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A novel two-step iterative method based on real interval arithmetic for finding enclosures of roots of systems of nonlinear equations

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Abstract

In the present paper, a novel two-step iterative method, based on real interval arithmetic, is produced. By using this method, we obtain enclosures of roots of systems of nonlinear equations. Discussion on the convergence analysis for the produced method is presented. The efficiency, accuracy, and validity of this method are demonstrated by its application to four implemented examples with INTLAB and by comparing our results with the results obtained by other methods available in the literature.

Keywords: Interval arithmetic; Iterative methods; Systems of nonlinear equations; Convergence analysis
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1 Introduction

The modern history of interval arithmetic can be traced back to the thesis of Moore [30] and the publication of his book Interval Analysis [31]. Following this, a number of conferences devoted to interval analysis were held with accompanying proceedings [33, 34, 35]. Further books on the subject have appeared [2, 23, 47] and bibliographies have been published [17, 18, 19].

In practice, based on interval arithmetic, we can provide rigorous enclosures for solutions of nonlinear equations.

Recently there has been a growing interest in interval arithmetic in a number of application areas [48]. In [5], Chen and Ward provided a tutorial on the use of interval arithmetic in design. Rao and Berke [46] discussed the uncertainty in engineering analysis/design problems and they mention a number of situations where an uncertain parameter can be modeled as an interval number. In manufacturing, the inspection of machine parts can be subject to variations leading naturally to an interval treatment as is discussed in [22]. Interval analysis, applied to vibrate systems, is considered in [7]. In [36, 37, 38], Oppenheimer and Michel provided a basis for interval analysis of linear electrical systems. The process of finding bounds on the parameters or state of a dynamical system is discussed in a series of papers in [27], where interval analytic methods form a central focus. In control systems and robotics, interval analysis

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has been applied to stability analysis [26, 42], model conversion [16], and robot motion planning [43, 44]. For more details concerning the application of interval arithmetic, see [39].

Let

$$F(X) = (f_1(X), f_2(X), \dots, f_n(X))^T = 0, \tag{1.1}$$

be the system of nonlinear equations, where functions $f_1, f_2, \dots, f_n$ are the coordinate functions of F and each of them maps a vector $X = (x_1, x_2, \dots, x_n)$ of the $n-$dimensional space $\mathbb{R}^n$ into the real line $\mathbb{R}$.

One of the basic problems in scientific and engineering applications is to solve systems of nonlinear equations and find solutions of these equations. This problem has fascinated mathematicians for centuries and the literature is full of ingenious methods, discussions of their merits, and shortcomings. For a review on iterative methods, see, for instance, [4, 6, 8, 9, 10, 13, 24, 25, 28, 29, 41, 45, 50, 54, 55, 56, 57].

In recent years, some iterative methods, based on real interval arithmetic, have been proposed and analyzed for solving nonlinear equations and systems of nonlinear equations (see, for instance, [3, 11, 12, 14, 15, 32, 40, 51, 52]). It has been shown that these methods are efficient in their performance.

In this paper, we restrict our attention to the problem of finding enclosures of roots of systems of nonlinear equations in $\mathbb{R}^n$, given by (1.1), one of the oldest and the most important problems in the theory and practice. To do this, we produce a new iterative method based on real interval arithmetic which we feel is more efficient than other methods currently found in the interval analysis literature. Furthermore, we provide error bounds of approximations to the solutions.

2 Preliminaries

2.1 Interval arithmetic

In the following, from [31, 32], we present an introduction about the interval arithmetic.
The set of real intervals are denoted by $\mathbb{IR}$ and each $\boldsymbol{x} \in \mathbb{IR}$ is as follows:

$$\boldsymbol{x} = [\underline{x}, \overline{x}] = \{x \in \mathbb{R} |\ \underline{x} \le x \le \overline{x}\}.$$

Also, $m(\boldsymbol{x}) = (\underline{x} + \overline{x})/2$ and $w(\boldsymbol{x}) = \overline{x} - \underline{x}$ are the mid-point and the width of $\boldsymbol{x} = [\underline{x}, \overline{x}]$, respectively. By using monotonicity properties, the following relations hold:

$$\begin{aligned}
&\boldsymbol{x} - \boldsymbol{y} = [\underline{x} - \overline{y}, \overline{x} - \underline{y}],\\
&\boldsymbol{x} + \boldsymbol{y} = [\underline{x} + \underline{y}, \overline{x} + \overline{y}],\\
&\boldsymbol{x}.\boldsymbol{y} = [\min\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}, \max\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}],\\
&\boldsymbol{x}/\boldsymbol{y} = \boldsymbol{x}.(1/\boldsymbol{y}),\\
&1/\boldsymbol{y} = [1/\overline{y}, 1/\underline{y}], \quad 0 \notin \boldsymbol{y}.
\end{aligned}$$

Note that subtraction and division are not the inverse operations of addition and multiplication, respectively.

Definition 2.1. If $\mathbf{x}^{(i+1)} \subseteq \mathbf{x}^{(i)}$, for all i, then the interval sequence $\mathbf{x}^{(i)}$ is nested.

Definition 2.2. $\mathbf{f}$ is an interval extension of f on $\mathbf{x}$, if it satisfy the following properties:

$$\begin{aligned}
&\mathbf{f}(\mathbf{x}) \supseteq \{f(x) \,|\, x \in \mathbf{x}\}, \quad (inclusion),\\
&\mathbf{f}([x, x]) = f(x), \quad (restriction).
\end{aligned}$$

Definition 2.3. An interval $\mathbf{f}$ is Lipschitz, if for every $\mathbf{x} \subseteq \mathbf{x}^{(0)}$, there is a constant L such that

$$w(\mathbf{f}(\mathbf{x})) \le L\, w(\mathbf{x}).$$

Lemma 2.4 (see [32]). A natural interval extension $\mathbf{f}$, defined on $\mathbf{x} \subseteq \mathbf{x}^{(0)}$, is Lipschitz, i.e.

$$w(\mathbf{f}(\mathbf{x})) \le L\, w(\mathbf{x}).$$

Lemma 2.5 (see [32]). For any $a, b \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{IR}$, the following relations hold:

$$\begin{aligned}
&w(a\mathbf{x} + b\mathbf{y}) = |a| w(\mathbf{x}) + |b| w(\mathbf{y}),\\
&w(\mathbf{x}\mathbf{y}) \le |\mathbf{x}| w(\mathbf{y}) + |\mathbf{y}| w(\mathbf{x}),\\
&w(1/\mathbf{y}) \le |1/\mathbf{y}|^2 w(\mathbf{y}) \quad if \quad 0 \notin \mathbf{y}.
\end{aligned}$$

2.2 Multivariate interval Newton method

Let $M(\mathbb{IR}^{n\times n})$ and $\mathbb{IR}^n$, be the sets of $n \times n$ interval matrices and $n \times 1$ interval vectors, respectively. The multivariate interval Newton method with the second-order convergence rate, introduced by Moore et al. [32] for finding enclosures of roots of systems of nonlinear equations (1.1), is as follows:

$$\boldsymbol{X}^{(i+1)} = \boldsymbol{X}^{(i)} \cap \left\{ m(\boldsymbol{X}^{(i)}) - \left(\boldsymbol{F}'(\boldsymbol{X}^{(i)})\right)^{-1} F(m(\boldsymbol{X}^{(i)}))\right\}, \qquad i = 0, 1, 2, \dots, \tag{2.1}$$

where $\boldsymbol{X}^{(0)} \in \mathbb{IR}^n$ is an initial vector and $\boldsymbol{F}'(\boldsymbol{X}^{(i)}) \in M(\mathbb{IR}^{n\times n})$ is an interval extension of Jacobian matrix of $\boldsymbol{F}$ over $\boldsymbol{X}^{(i)}$. The order of convergence of this sequence is two.

2.3 Singh et al.'s method

For solving system of nonlinear equations, defined by (1.1), Singh et al. [52] proposed the following interval iterative methods, denoted by PM1 and PM2:

PM1 method:

$$\begin{aligned}
&\boldsymbol{N}(\boldsymbol{X}^{(i)}) = m(\boldsymbol{X}^{(i)}) - \left(\boldsymbol{F}'(\boldsymbol{X}^{(i)})\right)^{-1} F(m(\boldsymbol{X}^{(i)})),\\
&\boldsymbol{Y}^{(i)} = \boldsymbol{N}(\boldsymbol{X}^{(i)}) \cap \boldsymbol{X}^{(i)},\\
&\boldsymbol{P}(\boldsymbol{X}^{(i)}) = m(\boldsymbol{Y}^{(i)}) - \left(\boldsymbol{F}'(\boldsymbol{X}^{(i)})\right)^{-1} F(m(\boldsymbol{Y}^{(i)})),\\
&\boldsymbol{X}^{(i+1)} = \boldsymbol{P}(\boldsymbol{X}^{(i)}) \cap \boldsymbol{X}^{(i)}.
\end{aligned} \tag{2.2}$$

PM2 method:

$$\begin{aligned}
&\boldsymbol{N}(\boldsymbol{X}^{(i)}) = m(\boldsymbol{X}^{(i)}) - \left(\boldsymbol{F}'(\boldsymbol{X}^{(i)})\right)^{-1} F(m(\boldsymbol{X}^{(i)})),\\
&\boldsymbol{Y}^{(i)} = \boldsymbol{N}(\boldsymbol{X}^{(i)}) \cap \boldsymbol{X}^{(i)},\\
&\boldsymbol{P}(\boldsymbol{X}^{(i)}) = m(\boldsymbol{Y}^{(i)}) - \left(\boldsymbol{F}'(\boldsymbol{X}^{(i)})\right)^{-1} F(m(\boldsymbol{Y}^{(i)})),\\
&\boldsymbol{Z}^{(i)} = \boldsymbol{P}(\boldsymbol{X}^{(i)}) \cap \boldsymbol{X}^{(i)},\\
&\boldsymbol{S}(\boldsymbol{X}^{(i)}) = m(\boldsymbol{Z}^{(i)}) - \left(\boldsymbol{F}'(\boldsymbol{X}^{(i)})\right)^{-1} F(m(\boldsymbol{Z}^{(i)})),\\
&\boldsymbol{X}^{(i+1)} = \boldsymbol{S}(\boldsymbol{X}^{(i)}) \cap \boldsymbol{X}^{(i)}.
\end{aligned} \tag{2.3}$$

The orders of convergence of these methods are three and four, respectively.

3 New interval iterative method

In this section, a new interval iterative method is produced to find enclosures of roots of systems of nonlinear equations.

Suppose that $X^* \in \boldsymbol{X}^{(0)}$ is a root of $\boldsymbol{F} \in C(\boldsymbol{X}^{(0)})$. Let $\boldsymbol{F}'(\boldsymbol{X}^{(0)})$ be nonsingular. Using (2.1) yields

$$\boldsymbol{X}^* \in \boldsymbol{Y}^{(0)} = \boldsymbol{X}^{(0)} \cap \left\{ m(\boldsymbol{X}^{(0)}) - \left(\boldsymbol{F}'(\boldsymbol{X}^{(0)})\right)^{-1} F(m(\boldsymbol{X}^{(0)}))\right\} \subset \boldsymbol{X}^{(0)}.$$

Let $F(X^*) = 0$ and $Y \in \boldsymbol{Y}^{(0)}$. Taylor expansion of $F(Y)$ about X^* gives

$$F(Y) = F(X^*) + (Y - X^*)F'(\xi),$$

$$F(Y) = F(X^*) + (Y - X^*)F'(\eta),$$

where $\xi, \eta \in \boldsymbol{Y}^{(0)} \subset \boldsymbol{X}^{(0)}$. Let $X \in \boldsymbol{X}^{(0)}$ and ξ, η be two vectors of the same length, parallel and with the same direction with vectors X, Y, respectively. Since f_k, for all k, is a smooth function, it can be supposed that $f_k'(\xi) \approx f_k'(X)$, $f_k'(\eta) \approx f_k'(Y)$ and therefore we have $F'(\xi) \approx F'(X)$, $F'(\eta) \approx F'(Y)$. Hence, we get

$$F(Y) = (Y - X^*)F'(X),$$

$$F(Y) = (Y - X^*)F'(Y).$$

From the above equations, we can write

$$2F(Y) = (Y - X^*)(F'(X) + F'(Y)).$$

Since $F'(X) \in F'(\boldsymbol{X}^{(0)})$, $F'(Y) \in F'(\boldsymbol{Y}^{(0)})$, by using (5), we obtain

$$X^* = Y - 2(F'(X) + F'(Y))^{-1}F(Y) \in Y - 2(F'(\boldsymbol{X}^{(0)}) + F'(\boldsymbol{Y}^{(0)}))^{-1}F(Y),$$

for any $Y \in \boldsymbol{Y}^{(0)}$, in particular for $Y = m(\boldsymbol{Y}^{(0)})$. Hence

$$X^* \in m(\boldsymbol{Y}^{(0)}) - 2(F'(\boldsymbol{X}^{(0)}) + F'(\boldsymbol{Y}^{(0)}))^{-1}F(m(\boldsymbol{Y}^{(0)})).$$

Since $X^* \in \boldsymbol{X}^{(0)}$, we can write

$$\boldsymbol{X}^* \in \boldsymbol{X}^{(1)} := \boldsymbol{X}^{(0)} \cap \left\{ m(\boldsymbol{Y}^{(0)}) - 2(F'(\boldsymbol{X}^{(0)}) + F'(\boldsymbol{Y}^{(0)}))^{-1}F(m(\boldsymbol{Y}^{(0)})) \right\} \subset \boldsymbol{X}^{(0)}.$$

Continuing this process gives

$$\begin{cases} \boldsymbol{Y}^{(i)} = \boldsymbol{X}^{(i)} \cap \left\{ m(\boldsymbol{X}^{(i)}) - \left(\boldsymbol{F}'(\boldsymbol{X}^{(i)})\right)^{-1} F(m(\boldsymbol{X}^{(i)})) \right\}, \\ \boldsymbol{X}^{(i+1)} = \boldsymbol{X}^{(i)} \cap \left\{ m(\boldsymbol{Y}^{(i)}) - 2\left(\boldsymbol{F}'(\boldsymbol{X}^{(i)}) + \boldsymbol{F}'(\boldsymbol{Y}^{(i)})\right)^{-1} F(m(\boldsymbol{Y}^{(i)})) \right\}. \end{cases} \tag{3.1}$$

Therefore, we have produced a new interval iterative method to fined enclosures of roots of systems of nonlinear equations.

4 Convergence analysis

Here, the convergence of the iterative method (3.1) is discussed.

Theorem 4.1. Let $\mathbf{F} \in C(\mathbf{X}^{(0)})$. Suppose that

$$0 \notin \left\{ \mathbf{F}'(\mathbf{X}^{(i)}) \right\}, \ \left\{ \mathbf{F}'(\mathbf{X}^{(i)}) + \mathbf{F}'(\mathbf{Y}^{(i)}) \right\}, \qquad i = 0, 1, 2, \ldots.$$

If $\mathbf{X}^{(0)}$ contains a root X^* of F, then do all intervals $\mathbf{X}^{(i)}, i = 1, 2, \ldots$. Therefore, the nested interval sequence $\{\mathbf{X}^{(i)}\}$, generated by (3.1), converges to X^*.

Proof . By induction, if $X^* \in \boldsymbol{X}^{(0)}$, then $X^* \in \boldsymbol{X}^{(i)}$ for $i = 0, 1, 2, \ldots$. Also, if there is an i such that $X^* = m(\boldsymbol{X}^{(i)})$, then we have $w(\boldsymbol{X}^{(i+1)}) = 0$ and therefore the convergence is proved. Now, let $X^* \neq m(\boldsymbol{X}^{(i)})$ for $i = 0, 1, 2, \ldots$. Since

$$0 \notin \left\{ \boldsymbol{F}'(\boldsymbol{X}^{(i)}) \right\}, \ \left\{ \boldsymbol{F}'(\boldsymbol{X}^{(i)}) + \boldsymbol{F}'(\boldsymbol{Y}^{(i)}) \right\}, \qquad i = 0, 1, 2, \ldots,$$

therefore

$$\left\{ 2\left(\boldsymbol{F}'(\boldsymbol{X}^{(i)}) + \boldsymbol{F}'(\boldsymbol{Y}^{(i)})\right)^{-1} F(m(\boldsymbol{Y}^{(i)})) \right\}, \qquad i = 0, 1, 2, \ldots,$$

consists entirely of elements of the same sign. Thus, the mid-point of $\boldsymbol{X}^{(i)}$ is not contained in $\boldsymbol{X}^{(i+1)}$. Therefore $w(\boldsymbol{X}^{(i+1)}) < \frac{1}{2} w(\boldsymbol{X}^{(i)})$ and the convergence is proved. □

Theorem 4.2. Let $\mathbf{F} \in C(\mathbf{X}^{(0)})$. Suppose that

$$0 \notin \left\{ \mathbf{F}'(\mathbf{X}^{(i)}) \right\}, \ \left\{ \mathbf{F}'(\mathbf{X}^{(i)}) + \mathbf{F}'(\mathbf{Y}^{(i)}) \right\}, \qquad i = 0, 1, 2, \ldots.$$

(i) If $\left\{ m(\mathbf{Y}^{(i)}) - 2\left(\mathbf{F}'(\mathbf{X}^{(i)}) + \boldsymbol{F}'(\mathbf{Y}^{(i)})\right)^{-1} F(m(\mathbf{Y}^{(i)})) \right\} \cap \mathbf{X}^{(i)} = \emptyset$, for $i = 0, 1, 2, \ldots$, then $\mathbf{X}^{(i)}$ contains no roots of F.

(ii) If $\left\{m(\mathbf{Y}^{(i)}) - 2\Big(\mathbf{F}'(\mathbf{X}^{(i)}) + \mathbf{F}'(\mathbf{Y}^{(i)})\Big)^{-1} F(m(\mathbf{Y}^{(i)}))\right\} \subset \mathbf{X}^{(i)}$, for $i = 0, 1, 2, \ldots$, then $\mathbf{X}^{(i)}$ contains exactly one root of F.

In this case, the method (3.1) has three order of convergence.

Proof . (i) Suppose that $\boldsymbol{X}^{(0)}$ contains a root X^*, then Theorem 4.1 results in

$$X^* \in \left\{m(\boldsymbol{Y}^{(i)}) - 2\Big(\boldsymbol{F}'(\boldsymbol{X}^{(i)}) + \boldsymbol{F}'(\boldsymbol{Y}^{(i)})\Big)^{-1} F(m(\boldsymbol{Y}^{(i)}))\right\}, \quad i = 0, 1, 2, \ldots,$$

which means that

$$X^* \in \boldsymbol{X}^{(i)} \cap \left\{m(\boldsymbol{Y}^{(i)}) - 2\Big(\boldsymbol{F}'(\boldsymbol{X}^{(i)}) + \boldsymbol{F}'(\boldsymbol{Y}^{(i)})\Big)^{-1} F(m(\boldsymbol{Y}^{(i)}))\right\}, \quad i = 0, 1, 2, \ldots.$$

Therefore, if

$$\boldsymbol{X}^{(i)} \cap \left\{m(\boldsymbol{Y}^{(i)}) - 2\Big(\boldsymbol{F}'(\boldsymbol{X}^{(i)}) + \boldsymbol{F}'(\boldsymbol{Y}^{(i)})\Big)^{-1} F(m(\boldsymbol{Y}^{(i)}))\right\} = \emptyset, \quad i = 0, 1, 2, \ldots,$$

then $\boldsymbol{X}^{(i)}$ cannot contains a root of F.
(ii) Since $0 \notin \boldsymbol{F}'(\boldsymbol{X}^{(i)})$, thus, for all $X \in \boldsymbol{X}^{(i)}$, $F'(X) \neq 0$ and F is monotonic on $\boldsymbol{X}^{(i)}$. Therefore, from the continuity of F, we can deduce that it has at most one root in $\boldsymbol{X}^{(0)}$. In other words, the function F has at most one root in $\boldsymbol{X}^{(i)}$. Now we try to find a root $X^* \in \boldsymbol{X}^{(i)}$. From Theorem 4.1, the function F has exactly one root in $\boldsymbol{X}^{(i)}$.
Now, we prove the order of convergence of (3.1). We know that the order of convergence of sequence $\{\boldsymbol{Y}^{(i)}\}$, generated by (2.1), is two.

Now consider the Mean Value Theorem as follows:

$$F(m(\boldsymbol{Y}^{(i)})) = (m(\boldsymbol{Y}^{(i)}) - X^*)F'(\delta), \tag{4.1}$$

where δ is between $m(\boldsymbol{Y}^{(i)})$ and X^*. Since

$$\left\{m(\boldsymbol{Y}^{(i)}) - 2\Big(\boldsymbol{F}'(\boldsymbol{X}^{(i)}) + \boldsymbol{F}'(\boldsymbol{Y}^{(i)})\Big)^{-1} F(m(\boldsymbol{Y}^{(i)}))\right\} \subset \boldsymbol{X}^{(i)},$$

from the formulas (3.1) and (4.1), we get

$$\boldsymbol{X}^{(i+1)} = m(\boldsymbol{Y}^{(i)}) - \lambda(m(\boldsymbol{Y}^{(i)}) - X^*)F'(\delta),$$

where

$$\lambda = 2\Big(\boldsymbol{F}'(\boldsymbol{X}^{(i)}) + \boldsymbol{F}'(\boldsymbol{Y}^{(i)})\Big)^{-1}.$$

Therefore

$$w(\boldsymbol{X}^{(i+1)}) = w(\lambda)\,|(m(\boldsymbol{Y}^{(i)}) - X^*)||F'(\delta)|. \tag{4.2}$$

Using Lemma 2.5 yields

$$\begin{aligned} w(\lambda) &= 2\left|\Big(\boldsymbol{F}'(\boldsymbol{X}^{(i)}) + \boldsymbol{F}'(\boldsymbol{Y}^{(i)})\Big)^{-1}\right|^2 w(\boldsymbol{F}'(\boldsymbol{X}^{(i)}) + \boldsymbol{F}'(\boldsymbol{Y}^{(i)})) \\ &= 2\left|\Big(\boldsymbol{F}'(\boldsymbol{X}^{(i)}) + \boldsymbol{F}'(\boldsymbol{Y}^{(i)})\Big)^{-1}\right|^2 \Big(w(\boldsymbol{F}'(\boldsymbol{X}^{(i)})) + w(\boldsymbol{F}'(\boldsymbol{Y}^{(i)}))\Big). \end{aligned} \tag{4.3}$$

Also, by using Lemma 2.4, we can write

$$\begin{aligned} w(\boldsymbol{F}'(\boldsymbol{X}^{(i)})) &\leq L_1 w(\boldsymbol{X}^{(i)}), \\ w(\boldsymbol{F}'(\boldsymbol{Y}^{(i)})) &\leq L_2 w(\boldsymbol{Y}^{(i)}). \end{aligned}$$

Let

$$2\left|\Big(\boldsymbol{F}'(\boldsymbol{X}^{(i)}) + \boldsymbol{F}'(\boldsymbol{Y}^{(i)})\Big)^{-1}\right|^2 \leq K_1, \tag{4.4}$$

$$|F'(\delta)| \leq K_2. \tag{4.5}$$

It is clear that

$$\left|m(\boldsymbol{Y}^{(i)}) - X^*\right| \leq w(\boldsymbol{Y}^{(i)}). \tag{4.6}$$

By using a monotone vector norm, Theorem 4.1, and from (4.3), (4.4), we get

$$\begin{aligned}\|w(\lambda)\| &\leq K_1 \left(L_1\|w(\boldsymbol{X}^{(i)})\| + L_2\|w(\boldsymbol{Y}^{(i)})\|\right)\\ &\leq K_1 \left(L_1\|w(\boldsymbol{X}^{(i)})\| + L_2\|w(\boldsymbol{X}^{(i)})\|^2\right)\\ &\leq K_1 \left(L_1 + L_2\|w(\boldsymbol{X}^{(i)})\|\right)\|w(\boldsymbol{X}^{(i)})\|. \end{aligned} \tag{4.7}$$

From (4.2), (4.5)-(4.7), and Theorem 4.1, we clearly have

$$\|w(\boldsymbol{X}^{(i+1)})\| \leq K_1K_2C\left(L_1 + L_2\|w(\boldsymbol{X}^{(i)})\|\right)\|w(\boldsymbol{X}^{(i)})\|^3.$$

Therefore, the sequence $\{\boldsymbol{X}^{(i+1)}\}$ converges to X^*. □

5 Numerical results

In this section, we consider four examples to demonstrate the efficiency and practicability of the new method. We compare the new method (3.1) with the methods of PM1 (2.2) and PM2 (2.3). All examples are tested using Matlab R2015a and version 8 of INTLAB toolbox, created by Rump [49]. The results are displayed in Tables 1,2,4, and 5. If $\boldsymbol{X}^{(i+1)} = \boldsymbol{X}^{(i)}$ or $\left\|w(\boldsymbol{X}^{(i)})\right\|_\infty = \max_{1\leq k\leq n} w(X^{(i)})_k \leq \varepsilon = 10^{-15}$, then we can determine a final interval vector $\boldsymbol{X}^*$ which contains X^*. The obtained results show that the new method with three order of convergence is faster and more efficient than the methods reported in [52] with three and four orders of convergence.

Example 5.1. Consider the following system of nonlinear equations, studied in [21]:

$$\begin{aligned}&F = (f_1, f_2)\\ &f_1(x_1, x_2) \equiv x_1^2 + x_2^2 - 1 = 0,\\ &f_2(x_1, x_2) \equiv x_1^2 - x_2 = 0,\end{aligned}$$

with

$$\begin{aligned}&X^* = \{0.786151377757423, 0.618033988749895\},\\ &\boldsymbol{X}^{(0)} = \{[0.7, 0.9], [0.5, 0.7]\},\\ &\boldsymbol{X}^* = \{[0.78615137775742, 0.78615137775743], [0.61803398874989, 0.61803398874990]\}.\end{aligned}$$

The obtained results, reported in Table 1, confirm the efficiency and accuracy of the new method.

Table 1: Computation of $\left\|w(\boldsymbol{X}^{(i)})\right\|_\infty$

Number of iterations i	Methods		
	(3.1)	PM1	PM2
1	3.6×10^{-7}	8.7×10^{-4}	5.3×10^{-5}
2	1.1×10^{-16}	1.0×10^{-11}	1.1×10^{-16}
3		1.1×10^{-16}	

Example 5.2. Consider the following system of nonlinear equations, studied in [53]:

$$\begin{aligned}&F = (f_1, f_2, f_3)\\ &f_1(x_1, x_2, x_3) \equiv 10x_1 + \sin(x_1 + x_2) - 1 = 0,\\ &f_2(x_1, x_2, x_3) \equiv 8x_2 - \cos^2(x_3 - x_2) - 1 = 0,\\ &f_3(x_1, x_2, x_3) \equiv 12x_3 + \sin(x_3) - 1 = 0,\end{aligned}$$

with

$$
\begin{aligned}
&X^* = \{0.068978349172667, 0.246442418609183, 0.076928911987537\},\\
&\boldsymbol{X}^{(0)} = \{[0,1],[0,1],[0,1]\},\\
&\boldsymbol{X}^* = \{[0.06897834917266, 0.06897834917267], [0.24644241860918, 0.24644241860919],\\
&\qquad [0.07692891198753, 0.07692891198754]\}.
\end{aligned}
$$

The obtained results, reported in Table 2, confirm the efficiency and accuracy of the new method.

Table 2: Computation of $\left\| w(\boldsymbol{X}^{(i)}) \right\|_\infty$

Number of iterations i	Methods		
	(3.1)	PM1	PM2
1	1.2×10^{-6}	1.5×10^{-2}	1.1×10^{-3}
2	2.7×10^{-17}	3.6×10^{-8}	2.3×10^{-15}
3		2.7×10^{-17}	2.7×10^{-17}

Example 5.3. Consider the following integral equation, studied in [52, 53]:

$$
F(x)(s) = -\frac{x(s)}{4}\int_0^1 \frac{s}{s+t}x(t)dt - 1 + x(s), \quad x \in C[0,1], \quad s \in [0,1].
$$

By using Gauss-Legendre quadrature formula, we transform the above equation into a finite dimensional equation. Therefore, we have

$$
\int_0^1 f(t)dt \approx \sum_{j=1}^{l} w_j f(t_j).
$$

Let $x(t_l) = x_l,\ l = 1, \ldots, 8.$ Therefore, the following system of nonlinear equations can be obtained:

$$
x_l - x_l \sum_{j=1}^{8} a_{lj}x_j - 1 = 0, \quad a_{lj} = \frac{t_l w_j}{4(t_l + t_j)}, \quad l = 1, 2, ..., 8.
$$

In Table 3, values of the abscissas t_j and the weights w_j of the eight-point Gauss-Legendre formula are reported. Let

$$
\begin{aligned}
&X^* = \{1.0217197314617227, 1.073186381733582, 1.125724893656528, 1.169753312169115,\\
&\qquad 1.203071751305358, 1.226490874633312, 1.241524600593500, 1.249448516693481\},\\
&\boldsymbol{X}^{(0)} = \{[0,2],[0,2],[0,2],[0,2],[0,2],[0,2],[0,2],[0,2]\},\\
&\boldsymbol{X}^* = \{[1.02171973146172, 1.02171973146173], [1.07318638173358, 1.07318638173359],\\
&\qquad [1.12572489365652, 1.12572489365653], [1.16975331216911, 1.16975331216912],\\
&\qquad [1.20307175130535, 1.20307175130536], [1.22649087463331, 1.22649087463332],\\
&\qquad [1.24152460059349, 1.24152460059351], [1.24944851669348, 1.24944851669349]\}.
\end{aligned}
$$

The obtained results, reported in Table 4, confirm the efficiency and accuracy of the new method.

Example 5.4. Consider the following boundary value problem, studied in [20, 52]:

$$
y'' = y + \sin(y), \quad y(0) = y(1) = 1.
$$

Let

$$
x_{i+1} = x_i + h, \quad h = \frac{1}{26}, \quad x_0 = 0, \quad x_{26} = 1,
$$

$$
y(x_i) = y_i, \quad y_j^{''} = \frac{y_{j-1} - 2y_j + y_{j+1}}{h^2}, \quad i = 0, 1, \ldots, 25, \quad j = 1, 2, \ldots, 25.
$$

Then, the following system can be obtained:

$$
y_{j-1} - 2y_j + y_{j+1} = h^2\left(\sin(y_j) + y_j\right), \quad j = 1, 2, \ldots, 25.
$$

Table 3: The values of t_j and w_j, for $j = 1, \ldots, 8$.

j	t_j	w_j
1	0.019855071751231884158219574	0.050614268145188129576265677
2	0.10166676129318663020422303	0.11119051722668723527217800
3	0.23723379504183550709113047	0.15685332293894364366898110
4	0.40828267875217509753026193	0.18134189168918099148257522
5	0.59171732124782490246973807	0.18134189168918099148257522
6	0.76276620495816449290886952	0.15685332293894364366898110
7	0.89833323870681336979577696	0.11119051722668723527217800
8	0.98014492824876811584178043	0.050614268145188129576265677

Table 4: Computation of $\left\|w(\boldsymbol{X}^{(i)})\right\|_\infty$

Number of iterations i	Methods		
	(3.1)	PM1	PM2
1	8.0×10^{-14}	1.9×10^{-4}	8.8×10^{-4}
2	2.2×10^{-16}	5.0×10^{-10}	2.2×10^{-16}
3		2.2×10^{-16}	

Also, let

$$\begin{aligned} X^* = \{&0.028276938174808, 0.056637530355823, 0.085165644408382, 0.113945575579911,\\ &0.143062259326381, 0.172601483033092, 0.202650096146265, 0.233296218130643,\\ &0.264629443538373, 0.296741043313635, 0.329724161263333, 0.363674004393813,\\ &0.398688025544152, 0.434866096434958, 0.472310668895130, 0.511126921625317,\\ &0.551422889404289, 0.593309571143194, 0.636901012644448, 0.682314359331687,\\ &0.729669873593532, 0.779090910740754, 0.830703846934660, 0.884637951833381,\\ &0.941025198162492\}, \end{aligned}$$

$$\boldsymbol{X}^{(0)} = \{[0,1], \ldots, [0,1]\},$$

$$\begin{aligned} \boldsymbol{X}^* = \{&[0.02827693817480, 0.02827693817481], [0.05663753035582, 0.05663753035583],\\ &[0.08516564440838, 0.08516564440839], [0.11394557557991, 0.11394557557992],\\ &[0.14306225932638, 0.14306225932639], [0.17260148303309, 0.17260148303310],\\ &[0.20265009614626, 0.20265009614627], [0.23329621813064, 0.23329621813065],\\ &[0.26462944353837, 0.26462944353838], [0.29674104331363, 0.29674104331364],\\ &[0.32972416126333, 0.32972416126334], [0.36367400439381, 0.36367400439382],\\ &[0.39868802554415, 0.39868802554416], [0.43486609643495, 0.43486609643496],\\ &[0.47231066889513, 0.47231066889514], [0.51112692162531, 0.51112692162532],\\ &[0.55142288940428, 0.55142288940429], [0.59330957114319, 0.59330957114320],\\ &[0.63690101264444, 0.63690101264445], [0.68231435933168, 0.68231435933169],\\ &[0.72966987359353, 0.72966987359354], [0.77909091074075, 0.77909091074076],\\ &[0.83070384693465, 0.83070384693467], [0.88463795183338, 0.88463795183339],\\ &[0.94102519816249, 0.94102519816250]\}. \end{aligned}$$

The obtained results, reported in Table 5, confirm the efficiency and accuracy of the new method.

6 Conclusion

In this research study, a novel iterative method was produced to find enclosures of roots of systems of nonlinear equations. Necessary and sufficient conditions about the convergency of the produced method were discussed. Also, the convergence rate was studied. Comparison of this method with the methods available in the literature was performed

Table 5: Computation of $\left\| w(\boldsymbol{X}^{(i)}) \right\|_\infty$

Number of iterations i	Methods		
	(3.1)	PM1	PM2
1	3.3×10^{-16}	8.7×10^{-5}	1.2×10^{-6}
2		1.1×10^{-16}	1.1×10^{-16}

in four examples. As a result, it was found that the best method to find enclosures of roots of systems of nonlinear equations is our new method. The main objective of this study was to try to get the correct enclosures of roots from a practical point of view.

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