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Some new Hermite-Hadamard type inequalities for functions whose absolute value of the third derivative are MT-convex functions with applications

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Abstract

This paper is concerned to establish new variants of the well-known Hermite-Hadamard (HH) inequality for 3-times differentiable functions. Under the utility of these identities, we establish some new inequalities for the class of functions whose absolute value of the third derivative are MT-convex. The results presented here would provide generalizations of those given in earlier works. Finally, we present applications of our findings for means of real numbers and applications for particular functions are pointed out.

Keywords: Hermite-Hadamard inequality, MT-convex function, Hölder inequality
2010 MSC: Primary 26D15;, Secondary 26A51, 26D10

1 Introduction

Many research papers have studied the properties of convex functions that make this concept interesting in mathematical analysis [5, 3].

Definition 1.1. A function $g : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if the following inequality holds

$$g(tx + (1 - t)y) \leq tg(x) + (1 - t)g(y), \quad x, y \in [a, b],\ t \in [0, 1].$$

The use of the convex function to study the integral inequalities have been deeply investigated, especially for the well-known inequality of Hermite-Hadamard type (HH-type inequality). The HH-type inequalities are one of the most important type inequalities and have a strong relationship to convex functions. In 1893 Hermite and Hadamard [6] found independently that for any convex function $g : [a, b] \to \mathbb{R}$, the inequality

$$g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b g(x)dx \leq \frac{g(a)+g(b)}{2}, \tag{1.1}$$

holds.

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The classical inequalities for means can be derived from the HH-type inequality (1.1) for appropriate particular choices of the function g. The reader is referred to [11, 12, 13, 14, 1] for the generalization, improvement and extension of the HH-type inequality (1.1).

In recent years, important generalizations have been made in the context of convexity: quasi-convex [8], pseudo-convex [9], strongly convex [23], strongly (s, m)-convex [19, 20, 22], invex and preinvex [10], approximately convex [7], and MT-convex [15].

In view of the above indices, We would like to extend the works done in [2] and [16] to establish some modified HH-type inequalities for the 3-times differentiable MT-convex functions. For this we recall the well-known AM-GM inequality for two positive real numbers which can be stated as follows: If $x, y \in \mathbb{R}^+$, then

$$\sqrt{xy} \leq \frac{x+y}{2}.$$

In [24], Tunç and Yidirim defined the so-called MT-convex function as follows

Definition 1.2. [24] A function $g : I \subset \mathbb{R} \to \mathbb{R}$ is said to be MT-convex on the interval I if the inequality

$$g(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} g(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} g(y), \tag{1.2}$$

holds for all $x, y \in I$ and $t \in (0, 1)$.

Remark 1.3. In (1.2), if we take $t = \frac{1}{2}$ inequality (1.2) reduce to Jensen convex.

Example 1.4. $f, g : (1, \infty) \to \mathbb{R},\ f(x) = x^p,\ g(x) = (1 + x)^p,\ p \in (1, \frac{1}{1000})$ are MT-convex functions, but they are not convex.

Remark 1.5. It is important to note that all of the positive convex functions is also an MT-convex function, but the reverse is not always true. Since g is MT-convex and $t \leq \frac{\sqrt{t}}{2\sqrt{1-t}},\ (1-t) \leq \frac{\sqrt{1-t}}{2\sqrt{t}}$ it is written

$$g(tx + (1-t)y) \leq tg(x) + (1-t)g(y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} g(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} g(y),$$

this indicates that each positive convex function is a MT-convex function.

In this paper, we discover novel integral identities for three times diferentiable functions. We use these identities to establish some general inequalities for functions whose third derivatives absolute values are MT-convex. These general inequalities give us some new estimates for the right-hand side of integrals inequalities of Hermite-Hadamard type. The main results are framed and justified in Section 2, followed by applications of our results to some special means and a particular function in Section 3.

2 Main Results

In this section we present new Hermite-Hadamard type inequalities for 3-times differentiable MT-convex functions. To prove our main results, we need the following lemma by Pshtiwan et al.[16]

Lemma 2.1. [16] Suppose that $g : J \subset \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $a, b \in J$ with $a < b$. If $g''' \in L[a, b]$, then we have

$$\begin{aligned} & g\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b g(x)dx + \frac{(b-a)^2}{24} g''\left(\frac{a+b}{2}\right) \\ &= \frac{(b-a)^3}{96}\left[\int_0^1 t^3 g'''\left(t\frac{a+b}{2} + (1-t)a\right)dt \right. \\ &\left. + \int_0^1 (t-1)^3 g'''\left(tb + (1-t)\frac{a+b}{2}\right)dt\right]. \end{aligned}$$

Next we establish the first result of this section, for this we assume throughout the paper that $g \in L[a,b]$ that the function g is differential and continuous on $[a,b]$

Theorem 2.2. Suppose that $g : J \subseteq [0,+\infty) \to \mathbb{R}$ is a differentiable function such that $g''' \in L[a,b]$, where $a, b \in J$ with $a < b$. If $|g'''|$ is MT-convex function on $[a,b]$, then we have

$$\begin{aligned} &\left| g\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b g(x)dx + \frac{(b-a)^2}{24} g''\left(\frac{a+b}{2}\right) \right| \\ &\le \frac{5\pi(b-a)^3}{24576}\left[14\left|g'''\left(\frac{a+b}{2}\right)\right| + |g'''(a)| + |g'''(b)|\right]. \end{aligned} \tag{2.1}$$

Proof . Making use of Lemma 2.1 and the MT-convexity $|g'''|$, we have that

$$\begin{aligned} &\left| g\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b g(x)dx + \frac{(b-a)^2}{24} g''\left(\frac{a+b}{2}\right) \right| \\ &\le \frac{(b-a)^3}{96}\left[\int_0^1 t^3 \left|g'''\left(t\frac{a+b}{2} + (1-t)a\right)\right| dt\right. \\ &\left.+ \int_0^1 (t-1)^3 \left|g'''\left(tb + (1-t)\frac{a+b}{2}\right)\right| dt\right] \\ &\le \frac{(b-a)^3}{96}\int_0^1 t^3\left[\frac{\sqrt{t}}{2\sqrt{1-t}}\left|g'''\left(\frac{a+b}{2}\right)\right| + \frac{\sqrt{1-t}}{2\sqrt{t}}|g'''(a)|\right] dt \\ &+ \frac{(b-a)^3}{96}\int_0^1 (1-t)^3\left[\frac{\sqrt{t}}{2\sqrt{1-t}}|g'''(b)| + \frac{\sqrt{1-t}}{2\sqrt{t}}\left|g'''\left(\frac{a+b}{2}\right)\right|\right] dt \\ &\le \frac{(b-a)^3}{96}\left[\frac{35\pi}{128}\left|g'''\left(\frac{a+b}{2}\right)\right| + \frac{5\pi}{256}|g'''(a)| + \frac{5\pi}{256}|g'''(b)|\right]. \end{aligned}$$

This rearranges to the desired result. □

Corollary 2.3. Let the assumptions of Theorem 2.2 be valid and let

$$H = \left| g\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b g(x)dx + \frac{(b-a)^2}{24} g''\left(\frac{a+b}{2}\right) \right|.$$

Then,

(i) if $|g'''|$ is increasing, then we have

$$H \le \frac{5\pi(b-a)^3}{1536}|g'''(b)|, \tag{2.2}$$

(ii) if $|g'''|$ is decreasing, then we have

$$H \le \frac{5\pi(b-a)^3}{1536}|g'''(a)|, \tag{2.3}$$

(iii) if $g'''\left(\frac{a+b}{2}\right) = 0$, then we have

$$H \le \frac{5\pi(b-a)^3}{24576}\Big[|g'''(a)| + |g'''(b)|\Big], \tag{2.4}$$

(iv) if $g'''(a) = g'''(b) = 0$, then we have

$$H \le \frac{35\pi(b-a)^3}{12288}\left|g'''\left(\frac{a+b}{2}\right)\right|, \tag{2.5}$$

Theorem 2.4. Suppose that $g : J \subseteq [0,+\infty) \to \mathbb{R}$ is a differentiable function such that $g''' \in L[a,b]$, where $a, b \in J$ with $a < b$. If $|g'''|^q$ is MT-convex function on $[a,b]$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we have

$$\left| g\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b g(x)dx + \frac{(b-a)^2}{24} g''\left(\frac{a+b}{2}\right) \right| \le \frac{(b-a)^3}{96(3p+1)^{\frac{1}{p}}}\left(\frac{\pi}{4}\right)^{\frac{1}{q}} H_q,$$

where $H_q = \left(\left|g'''\left(\frac{a+b}{2}\right)\right|^q + |g'''(a)|^q\right)^{\frac{1}{q}} + \left(|g'''(b)|^q + \left|g'''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}$.

Proof . Let $p>1$. Then from Lemma 2.1 and using the Hölder inequality, we can deduce

$$\begin{aligned}&\left|g\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b g(x)dx+\frac{(b-a)^2}{24}g''\left(\frac{a+b}{2}\right)\right|\\&\le\frac{(b-a)^3}{96}\left[\int_0^1 t^3\left|g'''\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt\right.\\&\left.+\int_0^1(1-t)^3\left|g'''\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right]\\&\le\frac{(b-a)^3}{96}\left(\int_0^1 t^{3p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|g'''\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\\&+\frac{(b-a)^3}{96}\left(\int_0^1(1-t)^{3p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|g'''\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

The MT-convexity of $|g'''|^q$ on $[a,b]$ implies that

$$\begin{aligned}&\int_0^1\left|g'''\left(t\frac{a+b}{2}+(1-t)b\right)\right|^q dt\\&\le\int_0^1\left[\frac{\sqrt{t}}{2\sqrt{1-t}}\left|g'''\left(\frac{a+b}{2}\right)\right|^q+\frac{\sqrt{1-t}}{2\sqrt{t}}|g'''(a)|^q\right]dt\\&=\frac{\pi}{4}\left|g'''\left(\frac{a+b}{2}\right)\right|^q+\frac{\pi}{4}|g'''(a)|^q,\end{aligned}$$

and

$$\begin{aligned}&\int_0^1\left|g'''\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\\&\le\int_0^1\left[\frac{\sqrt{t}}{2\sqrt{1-t}}|g'''(b)|^q+\frac{\sqrt{1-t}}{2\sqrt{t}}\left|g'''\left(\frac{a+b}{2}\right)\right|^q\right]dt\\&=\frac{\pi}{4}|g'''(b)|^q+\frac{\pi}{4}\left|g'''\left(\frac{a+b}{2}\right)\right|^q.\end{aligned}$$

Therefore, we obtain

$$\left|g\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b g(x)dx+\frac{(b-a)^2}{24}g''\left(\frac{a+b}{2}\right)\right|\le\frac{(b-a)^3}{96(3p+1)^{\frac{1}{p}}}\left(\frac{\pi}{4}\right)^{\frac{1}{q}}H_q,$$

where we used the identities

$$\int_0^1 t^{3p}dt=\int_0^1(1-t)^{3p}dt=\frac{1}{3p+1}.$$

Thus, our proof is completely done. □

Corollary 2.5. Let the assumptions of Theorem 2.4 be valid and let

$$H=\left|g\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b g(x)dx+\frac{(b-a)^2}{24}g''\left(\frac{a+b}{2}\right)\right|.$$

Then,

(i) if $|g'''|$ is increasing, then we have

$$H\le\frac{(b-a)^3}{96(3p+1)^{\frac{1}{p}}}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\left[\left|g'''\left(\frac{a+b}{2}\right)\right|+|g'''(b)|\right],$$

(ii) if $|g'''|$ is decreasing, then we have

$$H \leq \frac{(b-a)^3}{96(3p+1)^{\frac{1}{p}}} \left(\frac{\pi}{2}\right)^{\frac{1}{q}} \left[|g'''(a)| + \left|g'''\left(\frac{a+b}{2}\right)\right|\right],$$

(iii) if $g'''\left(\frac{a+b}{2}\right) = 0$, then we have

$$H \leq \frac{(b-a)^3}{96(3p+1)^{\frac{1}{p}}} \left(\frac{\pi}{4}\right)^{\frac{1}{q}} \left[|g'''(a)| + |g'''(b)|\right],$$

(iv) if $g'''(a) = g'''(b) = 0$, then we have

$$H \leq \frac{(b-a)^3}{48(3p+1)^{\frac{1}{p}}} \left(\frac{\pi}{4}\right)^{\frac{1}{q}} \left|g'''\left(\frac{a+b}{2}\right)\right|.$$

Theorem 2.6. Suppose that $g : J \subseteq [0, +\infty) \to \mathbb{R}$ is a differentiable function such that $g''' \in L[a, b]$, where $a, b \in J$ with $a < b$. If $|g'''|^q$ is MT-convex function on $[a, b]$ and $q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we have

$$\left|g\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b g(x)dx + \frac{(b-a)^2}{24} g''\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^3}{384} K_q,$$

where $K_q = \left(\frac{35\pi}{64}\left|g'''\left(\frac{a+b}{2}\right)\right|^q + \frac{5\pi}{64}|g'''(a)|^q\right)^{\frac{1}{q}} + \left(\frac{5\pi}{64}|g'''(b)|^q + \frac{35\pi}{64}\left|g'''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}$.

Proof . From Lemma 2.1, properties of modulus, and power mean inequality, we have

$$\begin{aligned}
&\left|g\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b g(x)dx + \frac{(b-a)^2}{24} g''\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{(b-a)^3}{96}\left[\int_0^1 t^3 \left|g'''\left(t\frac{a+b}{2} + (1-t)a\right)\right| dt\right. \\
&\left.+ \int_0^1 (1-t)^3 \left|g'''\left(tb + (1-t)\frac{a+b}{2}\right)\right| dt\right] \\
&\leq \frac{(b-a)^3}{96}\left(\int_0^1 t^3 dt\right)^{\frac{1}{p}} \left(\int_0^1 t^3 \left|g'''\left(t\frac{a+b}{2} + (1-t)a\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+ \frac{(b-a)^3}{96}\left(\int_0^1 (1-t)^3 dt\right)^{\frac{1}{p}} \left(\int_0^1 (1-t)^3 \left|g'''\left(tb + (1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Then, by using the MT-convexity of $|g'''|^q$ on $[a, b]$, we have

$$\begin{aligned}
&\int_0^1 t^3 \left|g'''\left(t\frac{a+b}{2} + (1-t)a\right)\right|^q dt \\
&\leq \int_0^1 t^3 \left[\frac{\sqrt{t}}{2\sqrt{1-t}}\left|g'''\left(\frac{a+b}{2}\right)\right|^q + \frac{\sqrt{1-t}}{2\sqrt{t}}|g'''(a)|^q\right] dt \\
&= \frac{35\pi}{256}\left|g'''\left(\frac{a+b}{2}\right)\right|^q + \frac{5\pi}{256}|g'''(a)|^q,
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 (1-t)^3 \left|g'''\left(tb + (1-t)\frac{a+b}{2}\right)\right|^q dt \\
&\leq \int_0^1 (1-t)^3 \left[\frac{\sqrt{t}}{2\sqrt{1-t}}|g'''(b)|^q + \frac{\sqrt{1-t}}{2\sqrt{t}}\left|g'''\left(\frac{a+b}{2}\right)\right|^q\right] dt \\
&= \frac{5\pi}{256}|g'''(b)|^q + \frac{35\pi}{256}\left|g'''\left(\frac{a+b}{2}\right)\right|^q.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&\left|g\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b g(x)dx+\frac{(b-a)^2}{24}g''\left(\frac{a+b}{2}\right)\right|\\&\le \frac{(b-a)^3}{96}\left(\frac{1}{4}\right)^{\frac{1}{p}}\left(\frac{1}{4}\right)^{\frac{1}{q}}K_q.\end{aligned}$$

Hence, our proof is completely done. □

Remark 2.7. If $q=\frac{p}{p-1}$ $(p>1)$, the constants of Theorem 2.4 are improved, since $\frac{1}{(3p+1)^{\frac{1}{p}}}<1$.

Corollary 2.8. Let the assumptions of Theorem 2.4 be valid and let

$$H=\left|g\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b g(x)dx+\frac{(b-a)^2}{24}g''\left(\frac{a+b}{2}\right)\right|.$$

Then,

(i) if $|g'''|$ is increasing, then we have

$$H \le \frac{(b-a)^3}{384}\left(\frac{5\pi}{8}\right)^{\frac{1}{q}}\left[\left|g'''\left(\frac{a+b}{2}\right)\right|+|g'''(b)|\right],$$

(ii) if $|g'''|$ is decreasing, then we have

$$H \le \frac{(b-a)^3}{384}\left(\frac{5\pi}{8}\right)^{\frac{1}{q}}\left[|g'''(a)|+\left|g'''\left(\frac{a+b}{2}\right)\right|\right],$$

(iii) if $g'''\left(\frac{a+b}{2}\right)=0$, then we have

$$H \le \frac{(b-a)^3}{384}\left(\frac{5\pi}{64}\right)^{\frac{1}{q}}\left[|g'''(a)|+|g'''(b)|\right],$$

(iv) if $g'''(a)=g'''(b)=0$, then we have

$$H \le \frac{(b-a)^3}{384}\left(\frac{35\pi}{64}\right)^{\frac{1}{q}}\left|g'''\left(\frac{a+b}{2}\right)\right|.$$

3 Applications

Thanks to the results that we have obtained in the previous section, we can establish applications for special means that will allow us to give an example of a particular function. We present these applications below.

3.1 Applications for special means

Consider the special means of positive real numbers $a>0$ and $b>0$, define by:

- Arithmetic Mean:

$$A(a,b)=\frac{a+b}{2}.$$

- Logarithmic mean:

$$L(a,b)=\frac{b-a}{\ln|b|-\ln|a|},\ |a|\neq|b|,\ a,b\neq 0.$$

- Generalized log-mean:

$$L_p(a,b)=\left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}},\ p\in\mathbb{Z}\setminus\{-1,0\},\ a\neq b.$$

Remark 3.1. Let $p \in (0, \frac{1}{1000})$ and $x > 1$. Then, we consider

$$g(x) = \frac{x^{p+3}}{(p+1)(p+2)(p+3)}, \quad g'''(x) = x^p,$$

therefore $g\left(\frac{a+b}{2}\right) = \frac{A^{p+3}(a,b)}{(p+1)(p+2)(p+3)}$. Furthermore, we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b g(x)dx &= \frac{1}{(p+1)(p+2)(p+3)}\left[\frac{b^{p+4}-a^{p+4}}{(p+4)(b-a)}\right] \\ &= \frac{1}{(p+1)(p+2)(p+3)}L_{p+3}^{p+3}(a,b).\end{aligned}$$

Proposition 3.2. Let $p \in (0, \frac{1}{1000})$ and $b > a > 1$, then we have

$$\begin{aligned}&\left|A^{p+3}(a,b) - L_{p+3}^{p+3}(a,b) + \frac{(b-a)^2(p+2)(p+3)}{24}A^{p+1}(a,b)\right| \\ &\leq \frac{5\pi(b-a)^3(p+1)(p+2)(p+3)}{24576 \times 2^p}\Big[14(a+b)^p + 2^p(a^p+b^p)\Big].\end{aligned}$$

Proof . Since x^p is MT-convex for each $x > 1$ and $p \in (0, \frac{1}{1000})$, so the assertion follows from inequality (2.1) with $g(x) = \frac{x^{p+3}}{(p+1)(p+2)(p+3)}$. □

Proposition 3.3. Let $a, b \in \mathbb{R}$ such that $a < b$ and $[a,b] \subset (0, +\infty)$, then we have

$$\left|A^{-1}(a,b) - L^{-1}(a,b) + \frac{(b-a)^2}{12}A^{-3}(a,b)\right| \leq \frac{5\pi(b-a)^3}{256 \times \beta_1^4}.$$

Proof . The assertion follows from inequality (2.3) with $g(x) = \frac{1}{x}$, $x \in [a,b]$, since $|g'''(x)| = \frac{6}{x^4}$ is decreasing and MT-convex. □

3.2 Application to a particular function

We define $g : \mathbb{R} \to \mathbb{R}$, by $g(x) = e^x$, then we have $g'''(x) = e^x$ is MT-convex.

Applying the inequality (2.2), we can deduce

$$\left|\left(1 + \frac{(b-a)^2}{24}\right)e^{\frac{a+b}{2}} - \frac{1}{b-a}\left(e^b - e^a\right)\right| \leq \frac{5\pi(b-a)^3}{1536}e^b.$$

Particularly for $a = 0$ and $b = x$, it follows that

$$\left|\left(1 + \frac{x^2}{24}\right)e^{\frac{x}{2}} - \frac{1}{x}(e^x - 1)\right| \leq \frac{5\pi x^3}{1536}e^x, \tag{3.1}$$

and for $x = 1$, it follows that

$$\left|\frac{25}{24}\sqrt{e} - e + 1\right| \leq \frac{5\pi e}{1536}.$$

For further illustration on the inequality (3.1), we present some plot example in the Figures 1.

4 Conclusion

A version of Hermite-Hadamard inequality via functions whose absolute value of the third derivative are MT-convex has been acquired successfully. This result combines several versions (new and old) of the Hermite-Hadamard inequality into a single form. We also established some special media applications that allowed us to give some examples for a particular function. Employing the method outlined in this paper, we anticipate that some other inequalities may be reestablished. We hope that the ideas used in this paper may inspire interested readers to explore some new applications.

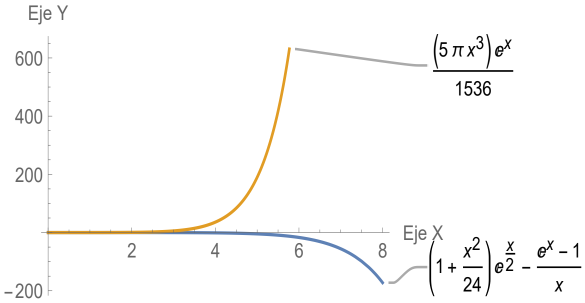

Figure 1: Plot illustration for inequality (3.1).

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