ of $S$, we obtain family $\{\phi_\alpha : \alpha \in \Lambda\}$ of $\pi$-group isomorphisms and a semilattice automorphism denoted by $\phi_\Lambda$. Thus we have $\phi_\Lambda$ and $\{\phi_\alpha : \alpha \in \Lambda\}$ determines $\phi$ completely.

Following lemma is due to Lallement; for the proof, one can see [2].

Lemma 2.2. Let $\varphi : S \to T$ be a homomorphism from a regular semigroup $S$ into a semigroup $T$. Then $\text{im}(\varphi)$ is regular. □
Let $S = \{\Lambda, \{S_\alpha\}_{\alpha \in \Lambda}, \{f_{\alpha, \beta}\}_{\alpha \geq \beta}\}$ be strong semilattices of $\pi$-groups. Note that $R^S = \bigcup_{\alpha \in \Lambda} R^{S_\alpha}$. Now, for any $s \in S$, we define a mapping $\psi : S \to R^S$ by

$$\psi(s) = e_\alpha s \text{ if } s \in S_\alpha.$$ 

Where $e_\alpha$ is the unique idempotent of the $\pi$-group $S_\alpha$. Since we know $R^S$ is an ideal of $S$. Thus the map $\psi$ is well defined. The following lemma shows that the map $\psi$ commutes with the linking homomorphisms.

**Lemma 2.3.** Let $S = \{\Lambda, \{S_\alpha\}_{\alpha \in \Lambda}, \{f_{\alpha, \beta}\}_{\alpha \geq \beta}\}$ be strong semilattices of $\pi$-groups. Then for any $\alpha, \beta \in \Lambda$ with $\alpha \geq \beta$ and for any $s \in S_\alpha$, we have

$$\psi f_{\alpha, \beta} = f_{\alpha, \beta} \psi.$$

**Proof.** Let $\alpha, \beta \in \Lambda$ with $\alpha \geq \beta$. Then for any $s \in S_\alpha$ we have

$$f_{\alpha, \beta} \psi(s) = f_{\alpha, \beta}(e_\alpha s)$$

$$= f_{\alpha, \beta}(e_\alpha) f_{\alpha, \beta}(s)$$

$$= e_\beta(f_{\alpha, \beta}(s))$$

$$= \psi f_{\alpha, \beta}(s).$$

Thus we have

$$\psi f_{\alpha, \beta} = f_{\alpha, \beta} \psi.$$

Next, we start from semilattices automorphism and a family of $\pi$-group isomorphisms satisfying a condition under which an automorphism of strong semilattices of $\pi$-groups can be constructed.

**Theorem 2.4.** Let $S = \{\Lambda, \{S_\alpha\}_{\alpha \in \Lambda}, \{f_{\alpha, \beta}\}_{\alpha \geq \beta}\}$ be strong semilattices of $\pi$-groups. Let $\phi : \Lambda \to \Lambda$ and for each $\alpha \in \Lambda$, $\phi_\alpha : S_\alpha \to S_{\phi_\alpha(\alpha)}$ be an isomorphism of $\pi$-groups. Also, assume that the following conditions are satisfied.

1. $\phi|N^S$ is a partial automorphism of $N^S$, and for any $s, s' \in N^S$, if $ss' \notin N^S$, then $\phi(s)\phi(s') \notin N^S$.
2. $\psi \phi_\beta f_{\alpha, \beta} = \psi f_{\phi_\alpha(\alpha), \phi_\beta(\beta)} f_\phi.$

Define a mapping $\phi$ on $S$ by $\phi(s) = \phi_\alpha(s)$ if $s \in S_\alpha$. Then $\phi$ is an automorphism of $S$. Conversely, every automorphism of strong semilattices of $\pi$-groups satisfies the conditions.

**Proof.** Suppose there exists a semilattice automorphism $\phi : \Lambda \to \Lambda$ and a family of $\pi$-group isomorphisms $\{\phi_\alpha : \alpha \in \Lambda\}$ where $\phi_\alpha : S_\alpha \to S_{\phi_\alpha(\alpha)}$ satisfying the above two conditions. Let $\phi : S \to S$ be a map defined by $\phi(s) = \phi_\alpha(s)$ if $s \in S_\alpha$. We show that $\phi \in \text{Aut}(S)$. Let $s_1, s_2 \in S$. If $s_1 = s_2$, then there exists $\alpha \in \Lambda$ such that $s_1, s_2 \in S_\alpha$. Since $\phi_\alpha : S_\alpha \to S_{\phi_\alpha(\alpha)}$ is an isomorphism, therefore we have

$$s_1 = s_2 \iff \phi_\alpha(s_1) = \phi_\alpha(s_2) \iff \phi(s_1) = \phi(s_2).$$

That is, $\phi$ is well defined and injective. Now for any $t \in S$, there exists some $\alpha \in \Lambda$ with $\phi_\alpha(\alpha) = \delta \in \Lambda$ such that $t \in S_\delta$. As $\phi_\alpha : S_\alpha \to S_{\phi_\alpha(\alpha)}$ is an isomorphism. Therefore there exists some $s \in S_\alpha$ such that $t = \phi_\alpha(s) = \phi(s)$, and so $\phi$ is surjective. Hence $\phi$ is bijective.

Now we need to show $\phi$ is a homomorphism. For this, let $s_\alpha \in S_\alpha$ and $s_\beta \in S_\beta$. Then we have the following cases.

**Case 1:** If $s_\alpha s_\beta \in N_{\alpha \beta}$, then $s_\alpha \in N_\alpha$ and $s_\beta \in N_\beta$, hence by condition (1), we have $\phi(s_\alpha s_\beta) = \phi(s_\alpha)\phi(s_\beta)$.

**Case 2:** If $s_\alpha s_\beta \notin N_{\alpha \beta}$, then we have $\phi(s_\alpha s_\beta) \notin N^S$, therefore we have

$$\phi(s_\alpha s_\beta) = \phi_\alpha(s_\alpha s_\beta)$$

$$= \phi_\alpha(f_{\alpha, \beta}(s_\alpha)f_{\beta, \alpha}(s_\beta))$$

$$= \phi_\alpha(f_{\alpha, \beta}(s_\alpha)) \phi_\beta(f_{\beta, \alpha}(s_\beta))$$

$$= e_{\phi_\alpha(\alpha)} \phi_\alpha(f_{\alpha, \beta}(s_\alpha)) \phi_\beta(f_{\beta, \alpha}(s_\beta))$$

$$= (\psi \phi_\alpha f_{\alpha, \beta}(s_\alpha))(\psi \phi_\beta f_{\beta, \alpha}(s_\beta)).$$
On the other hand, we have
\[
\phi(s_\alpha) \phi(s_\beta) = \phi_\alpha(s_\alpha) \phi_\beta(s_\beta)
\]
\[
= f_{\phi_\alpha(\alpha),\phi_\lambda(\alpha)}(\phi_\alpha(s_\alpha)) f_{\phi_\lambda(\beta),\phi_\lambda(\alpha)}(\phi_\beta(s_\beta))
\]
\[
= (e_{\phi_\lambda(\alpha)} f_{\phi_\lambda(\alpha),\phi_\lambda(\alpha)}(\phi_\alpha(s_\alpha))) (e_{\phi_\lambda(\beta)} f_{\phi_\lambda(\beta),\phi_\lambda(\alpha)}(\phi_\beta(s_\beta))) \quad \text{(by condition (1))}
\]
\[
= (\psi f_{\phi_\lambda(\alpha),\phi_\lambda(\alpha)} \phi_\alpha(s_\alpha)) (\psi f_{\phi_\lambda(\beta),\phi_\lambda(\alpha)} \phi_\beta(s_\beta))
\]
\[
= (\psi \phi_\alpha \beta f_{\alpha,\beta}(s_\alpha)) (\psi \phi_\alpha \beta f_{\beta,\alpha}(s_\beta)) \quad \text{(by condition (2))}.
\]

Hence we have \(\phi(s_\alpha s_\beta) = \phi(s_\alpha) \phi(s_\beta)\). Thus \(\phi\) is an automorphism of \(S\).

Conversely, suppose \(\phi\) is an automorphism of \(S\). By Lemma 2.1, we have the existence of semilattice automorphism \(\phi_\lambda\) and a family \(\{\phi_\alpha : S_\alpha \to S_{\phi_\lambda(\alpha)}\}\) of \(\pi\)-group isomorphisms. Since \(\phi \in \text{Aut}(S)\), then image of \(N^S\) is \(N^S\), by Lemma 2.2. Therefore, condition (1) holds clearly.

Now for any \(\alpha \geq \beta\), then \(\alpha \beta = \beta\). For \(s \in S_\alpha\), we have \(e_\beta s = f_{\beta,\beta}(e_\beta) f_{\alpha,\beta}(s) = e_\beta f_{\alpha,\beta}(s) = \psi f_{\alpha,\beta}(s)\). Thus we have,
\[
\phi(e_\beta s) = \phi(\psi f_{\alpha,\beta}(s))
\]
\[
= \phi_\beta(\psi f_{\alpha,\beta}(s))
\]
\[
= \phi_\beta(\psi f_{\alpha,\beta}(s)).
\]

Also, we have
\[
\phi(e_\beta) \phi(s) = e_{\phi_\lambda(\beta)} \phi_\alpha(s)
\]
\[
= (e_{\phi_\lambda(\beta)})(e_{\phi_\lambda(\alpha)})(\phi_\alpha(s))
\]
\[
= e_{\phi_\lambda(\beta)} \psi \phi_\alpha(s)
\]
\[
= f_{\phi_\lambda(\alpha),\phi_\lambda(\beta)} \psi \phi_\alpha(s).
\]

Thus we have
\[
\phi_\beta \psi f_{\alpha,\beta} = f_{\phi_\lambda(\alpha),\phi_\lambda(\beta)} \psi \phi_\alpha. \tag{1}
\]

Now for any \(\alpha \in \Lambda\) and \(s \in S_\alpha\) we have
\[
\phi_\alpha \psi(s) = \phi_\alpha(e_\alpha s)
\]
\[
= \phi_\alpha(e_\alpha) \phi_\alpha(s)
\]
\[
= e_{\phi_\lambda(\alpha)} \phi_\alpha(s)
\]
\[
= \psi \phi_\alpha(s).
\]

Therefore, we have
\[
\phi_\alpha \psi = \psi \phi_\alpha. \tag{2}
\]

Hence we have
\[
\psi \phi_\beta f_{\alpha,\beta} = \psi \phi_\beta f_{\alpha,\beta} \quad \text{(by equation (2))}
\]
\[
= f_{\phi_\lambda(\alpha),\phi_\lambda(\beta)} \psi \phi_\alpha \quad \text{(by equation (1))}
\]
\[
= \psi f_{\phi_\lambda(\alpha),\phi_\lambda(\beta)} \phi_\alpha \quad \text{(by Lemma (2.3))}.
\]

That is,
\[
\psi \phi_\beta f_{\alpha,\beta} = \psi f_{\phi_\lambda(\alpha),\phi_\lambda(\beta)} \phi_\alpha.
\]

Thus the proof is completed. \(\Box\) In the following theorem, we provide a construction for the automorphisms of \(S\) from the automorphisms of underlying \(\pi\)-groups \(S_\alpha\).

**Theorem 2.5.** Suppose all the linking homomorphisms are bijective and \(\Lambda = \{\alpha, \beta\}_{\alpha \leq \beta}\). Consider \(S = S_\alpha \cup S_\beta\), then every automorphism of \(S_\alpha\) or \(S_\beta\) gives rise to an automorphism of \(S\).
Proof. Suppose all the linking homomorphisms be bijective. Then $S_\alpha \cong S_\beta \cong G$. Let $\theta \in \operatorname{Aut}(G)$ be the arbitrary automorphism of $G$. Since $S_\alpha \cong S_\beta \cong G$, therefore we have the isomorphisms $\phi_\alpha : G \to S_\alpha$ and $\phi_\beta : G \to S_\beta$.

Let $\theta^\phi : S \to S$ be the map defined by

$$\theta^\phi(s) = \begin{cases} \phi_\alpha \theta_\phi^{-1}(s) & \text{if } s \in S_\alpha \\ \phi_\beta \theta_\phi^{-1}(s) & \text{if } s \in S_\beta. \end{cases}$$

We show that $\theta^\phi \in \operatorname{Aut}(S)$. For this, we first show that for all $s \in S_\beta$,

$$\theta^\phi f_{\beta,\alpha}(s) = f_{\beta,\alpha} \theta^\phi(s). \quad (3)$$

Since $\phi_\beta : G \to S_\beta$ and $f_{\beta,\alpha} : S_\beta \to S_\alpha$ are isomorphisms, we can define $\phi_\alpha = f_{\beta,\alpha} \phi_\beta$. Therefore, we have

$$\phi_\alpha^{-1} = (f_{\beta,\alpha} \phi_\beta)^{-1} \Rightarrow \phi_\alpha^{-1} = \phi_\beta^{-1} f_{\beta,\alpha}^{-1} \Rightarrow \phi_\alpha \theta_\phi^{-1} = \phi_\alpha \psi_\phi^{-1} f_{\beta,\alpha}^{-1} \Rightarrow \phi_\alpha \theta_\phi^{-1} f_{\beta,\alpha} = f_{\beta,\alpha} \phi_\beta \phi_\beta^{-1}.$$

Now for any $s \in S_\beta$, we have

$$\phi_\alpha \theta_\phi^{-1} f_{\beta,\alpha}(s) = f_{\beta,\alpha} \phi_\beta \phi_\beta^{-1}(s) \Rightarrow \phi_\alpha \theta_\phi^{-1}(f_{\beta,\alpha}(s)) = f_{\beta,\alpha} \theta^\phi(s) \Rightarrow \theta^\phi(f_{\beta,\alpha}(s)) = f_{\beta,\alpha} \theta^\phi(s).$$

Hence for all $s \in S_\beta$ we have $\theta^\phi f_{\beta,\alpha}(s) = f_{\beta,\alpha} \theta^\phi(s)$.

It is clear that $\theta^\phi$ is bijective. Now we show that $\theta^\phi$ is a homomorphism. For this, we have the following cases.

Case(i). If $s, t \in S_\alpha$ or $S_\beta$, then we have

$$\psi^\phi(st) = \phi_\alpha \theta_\phi^{-1}(st) = \phi_\alpha \theta_\phi^{-1}(s) \phi_\alpha \theta_\phi^{-1}(t) = \theta^\phi(s) \theta^\phi(t).$$

Case(ii). If $s \in S_\alpha$ and $t \in S_\beta$, then we have

$$\theta^\phi(st) = \theta^\phi(f_{\alpha,\alpha}(s) f_{\beta,\alpha}(t)) \quad \Rightarrow \quad \theta^\phi(s f_{\beta,\alpha}(t)) \quad \Rightarrow \quad \theta^\phi(s s_\alpha) \quad \text{(where } s_\alpha = f_{\beta,\alpha}(t)) \quad \Rightarrow \quad \phi_\alpha \theta_\phi^{-1}(s_\alpha) \quad \Rightarrow \quad \phi_\alpha \theta_\phi^{-1}(s) \phi_\alpha \theta_\phi^{-1}(s) \quad \Rightarrow \quad \theta^\phi(s) \theta^\phi(t) \quad \text{(by equation (3))} \quad \Rightarrow \quad \theta^\phi(s) \theta^\phi(t).$$

Hence we have $\theta^\phi \in \operatorname{Aut}(S)$ and every automorphism of $S$ can be constructed in this way. □

The next lemma helps us to prove the above theorem for arbitrary semilattices.

Lemma 2.6. Let $S = (\Lambda, \{S_\alpha\}_{\alpha \in \Lambda}, \{f_{\alpha,\beta}\}_{\alpha \geq \beta})$ be strong semilattices of $\pi$-groups with all the linking homomorphisms bijective, then for any $\lambda \in \Lambda$, we have $S \cong \Lambda \times S_\lambda \cong \Lambda \times G$.

Proof. Fix $\lambda \in \Lambda$, then for each $\alpha \in \Lambda$ we have an isomorphism

$$\sigma_\alpha = f_{\lambda,\lambda}^{-1} f_{\alpha,\lambda} : S_\alpha \to S_\lambda.$$
Now define a map $\chi : S \to \Lambda \times S_{\lambda}$ by $\chi(s) = (\alpha, \sigma_\alpha(s))$ if $s \in S_{\alpha}$. We show $\chi$ is an isomorphism. Let $s_1, s_2 \in S$. If $s_1 = s_2$, then there exists $\alpha \in \Lambda$ such that $s_1, s_2 \in S_{\alpha}$. Since $\sigma_\alpha : S_{\alpha} \to S_{\lambda}$ is an isomorphism, therefore we have

$$s_1 = s_2 \iff \sigma_\alpha(s_1) = \sigma_\alpha(s_2) \iff (\alpha, \sigma_\alpha(s_1)) = (\alpha, \sigma_\alpha(s_2)).$$

That is, $\chi$ is well defined and injective. Now for any $(\alpha, t) \in \Lambda \times S_{\lambda}$, there exists some $s' \in S_{\alpha}$ such that $t = \sigma_\alpha(s')$ as $\sigma_\alpha$ is surjective, therefore we have $(\alpha, t) = (\alpha, \sigma_\alpha(s')) = \chi(s')$, that is, $\chi$ is surjective.

Now for any $s, t \in S$ then $s \in S_{\alpha}$ and $t \in S_{\beta}$ for some $\alpha, \beta \in \Lambda$. If $\alpha = \beta$, then there is nothing to prove. Now suppose $\alpha \neq \beta$, we have

$$\chi(st) = (\alpha \beta, \sigma_{\alpha \beta}(st)) = (\alpha \beta, \sigma_\alpha(s) \sigma_\beta(t)) \quad \text{(as all the linking homomorphisms are bijective)}$$

$$= (\alpha, \sigma_\alpha(s))(\beta, \sigma_\beta(t)) = \chi(s)\chi(t).$$

Therefore $\chi$ is an isomorphism. □

**Corollary 2.7.** Let $S = (\Lambda, \{S_{\alpha}\}_{\alpha \in \Lambda}, \{f_{\alpha, \beta}\}_{\alpha \geq \beta})$ be strong semilattices of $\pi$-groups with all the linking homomorphisms bijective, then every automorphism of $S_{\alpha}$ for some $\alpha \in \Lambda$ gives rise to an automorphism of $S$.

**Proof.** The proof follows from Lemma 2.6 and Theorem 2.5. □

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