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Application of reproducing kernel Hilbert space method for generalized 1-D linear telegraph equation

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Abstract

This paper presents a generalized 1-D linear telegraph equation. We have solved this equation by the Reproducing Kernel Hilbert Space (RKHS) method and compared it with other methods such as fourth-order compact difference and alternating direction implicit schemes and meshless local radial point interpolation (MLRPI). Comparing the results of these three methods and comparing the exact solution, indicating the efficiency and validity of RKHS. The uniformly converges of the computed solution to the analytical solution are proved. Note that the procedure is easy to implement, and it no need discretization, and is mesh-free too.

Keywords: Reproducing Kernel Hilbert Space (RKHS), Telegraph equations, Gram-Schmidt orthogonalization process, Differential equation
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1 Introduction

The telegraph equation plays an important role in mathematics and physics in various fields such as the vibrational systems [3], the propagation of electrical signals and transmission[11, 14]. Some of the particular cases of the telegraph equation are heat diffusion and wave propagation equation. Recently, many authors have studied second–order hyperbolic differential equations by applying several techniques and they have done many attempts to develop and implementation of stable numerical schemes for solving these equations. Many numerical schemes have been proposed to solve the linear hyperbolic telegraph type equations, such as Chebyshev Tau method [23, 2], Combination of boundary knot method and analog equation method [5], fourth-order compact difference, and alternating direction implicit scheme [27] and also in [26], the meshless local radial point interpolation method has been applied on generalized 1-D linear telegraph equation.

Besides, the literature on the development of numerical methods for the nonlinear hyperbolic telegraph type equation has attracted a lot of attention, but we refuse to provide it. The interested reader can see [13].

Suppose $\Omega = (0\,,\,X\,) \subset \mathbb{R}^+$, and

$$\frac{\partial^2 v}{\partial t^2} + c\frac{\partial v}{\partial t} + bv - p\frac{\partial^2 v}{\partial x^2} = f(x,t), \quad (x,t) \in \Omega \times (0,T], \tag{1.1}$$

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with the following conditions

$$v(x,0) = \phi(x), \qquad \frac{\partial v}{\partial t}(x,0) = \psi(x), \tag{1.2}$$

$$v(0,t) = g_l(t), \qquad v(X,t) = g_r(t), \tag{1.3}$$

wherever $b \geq 0$ and $p > 0$ and $c \geq 0$ are all fixed, is the equation for 1-D linear telegraph equation. Assume that the functions ψ and f are sufficiently smooth, furthermore the functions ϕ, g_l, g_r and ψ are given and v is unknown and should be determined.

Note that mesh-based techniques such as the Finite Difference Method (FDM) and Finite Element Method (FEM) use the mesh. Recently (about 20 years), meshless schemes have been created to defeat the difficulties of some mesh-based techniques [16]. For further reading, see [9, 18, 10, 21, 19].

One of the meshfree methods is the Reproducing Kernel Hilbert Space (RKHS) method. In the current work, we focus on the numerical solution of the equation (1.1) with the conditions (1.2)-(1.3) by applying the RKHS method. The theory of this scheme goes back to the paper [29], in which S. Zaremba used this method in the early twentieth century for the first time in his work. His work focused on the boundary value problem (BVP) with Dirichlet condition about biharmonic and harmonic functions [22]. Reproducing kernel Hilbert space method gives the solution of differential equations in convergent series form. Recently, the RKHS method is used to solve some types of linear and nonlinear problems with partial and ordinary differential equations. For example, multiple solutions of nonlinear boundary value problems [1], solving a class of nonlinear integral equations [6], solving a class of singular weakly nonlinear boundary value problems [28], solving non-local fractional heat equations [15], 2-D nonlinear coupled Burgers equations [20], solving system of second-order BVPs using a new algorithm based on reproducing kernel Hilbert space [24], Generalized Jacobi reproducing kernel method in Hilbert spaces for solving the Black-Scholes option pricing problem arising in financial modeling [7], Reproducing kernel method in Hilbert spaces for solving the linear and nonlinear four-point boundary value problems [8] and etc.

In this paper, the algorithm of RKHS for determining the unknown solution of the 1-D linear telegraph equation based on the Reproducing Kernel Hilbert Space, is presented. This scheme has some main advantages such as this scheme is mesh-free, easy to implement, it needs no time discretization and the estimated solution by this technique converges to the exact solution.

The numerical results obtained for the mentioned linear equation in this study reveal that the present method is high accuracy. By focusing on numerical experiments gained by this method with other available methods, and comparing them, we can find that the proposed scheme capable of solving the telegraph equation, playing the role of a powerfully effective, and practical numerical technique.

2 Governing equation

In order to apply the RKHS technique on equation (1.1), A new function space must be constructed, each member of this space satisfies (1.2) -(1.3), then the problem can be solved in the new space. After that, all conditions must be homogeneous. Let us assume

$$w(x\,,\,t) = v(x\,,\,t) + Z(x\,,\,t) - \psi(\,x\,)t - v_0(\,x\,) - Z_0(\,x\,) - \frac{\partial Z_0(\,x\,)}{\partial t}t,$$

where

$$Z(x\,,\,t) = (g_l(\,t\,) - g_r(\,t\,))x - g_l(\,t\,),\;\; Z_0(\,x\,) = Z(x\,,\,0).$$

By this, we have the following new problem

$$\frac{\partial^2 w}{\partial t^2} + c\frac{\partial w}{\partial t} + bw - p\frac{\partial^2 w}{\partial x^2} = F(x,\,t)\,, \quad (x\,,\,t) \in \Omega \times (0\,,\,T], \tag{2.1}$$

with initial conditions

$$w(x\,,\,0) = 0\,, \quad \frac{\partial w}{\partial t}(x\,,\,0) = \,0\,, \tag{2.2}$$

and boundary conditions

$$w(0\,,\,t) = 0\,, \quad w(X\,,\,t) = 0\,, \tag{2.3}$$

that

$$F(x,\ t)=\frac{\partial^2 Z(x\ ,\ t)}{\partial t^2}+c\left(\frac{\partial Z(x\ ,\ t)}{\partial t}-\psi(\ x\ )-\frac{\partial Z_0(\ x\ )}{\partial t}\right)$$
$$+\,b\left(Z(x\ ,\ t)-\psi(\ x\ )\,t\ -v_0(\ x\ )-Z_0(\ x\ )-\frac{\partial Z_0(\ x\ )}{\partial t}\ t\right)+p\left[\frac{\partial^2\psi(x)}{\partial x^2}t+\frac{\partial^2 v_0(\ x\ )}{\partial x^2}\right]\ +\ f(x\ ,\ t)\ .$$

3 Preliminaries

We review some of the important properties of the RKHS method and its basic definition in this section, so we refuse to include duplicate concepts in other related articles and unnecessary content but can refer to [6, 17, 25] for more information.

Definition 3.1. [4] For an nonempty set $\Gamma\subset\mathbb{R}$, let H is a Hilbert space of real functions on Γ. We state that H be a RKHS if $\forall\, y\in\Gamma$ there exist a unique function $K_y\in H$, which

$$g(y)=\langle g(.),K_y(.)\rangle\ ,\ \forall g\in H, \tag{3.1}$$

that $\langle .\ ,\ .\rangle_{\mathbf{H}}$ indicates the inner product of the Hilbert space H, also K_y is named the reproducing kernel function.

Definition 3.2. [4] The RKHS $W_2^m[a\ ,\ b]$ is provided as $W_2^m[a\ ,\ b]=\{w|w,\ \ w',\ \ldots,\ w^{(m-1)}$ are functions that are absolutely continuous with real values in $[a\ ,\ b]$ and $w^{(m-1)}\in L^2[a\ ,\ b]\}$.

The inner product in $W_2^m[a,b]$ as follows

$$\langle w_1(.),w_2(.)\rangle_{W_2^m}=\sum_{i=0}^{m-1}w_1^{(i)}(a)w_2^{(i)}(a)+\int_a^b w_1^{(m)}(\eta)w_2^{(m)}(\eta)d\eta, \tag{3.2}$$

for every $w_1,\ w_2\in W_2^m[a,b]$.

It is taken away that the presentation form of the Reproducing Kernel function $K_y\in W_2^m[a,b]$ is obtained by solving the following generalized system

$$(-1)^m\frac{\partial^{2m}K_y(x)}{\partial x^{2m}}=\delta(x-y),$$
$$\frac{\partial^i K_y(a)}{\partial x^i}-(-1)^{m\,-\,i\,-\,1}\frac{\partial^{2m\,-\,i\,-\,1}K_y(a)}{\partial x^{2m\,-\,i\,-\,1}}=0\ ,\ i=\ 0,\ 1,\ \ldots,\ m-1,$$
$$\frac{\partial^{2m\,-\,i\,-\,1}K_y(b)}{\partial x^{2m\,-\,i\,-\,1}}=0,\ i=\ 0\ ,\ 1,\ \ldots,\ m-1,$$

that δ represents the Dirac's delta function, [6].

Definition 3.3. The RKHS ${}^0W_2^3[0,X]$ is given as ${}^0W_2^3[0,X]=\{w|w\in W_2^3[0,X]\ \mathit{and}\ w(0)=w(X)=0\}$.

The reproducing kernel function R_η of ${}^0W_2^3[0\ ,\ X]$ is defined as

$$R_\eta(x)=\begin{cases}\mathbf{P}_\eta(x),\ x\leqslant\eta,\\ \mathbf{P}_x(\eta),\ x>\eta,\end{cases} \tag{3.3}$$

where

$$\mathbf{P}_\eta(x)=\Big(156x^5+4320x\eta-3600x^2\eta-1200x^3\eta-180x^4\eta-120x^5\eta-3600x\eta^2+3780x^2\eta^2+$$
$$1260x^3\eta^2+150x^4\eta^2-30x^5\eta^2-1200x\eta^3-300x^2\eta^3-100x^3\eta^3+50x^4\eta^3-10x^5\eta^3+$$
$$600x\eta^4+150x^2\eta^4+50x^3\eta^4-25x^4\eta^4+5x^5\eta^4-120x\eta^5-30x^2\eta^5-10x^3\eta^5+$$
$$5x^4\eta^5-x^5\eta^5\Big)/18720. \tag{3.4}$$

Definition 3.4. The RKHS ${}^0\overline{W}_2^3[0,T]$ is presented as ${}^0\overline{W}_2^3[0,T] = \{w | w \in W_2^3[0,T] \ and \ w(0) = w'(T) = 0\}$.

The reproducing kernel function r_ξ of ${}^0\overline{W}_2^3[0,T]$ is defined as

$$r_\xi(t) = \begin{cases} \mathbf{p}_\xi(t), & t \leqslant \xi, \\ \mathbf{p}_t(\xi), & t > \xi, \end{cases} \tag{3.5}$$

where

$$\mathbf{p}_\xi(t) = \frac{(t^5 - 5t^4\xi + 30t^2\xi^2 + 10t^3\xi^2)}{120}. \tag{3.6}$$

Definition 3.5. [4] Suppose $\overline{\Omega} = \Omega \times (0\ ,\ T]$. The binary function space $W_2^{(m,n)}(\overline{\Omega})$ is defined as $W_2^{(m,n)}(\overline{\Omega}) = \{w | \frac{\partial^{m+n-2} w(x\ ,\ t)}{\partial x^{m-1}\partial t^{m-1}}$ is completely continuous in $\overline{\Omega}$ and $\frac{\partial^{(m+n)} w(x\ ,\ t)}{\partial x^m \partial t^n} \in L^2(\overline{\Omega})\}$.

The inner product in $W_2^{(m,n)}(\overline{\Omega})$ as follows

$$\langle w_1(.,:), w_2(.,:)\rangle_{W_2^{(m,n)}} = \sum_{i=0}^{m-1} \int_0^T \left[\frac{\partial^n}{\partial t^n}\frac{\partial^i}{\partial x^i} w_1(x,t) \frac{\partial^n}{\partial t^n}\frac{\partial^i}{\partial x^i} w_2(x,t)\right]_{x=0} dt +$$
$$\sum_{j=0}^{n-1} \left\langle \frac{\partial^j}{\partial t^j} w_1(x,t)|_{t=0}, \frac{\partial^j}{\partial t^j} w_2(x,t)|_{t=0} \right\rangle_{W_2^m} + \int_0^T \int_0^X \left[\frac{\partial^m}{\partial x^m}\frac{\partial^n}{\partial t^n} w_1(x,t) \frac{\partial^m}{\partial x^m}\frac{\partial^n}{\partial t^n} w_2(x,t)\right] dxdt, \tag{3.7}$$

for every binary functions $w_1, w_2 \in W_2^{(m,n)}(\overline{\Omega})$. The binary function space ${}^0W_2^{(m,n)}(\overline{\Omega})$ is defined as

$${}^0\mathbf{W}_2^{(m,n)}(\overline{\Omega}) = \{u | w \in W_2^{(m,n)}(\overline{\Omega}), \ and \ w(x,\ 0) = w(0,\ t) = w(X,\ t) = w_t(x,\ 0) = 0\}.$$

It can be proved that, $W_2^{(m,n)}(\overline{\Omega})$ and ${}^0W_2^{(m,n)}(\overline{\Omega})$ are reproducing kernel Hilbert spaces and in our special case we put $m = n = 3$, the reproducing kernel function $\mathbf{K}_{(\eta,\ \xi)}$ of ${}^0W_2^{(3,3)}(\overline{\Omega})$ is

$$\mathbf{K}_{(\eta,\xi)}(x,\ t) = R_\eta(x) r_\xi(t).$$

4 Main Idea and Theoretical Discussion

To solve our problem, we choose the operator $L :\ {}^0W_2^{(3,\ 3)}(\overline{\Omega}) \longrightarrow W_2^{(1,\ 1)}(\overline{\Omega})$ by

$$Lw = \frac{\partial^2 w}{\partial t^2} + c\,\frac{\partial w}{\partial t} + b\,w - p\,\frac{\partial^2 w}{\partial x^2},$$

then the problem (2.1)-(2.3) transformed into

$$\begin{cases} Lw(x\ ,\ t) = F(x,\ t)\ , & (x\ ,\ t) \in \Omega \times (0\ ,\ T], \\ w(x\ ,\ 0) = 0\ , & \frac{\partial w}{\partial t}(x\ ,\ 0) = 0\ , \\ w(0\ ,\ t) = 0\ , & w(X\ ,\ t) = 0. \end{cases} \tag{4.1}$$

Theorem 4.1. The operator L is a linear and bounded operator.

Proof . The linearity of L is obvious, then we will prove it is bounded as follow

$$\|Lw\|^2_{W_2{}^{(1,1)}} = \langle Lw(x\ ,\ t), Lw(x\ ,\ t)\rangle_{W_2{}^{(1,1)}} =$$
$$\int_0^T [Lw(0\ ,\ t)]^2\, dt + \langle Lw(x\ ,\ 0), Lw(x\ ,\ 0)\rangle_{W_2^1} + \int_0^T \int_0^X \left[\frac{\partial}{\partial x}\frac{\partial}{\partial t} Lw(x\ ,\ t)\right]^2 dxdt =$$
$$\int_0^T [Lw(0\ ,\ t)]^2\, dt + [Lw(0\ ,\ 0)]^2 + \int_0^X \left[\frac{\partial}{\partial x} Lw(x\ ,\ 0)\right]^2 dx + \int_0^T \int_0^X \left[\frac{\partial}{\partial x}\frac{\partial}{\partial t} Lw(x\ ,\ t)\right]^2 dxdt.$$

Since $w(x\,,\,t)=\langle w(.\,,\,:),\mathbf{K}_{(x\,,\,t)}(.\,,\,:)\rangle_{{}^0W_2^{(3,3)}}\,,Lw(x\,,\,t)=\langle w(.\,,\,:),L\mathbf{K}_{(x\,,\,t)}(.\,,\,:)\rangle_{{}^0W_2^{(3,3)}}$, because the reproducing function $\mathbf{K}_{(x,t)}$ is continuous, thus

$$|Lw(x\,,\,t)|\le\|w\|_{{}^0W_2^{(3,3)}}\|L\mathbf{K}_{(x\,,\,t)}\|_{{}^0W_2^{(3,3)}}\le C_0\|w\|_{{}^0W_2^{(3,3)}}.$$

Similarly

$$\frac{\partial}{\partial x}Lw(x\,,\,t)=\left\langle w(\eta\,,\,\xi),\frac{\partial}{\partial x}L\mathbf{K}_{(x\,,\,t)}(\eta\,,\,\xi)\right\rangle_{{}^0W_2^{(3,3)}},$$

$$\frac{\partial}{\partial t}\frac{\partial}{\partial x}Lw(x\,,\,t)=\left\langle w(\eta\,,\,\xi),\frac{\partial}{\partial t}\frac{\partial}{\partial x}L\mathbf{K}_{(x\,,\,t)}(\eta\,,\,\xi)\right\rangle_{{}^0W_2^{(3,3)}},$$

and then

$$|\frac{\partial}{\partial x}Lw(x\,,\,t)|\le C_1\|w\|_{{}^0W_2^{(3,3)}},$$

$$|\frac{\partial}{\partial t}\frac{\partial}{\partial x}Lw(x\,,\,t)|\le C_2\|w\|_{{}^0W_2^{(3,3)}}.$$

Therefore

$$\|Lw(x\,,\,t)\|^2_{W_2^{(1,1)}}\le(C_0^2+C_1^2+C_2^2)\|w\|_{{}^0W_2^{(3,3)}}\le C^2\|w\|_{{}^0W_2^{(3,3)}}.$$

□

Now, we select a subset that dense and countable, such that $\{(x_1\,,\,t_1),(x_2\,,\,t_2),\ldots\}\ in\ \overline{\Omega}$, then assume

$$\phi_i(x\,,\,t)=\mathbf{S}_{(x_i\,,\,t_i)}(x\,,\,t),\ \psi_i(x\,,\,t)=L^*\phi_i(x\,,\,t),$$

that L^* and L are the adjoint operator of each other and the binary function $\mathbf{S}_{(\eta,\xi)}$ is the reproducing kernel function of $W_2^{(1,1)}(\overline{\Omega})$.

Lemma 4.2. [4] Let $\{(x_i,t_i)\}_{i=1}^{\infty}$ on ${}^0W_2^{(3,3)}(\overline{\Omega})$ be dense, so the complete system of ${}^0W_2^{(3,3)}(\overline{\Omega})$ is $\{\psi_i(x,t)\}_{i=1}^{\infty}$ and $\psi_i(x\,,\,t)=L_{(y,s)}\mathbf{K}_{(y,\,s)}(x\,,\,t)|_{(y\,,\,s)=(x_i\,,\,t_i)}$.

We apply the Gram-Schmidt orthogonalization process on the complete system $\{\psi_i(x,t)\}_{i=1}^{\infty}$ and derived the system $\{\overline{\psi_i}(x,t)\}_{i=1}^{\infty}$, which is orthonormal, as

$$\overline{\psi_i}(x\,,\,t)=\sum_{k=1}^{i}\beta_{ik}\psi_k(x\,,\,t),\qquad(i=\,1,\,2,\,\ldots).\tag{4.2}$$

To orthonormalize the sequence $\{\overline{\psi_i}(x,t)\}_{i=1}^{\infty}$ in the reproducing kernel space by Gram-Schmidt process, orthogonal coefficients β_{ik} given by

$$\beta_{11}=\frac{1}{\|\psi_1\|},\qquad\beta_{ii}=\frac{1}{\sqrt{\|\psi_i\|^2-\sum\limits_{k=1}^{i-1}b_{ik}^2}},\qquad\beta_{ij}=\frac{-\sum\limits_{k=j}^{i-1}b_{ik}\beta_{kj}}{\sqrt{\|\psi_i\|^2-\sum\limits_{k=1}^{i-1}b_{ik}^2}},\tag{4.3}$$

where $b_{ik}=\langle\psi_i\,,\,\overline{\psi_k}\rangle_{{}^0W_2^{(3,3)}}$.

Theorem 4.3. If $\{(x_i,t_i)\}_{i=1}^{\infty}$ be dense in $\overline{\Omega}$ and the problem (1.1)-(1.3) has a unique solution, then (4.1) in ${}^0W_2^{(3,3)}(\overline{\Omega})$ as follows

$$w(x\,,\,t)=\sum_{i=1}^{\infty}\sum_{j=1}^{i}\beta_{ij}\,F(x_j,\,t_j)\,\overline{\psi_i}(x\,,\,t).\tag{4.4}$$

So, we can estimate the approximate solution by taking N-terms intercept of the series display of w as the following

$$w_N(x\,,\,t)=\sum_{i=1}^{N}\sum_{j=1}^{i}\beta_{ij}\,F(x_j\,,\,t_j)\,\overline{\psi_i}(x\,,\,t),\tag{4.5}$$

which $w_N(x\,,\,t)$ converges to the exact solution when $N\to\infty$.

Proof . proved in section 5. □ **Algorithm** 4.1**.** The following steps exist for approximating the solution by applying Gram-Schmidt orthogonal process:

Step1. Set $\mathbf{K}_{(y,\ s)}(x\ ,\ t)$ according to $(x,t)\in\Omega\times(0,T]$.

Step2. For $i=1,2,...,N$ set $x_i\ =\ t_i\ =\ \dfrac{i}{N}$.

Set $\psi_i(x\ ,\ t)=L_{(y,s)}\mathbf{K}_{(y,\ s)}(x\ ,\ t)|_{(y\ ,\ s)=(x_i\ ,\ t_i)}$.

Step3. Set $\beta_{11}=\frac{1}{\|\psi_1\|}$, Then for $i=1,2,...,N^2$ and $i=1,2,...,N^2$ if $i\neq j$ then set $\beta_{ij}=\dfrac{-\sum_{k=j}^{i-1}\langle\psi_i\ ,\ \overline{\psi_k}\rangle_{{}_0W_2^{(3,3)}}\beta_{kj}}{\sqrt{\|\psi_i\|^2-\sum_{k=1}^{i-1}b_{ik}^2}}$

and Else $\beta_{ii}=\dfrac{1}{\sqrt{\|\psi_i\|^2-\sum_{k=1}^{i-1}b_{ik}^2}}$.

Step4. For $i=1,2,...,N^2$ set $\overline{\psi_i}(x\ ,\ t)=\sum_{k=1}^{i}\beta_{ik}\psi_k(x\ ,\ t)$.

Step5. Set $w_0(x_1,\ t_1)\ =\ w(x_1,\ t_1)$.

Step6. Set $n\ =\ 1$.

Step7. Set $B_n\ =\ \sum_{j=1}^{n}\beta_{nj}\ F(x_j\ ,\ t_j)$.

Step8. Set $w_n(x_1,\ t_1)\ =\ \sum_{j=1}^{n}B_j\ \overline{\psi_j}(x\ ,\ t)$.

Step9. If $n\leq N^2$ then set $n\ =\ n\ +1$ and go to step 7.

Else stop.

5 Convergence analysis

We discuss the convergence features of the mentioned method in this section and show convergence of the estimated solution $w_N(x\ ,\ t)$ to the exact solution $w(x\ ,\ t)$. We present lemmas and theorems related to the subject of discussion as follow:

Lemma 5.1. If $\|w_N-\overline{w}\|_{{}_0W_2^{(3,3)}}\to 0$, $\|w_N\|_{{}_0W_2^{(3,3)}}$ is bounded, $(x_N\ ,\ t_N)\to(\eta\ ,\ \xi)$ and $F(x,\ t)$ is continuous, thus

$$F(x_N,\ t_N\ )\to F(\eta\ ,\ \xi).$$

Proof . Note that

$$\begin{aligned}|w(x,t)|&=|\left\langle w(\eta,\xi),\mathbf{K}_{(x,t)}(\eta,\xi)\right\rangle_{{}_0W_2^{(3,3)}}|\leq\|w(\eta,\xi)\|_{{}_0W_2^{(3,3)}}\|\mathbf{K}_{(x,t)}(\eta,\xi)\|_{{}_0\mathbf{W}_2^{(3,3)}}\\&\leq c_1\|w(\eta,\xi)\|_{{}_0W_2^{(3,3)}},\\|w_t(x,t)|&=|\left\langle w(\eta,\xi),\frac{\partial\mathbf{K}_{(x,t)}(\eta,\xi)}{\partial t}\right\rangle_{{}_0W_2^{(3,3)}}|\leq\|w(\eta,\xi)\|_{{}_0W_2^{(3,3)}}\|\frac{\partial\mathbf{K}_{(x,t)}(\eta,\xi)}{\partial t}\|_{{}_0W_2^{(3,3)}}\\&\leq c_2\|w(\eta,\xi)\|_{{}_0W_2^{(3,3)}},\\|w_x(x,t)|&=|\left\langle w(\eta,\xi),\frac{\partial\mathbf{K}_{(x,t)}(\eta,\xi)}{\partial x}\right\rangle_{{}_0W_2^{(3,3)}}|\leq\|w(\eta,\xi)\|_{{}_0W_2^{(3,3)}}\|\frac{\partial\mathbf{K}_{(x,t)}(\eta,\xi)}{\partial x}\|_{{}_0W_2^{(3,3)}}\\&\leq c_3\|w(\eta,\xi)\|_{{}_0W_2^{(3,3)}},\\|w_{tt}(x,t)|&=|\left\langle w(\eta,\xi),\frac{\partial^2\mathbf{K}_{(x,t)}(\eta,\xi)}{\partial t^2}\right\rangle_{{}_0W_2^{(3,3)}}|\leq\|w(\eta,\xi)\|_{{}_0W_2^{(3,3)}}\|\frac{\partial^2\mathbf{K}_{(x,t)}(\eta,\xi)}{\partial t^2}\|_{{}_0W_2^{(3,3)}}\\&\leq c_4\|w(\eta,\xi)\|_{{}_0W_2^{(3,3)}},\end{aligned}$$

$$\begin{aligned}|w_{xx}(x,t)|&=|\left\langle w(\eta,\xi),\frac{\partial^2\mathbf{K}_{(x,t)}(\eta,\xi)}{\partial x^2}\right\rangle_{{}_0W_2^{(3,3)}}|\leq\|w(\eta,\xi)\|_{{}_0W_2^{(3,3)}}\|\frac{\partial^2\mathbf{K}_{(x,t)}(\eta,\xi)}{\partial x^2}\|_{{}_0W_2^{(3,3)}}\\&\leq c_5\|w(\eta,\xi)\|_{{}_0W_2^{(3,3)}},\end{aligned}$$

and

$$\begin{aligned}|w_{N-1}(\eta,\xi)-\overline{w}(\eta,\xi)| &= |\left\langle w_{N-1}(\eta,\xi)-\overline{w}(\eta,\xi),\mathbf{K}_{(x,t)}(\eta,\xi)\right\rangle_{{}^0W_2^{(3,3)}}|\\ &\le \|w_{N-1}(\eta,\xi)-\overline{w}(\eta,\xi)\|_{{}^0W_2^{(3,3)}}\|\mathbf{K}_{(x,t)}(\eta,\xi)\|_{{}^0W_2^{(3,3)}}\\ &\le c_6\|w_{N-1}(\eta,\xi)-\overline{w}(\eta,\xi)\|_{{}^0W_2^{(3,3)}}.\end{aligned}$$

Again

$$\begin{aligned}|w_{N-1}(x_N,t_N)-\overline{w}(\eta,\xi)| &= |w_{N-1}(x_N,t_N)-w_{N-1}(\eta,\xi)+w_{N-1}(\eta,\xi)-\overline{w}(\eta,\xi)|\\ &\le |\nabla w_{N-1}(\eta,\xi)|.\|(x_N,t_N)-(\eta,\xi)\|+|w_{N-1}(\eta,\xi)-\overline{w}(\eta,\xi)|.\end{aligned}$$

From assumptions, we have

$$|w_{N-1}(\eta,\xi)-\overline{w}(\eta,\xi)|\to 0,\ \mathit{and}\ |\nabla w_{N-1}(\eta,\xi)|\le\sqrt{c_2^2+c_3^2}\|w\|_{{}^0W_2^{(3,3)}}.$$

Thus

$|w_{N-1}(x_N,t_N)-\overline{w}(\eta,\xi)|\to 0$, as $N\to\infty$.
Since $F(x\ ,\ t\ )$ is continuous, then $F(x_N,t_N)\to F(\eta,\xi)$, as $N\to\infty$. □

Theorem 5.2. Suppose that in (4.5), $\|w_N\|_{{}^0W_2^{(3,3)}}$ is bounded and the problem (4.1) has a unique solution. If $\{(x_i,t_i)\}_{i=1}^{\infty}$ be dense in $\overline{\Omega}$, then the w_N converges to the w of the problem(Note that w is the exact solution and w_N is the N-terms approximate solution). and

$$w(x,t)=\sum_{i=1}^{\infty}\sum_{j=1}^{i}\beta_{ij}\,F(\,x_j,\,t_j)\,\overline{\psi_i}(x\ ,\ t). \tag{5.1}$$

Proof . At first, the proof of the convergence of $\|w_N\|_{{}^0W_2^{(3,3)}}$ is done. According to (4.5), we conclude that

$$w_{N+1}(x\ ,\ t)=w_N(x\ ,\ t)+B_{N+1}\overline{\psi}_{N+1}(x\ ,\ t),$$

where $B_i=\sum\limits_{j=1}^{i}\beta_{ij}\,F(x_j\ ,\ t_j\ )$. By orthonormality of the $\{\overline{\psi_i}\}_{i=1}^{\infty}$, we have

$$\|w_{N+1}\|^2_{{}^0W_2^{(3,3)}}=\|w_N\|^2_{{}^0W_2^{(3,3)}}+B^2_{N+1}. \tag{5.2}$$

In (5.2) , it holds that $\|w_N\|_{{}^0W_2^{(3,3)}}\le\|w_{N+1}\|_{{}^0W_2^{(3,3)}}$. Due to $\|w_N\|_{{}^0W_2^{(3,3)}}$ being bounded, $\|w_N\|_{{}^0W_2^{(3,3)}}$ is convergent as $N\to\infty$ and we have $\sum\limits_{j=1}^{\infty}B_j^2=c$ where c is a constant. As a result $\{B_j\}_{j=1}^{\infty}\in l^2$.

If $M>N$, so

$$\begin{aligned}\|w_M-w_N\|^2_{{}^0W_2^{(3,3)}} &= \|w_M-w_{M-1}+w_{M-1}-w_{M-2}+...+w_{N+1}-w_N\|^2_{{}^0W_2^{(3,3)}}\\ &= \|w_M-w_{M-1}\|^2_{{}^0W_2^{(3,3)}}+\|w_{M-1}-w_{M-2}\|^2_{{}^0W_2^{(3,3)}}+...+\|w_{N+1}-w_N\|^2_{{}^0W_2^{(3,3)}}.\end{aligned}$$

Because $\|w_M-w_{M-1}\|^2_{{}^0W_2^{(3,3)}}=B_M^2$, consequently, $\|w_M-w_N\|^2_{{}^0W_2^{(3,3)}}=\sum\limits_{j=N+1}^{M}B_j^2\to 0$ as $N\to\infty$. Since ${}^0W_2^{(3,3)}(\overline{\Omega})$ is the the complete system, shows that $w_N\to\overline{w}$ as $N\to\infty$. In the following, we will prove that $\overline{w}$ is the solution of (4.1). Taking limits in (4.5) , we get $\overline{w}(x,t)=\sum\limits_{i=1}^{\infty}B_i\overline{\psi_i}(x,t)$. Note that $L\overline{w}(x,t)=\sum\limits_{i=1}^{\infty}B_iL\overline{\psi_i}(x,t)$, and

$$\begin{aligned}L\overline{w}(x_l,t_l) &= \sum_{i=1}^{\infty}B_iL\overline{\psi_i}(x_l,t_l)=\sum_{i=1}^{\infty}B_i\left\langle L\overline{\psi_i}(x,t),\phi_l(x,t)\right\rangle_{W_2{}^{(1,1)}}\\ &= \sum_{i=1}^{\infty}B_i\left\langle\overline{\psi_i}(x,t),L^*\phi_l(x,t)\right\rangle_{{}^0W_2^{(3,3)}}=\sum_{i=1}^{\infty}B_i\left\langle\overline{\psi_i}(x,t),\psi_l(x,t)\right\rangle_{{}^0W_2^{(3,3)}}.\end{aligned}$$

Thus

$$\sum_{l=1}^{i}\beta_{il}L\overline{w}(x_l,t_l)=\sum_{j=1}^{\infty}B_j\left\langle\overline{\psi_j}(x,t),\sum_{l=1}^{i}\beta_{il}\psi_l(x,t)\right\rangle_{{}^0W_2^{(3,3)}}=\sum_{j=1}^{\infty}B_j\left\langle\overline{\psi_j}(x,t),\overline{\psi_i}(x,t)\right\rangle_{{}^0W_2^{(3,3)}}=B_i.$$

Hence by definition of B_i, we have

$$L\overline{w}(x_l\ ,\ t_l) = F(x_l\ ,\ t_l\ ).$$

Since $\{(x_i, t_i)\}_{i=1}^{\infty}$ be dense in $\overline{\Omega}$, $\forall\,(\eta,\xi) \in \overline{\Omega}$, there exists a subsequence $\{(x_{N_i}, t_{N_i})\}_{i=1}^{\infty}$ such that $(x_{N_i}, t_{N_i}) \to (\eta, \xi)$ when $i \to \infty$. According to lemma 5.1 and since the function F is continuous, will have

$$L\overline{w}(\eta\ ,\ \xi) = F(\eta\ ,\ \xi\ ),$$

which indicates that $\overline{w}(x, t)$ satisfies (4.1). □

6 Numerical experiences

In this section, based on the previous discussion, some numerical examples are given to illustrate the accuracy, validity of the proposed technique. In all examples, the results of the present method are computed by Mathematica 10 software and are compared with methods in [26, 27] and with exact solution. In order to test our scheme, we estimate the error, the computational order of convergence (COC) by the following formula [12]

$$COC\ =\ \frac{ln\ |\ \dfrac{e_{n+1}}{e_n}\ |}{ln\ |\ \dfrac{e_n}{e_{n-1}}\ |}\ ,$$

where e_n is the absolute error of $n-th$ order approximation. We have used COC to check the accuracy and validity of the proposed method. Results which are obtained by this scheme show a proper agreement with exact solution. The stability and consistency of this technique cause this method to be more applicable and reliable.

Example 1. Consider (1.1) with $p = 1$, $b = 25$, $c =\ 20$ then

$$f(x\ ,\ t) = (\mathbf{e}^t +\ c\,\mathbf{e}^t\ )(x^2(1\ - x\ )^2) + b\,\mathbf{e}^t\ (x^2(1\ -\ x)^2) - p\,\mathbf{e}^t\ (12x^2 - 12x + 2),$$

with the following conditions

$$w(x\ ,\ 0) =\ x^2(1 -\ x)^2,\quad \frac{\partial w}{\partial t}(x, 0) = x^2(1 -\ x)^2,$$

$$w(0\ ,\ t) =\ w(1, t) = 0,\qquad t \geq 0,$$

on $\overline{\Omega} = [0,\ 1] \times [0,\ 1]$. And $w(x\ ,\ t) =\ \mathbf{e}^t\ x^2\ (1 -\ x)^2$ is the exact solution.

By using the RKSH method, taking $x_i\ = t_i\ = \dfrac{i}{N}$, $i = 1, 2, ..., N$, with the reproducing kernel function $\mathbf{K}_{(\eta,\xi)}(x, t)$ on region $\overline{\Omega}$, the approximate solution w_n(that $n = N^2$) is obtained by (4.5). The exact and approximated solutions, the computational order of convergence (COC) along with the absolute errors at some nodal points for $N\ =\ 9$ with $T = 1$ have been shown in Table (1). From this Table, it is observed that the numerical results are very close to the exact solution and we achieved an excellent approximation for the exact solution by employing current scheme. Note that according to Table (1), it is seen that in some points such as $(0.72, 0.72)$, the computational order of convergence gets a negative value for the following reasons:

1. $e_{n-1} < e_n$ and $e_{n+1} < e_n$.
2. $e_n < e_{n-1}$ and $e_n < e_{n+1}$.

We also see that COC has taken a large amount in some points such as $(0.3, 0.3)$ or $(0.36, 0.36)$, because

$e_{n+1} << e_n$ or $e_n << e_{n+1}$.

Table (2), compares the L^{∞} and L^2 errors computed by three schemes for this example with $\Delta x\ =\ \Delta t = \dfrac{1}{10}$ with $T = 1$. From Table (2), it was found the our method in comparison with mentioned methods is better with view to utilization and accuracy. Figure (1) compares the exact and approximated solution which confirms the reliability of RKHS method. Moreover, in Figure (2) we plot the absolute error for this example. Not that the Figures (1) and (2) show a proper agreement between approximate and exact solution.

Table 1: The absolute errors of w for Ex.1; ($n = 81$, at $T = 1.0$).

t	x	Exact solution w	Approximate solution w_{81}	Absolute error	COC
0.06	0.06	0.00337766	0.00337778	1.25494E-007	6.3446
0.12	0.12	0.01257312	0.01257025	2.86817E-006	4.5986
0.18	0.18	0.02608229	0.02607668	5.61196E-006	8.3475
0.24	0.24	0.04229415	0.04228894	5.20938E-006	10.5223
0.3	0.3	0.05952877	0.05952122	7.57527E-006	45.5136
0.36	0.36	0.07608709	0.07607117	1.59231E-005	-34.5644
0.42	0.42	0.09031466	0.09029223	2.24395E-005	-1.7198
0.48	0.48	0.10068169	0.10065702	2.46729E-005	0.7006
0.54	0.54	0.10588202	0.10585056	3.14666E-005	1.2566
0.6	0.6	0.10495404	0.10491579	3.82564E-005	1.8185
0.66	0.66	0.09742716	0.09738933	3.78319E-005	-2.5356
0.72	0.72	0.08349743	0.08346029	3.71462E-005	-2.8373
0.78	0.78	0.06423685	0.06420914	2.77135E-005	0.9251
0.84	0.84	0.04184137	0.04181375	2.76203E-005	5.9961
0.9	0.9	0.01992278	0.01989678	2.60060E-005	4.7152

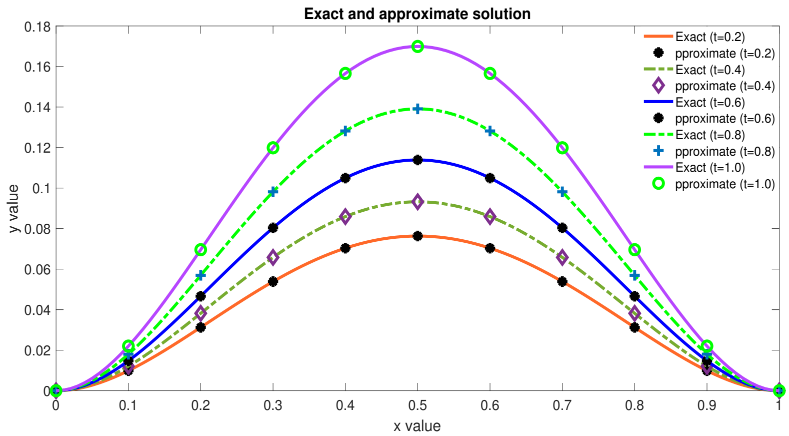

Fig. 1: Comparison of between approximate solution(w_{81}) of RKHS method and exact solution at times $T = 0.2\,, 0.4\,, 0.6\,, 0.8$ and 1.0 for example 1.

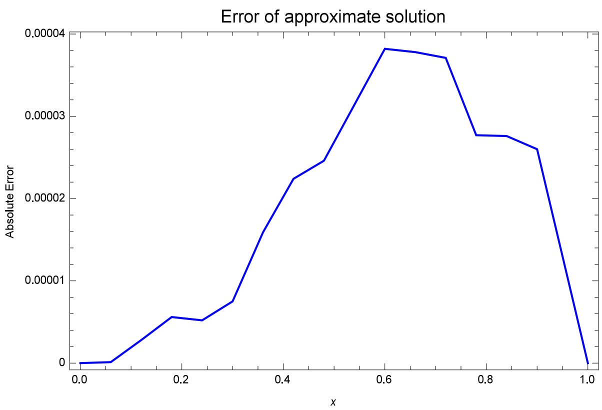

Fig. 2: The absolute errors comparison between numerical solution(w_{81}) and exact solution for Example.1 .

Table 2: The L^∞ and L^2 errors computed by three schemes for Ex.1 ($\Delta x = \Delta t = \frac{1}{10}$ at $t = 1.0$).

$N = 100$	$\parallel E \parallel_\infty$	$\parallel E \parallel_2$
Fourth-order compact difference	$1.755578E-003$	$3.363310E-003$
Meshless Local Radial Point Interpolation	$3.288094E-004$	$5.101931E-004$
Reproducing Kernel Hilbert Space	$3.275141E-005$	$3.750492E-005$

Example 2. Consider (1.1) with $p = 1$, $b = 25$, $c = 20$ then

$$f(x\,,\,t) = (6\,t + 3\,c\,t^2\,)(x^2(1\,-x\,)^2) + b\,t^3\,(x^2(1\,-\,x)^2) - p\,t^3\,(12x^2 - 12x + 2),$$

with the following conditions

$$w(x\,,\,0) = 0\,,\quad \frac{\partial w}{\partial t}(x, 0) = 0\,,$$

$$w(0\,,\,t) = w(1, t) = 0, \qquad t \geq 0,$$

on $\overline{\Omega} = [0,\ 1] \times [0,\ 1]$. And the exact solution is $w(x\,,\,t) = t^3\,x^2\,(1\,-\,x\,)^2$.

Like pervious example, by using the RKHS method, taking $x_i = t_i = \frac{i}{N}$, $i = 1, 2, \ldots, N$, with the reproducing kernel function $\mathbf{K}_{(\eta,\xi)}(x,\,t)$ on region $\overline{\Omega}$, the approximate solution w_n(that $n = N^2$) is obtained by (4.5). The exact and approximated solutions, the computational convergence order (COC) along with the absolute errors at some nodal points for $N = 9$ with $T = 1$ have been shown in Table (3). From this Table, it is observed that the numerical results are very close to the exact solution and we achieved an excellent approximation for the exact solution by employing current scheme. Note that according to Table (3), it is seen that in some points such as $(0.24, 0.24)$, the computational order of convergence gets a negative value for the following reasons:

1. $e_{n-1} < e_n$ and $e_{n+1} < e_n$.

2. $e_n < e_{n-1}$ and $e_n < e_{n+1}$.

We also see that COC has taken a large amount in some points such as $(0.6, 0.6)$, because

$e_{n+1} << e_n$ or $e_n << e_{n+1}$.

Table (4), compares the L^∞ and L^2 errors computed by three schemes for this example with $\Delta x = \Delta t = \frac{1}{10}$ with $T = 1$. From Table (4), it was found the our method in comparison with mentioned methods is better with view to utilization and accuracy. Figure (3) compares the exact and approximated solution which confirms the reliability of RKHS method. Moreover, in Figure (4) we plot the absolute error for this example. Not that the Figures (3) and (4) show a proper agreement between approximate and exact solution.

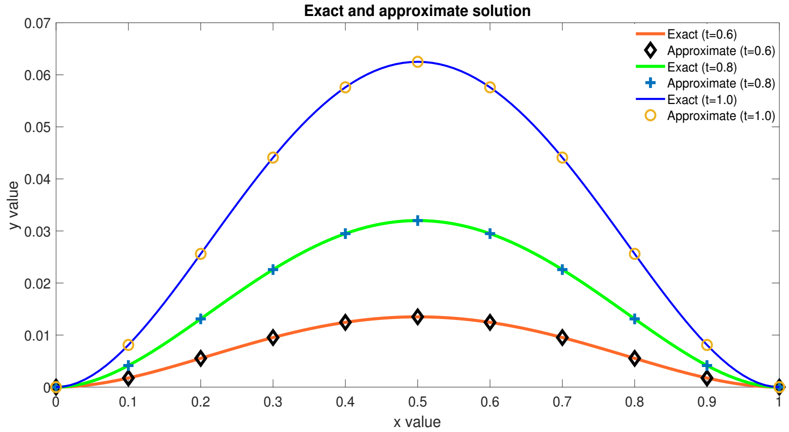

Table 3: The absolute errors of w for Ex.2; ($n = 81$, at $T = 1.0$).

t	x	Exact solution w	Approximate solution w_{81}	Absolute error	COC
0.06	0.06	0.000000687	0.000015624	1.49369E-005	7.1456
0.12	0.12	0.000019270	0.000091515	7.22452E-005	4.0321
0.18	0.18	0.000127055	0.000260388	1.33334E-004	0.8127
0.24	0.24	0.000459921	0.000641067	1.81146E-004	-2.0583
0.3	0.3	0.001190700	0.001390491	1.99788E-004	-4.7123
0.36	0.36	0.002476695	0.002673621	1.96926E-004	4.9037
0.42	0.42	0.004396453	0.004570320	1.73672E-004	1.0071
0.48	0.48	0.006889900	0.007032000	1.42098E-004	0.9326
0.54	0.54	0.009715932	0.009817261	1.01439E-004	2.6429
0.6	0.6	0.012441600	0.012499854	5.82542E-005	14.4135
0.66	0.66	0.014476965	0.014494569	1.76048E-005	-2.3143
0.72	0.72	0.015169754	0.015135627	3.41275E-005	-1.2164
0.78	0.78	0.013973239	0.013931273	4.19661E-005	5.4732
0.84	0.84	0.010706225	0.010645442	6.07803E-005	3.9128
0.9	0.9	0.005904900	0.005855634	4.92668E-005	4.2194

Table 4: The L^{∞} and L^2 errors computed by three schemes for Ex.2 ($\Delta x = \Delta t = \frac{1}{10}$ at $t = 1.0$).

$N = 100$	$\parallel E \parallel_{\infty}$	$\parallel E \parallel_2$
Fourth-order compact difference	$5.210669E - 003$	$1.084486E - 003$
Meshless Local Radial Point Interpolation	$2.934016E - 004$	$6.075916E - 004$
Reprducing Kernel Hilbert Space	$2.813842E - 005$	$7.850749E - 005$

Fig. 3: Comparison of between approximate solution(w_{81}) of RKHS method and exact solution at times $T =$ 0.6 , 0.8 and 1.0 for example 2.

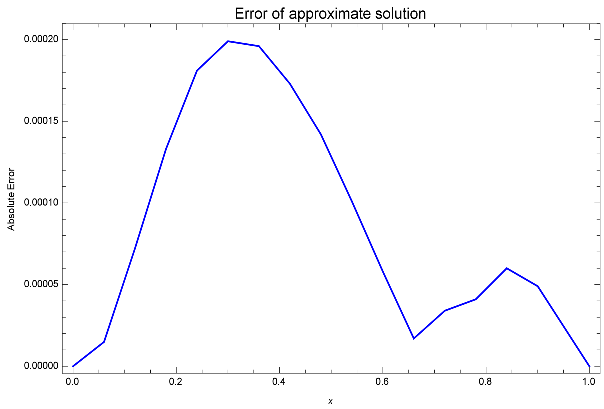

Fig. 4: The absolute errors comparison between numerical solution(w_{81}) and exact solution for Example.2 .

7 Conclusions

In the current paper, we have proposed the Reproducing Kernel Hilbert Space (RKHS) scheme for 1-dimensional linear telegraph equations. We have proved that the proposed method has good convergence, is easy to implement and its concepts are simple. The obtained results are excellent compared to other methods. Finally, the numerical results have been presented to clarify the effectiveness and accuracy of this technique.

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