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A state-dependent Chandrasekhar integral equation

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Abstract

Phenomena depending on their past history or their past state have received more importance. The mathematical models of these phenomena can be described by differential equations of a hereditary or a self-referred type. This paper is devoted to study the solvability of a state-dependent or self-referred integral equation via Chandrasekhar kernel. The investigation of this problem is motivated by the results from, Eder [10], Fečkan [11] and Buica [3] who initiated the study of state dependent differential equations. Here, the existence and the uniqueness of the solution of this state-dependent integral equation via Chandrasekhar kernel have been discussed. The data dependency of the solution on some functions has been studied.

Keywords: Chandrasekhar kernel, Existence and Uniqueness of the solution, Continuous dependence
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1 Introduction

The state-dependent differential and integral equations are one of the recent kind of functional differential equations ([14]-[29]). Most of the differential and integral equations with deviating arguments that appear in many literature, the deviation of the argument usually involves only the time itself, However, another case, in which the deviating arguments depend on both the state variable x and the time t, is of importance in theory and practice. Several papers have appeared recently that are devoted to such kind of differential equations [1], [2], [3] and references therein.

In [24], some existence results for equations with state dependent delays, with emphasis on particular models and on the emerging theory from the dynamical systems point of view are illustrated. Several new results are presented. Various examples of differential equations with state dependent delays which arise in physics, automatic control, neural networks, infectious diseases, population growth, and cell production are described. Some of these models differ considerably from others, and most of them do not look simple. Typically the delay is not given explicitly as a function of what seems to be the natural state variable; the delay may be defined implicitly by a functional, integral or differential equation and should often be considered as part of the state variables.

One of the first papers studying this class of functional equations is the one by Eder [10] who considered the functional differential equation

$$x'(t) = x(x(t)), \quad t \in A \subset \mathbb{R},$$

while Fečkan [11] studied a functional differential equation of the form

$$x'(t) = f(x(x(t))) \tag{1.1}$$

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with $f \in C^1(\mathbb{R})$ by applying the Leray-Schauder theorem. Also, the existence of a Picard iteration method enabled him to approximate a solution of (1.1).

This kind of problems has been studied by Coman Pravel [4], Oberg [25] and [3]. In [3] existence, uniqueness and continuous dependence theorems were proved for the differential equation $x'(t) = f(t, x(x(t)))$ with an initial condition.

Recently, the state-dependent or self- referred differential and integral equations have been gained a great attention by many authors (see [13]-[23] and [26] -[27]).

Phenomena depending on their past history or hereditary problems may be served as a mathematical model of a self-referred problem [27]and [28]. In the aim of proving the existence of solutions for these problems, they have applied different techniques such as direct application of some fixed point theorems or considering an iterative scheme (sequence of functions), for which the uniform limit exists and this limit is a solution of their problems.

Let $I = [0, 1]$. In this paper, we investigate the following state-dependent integral equation via Chandrasekhar kernel

$$x(t) = b(t) + \lambda\, x\Big(\int_0^1 \frac{t}{t+s}\, a(s)x(s)\, ds\Big), \quad t \in I, \tag{1.2}$$

where $a \in L^1(I)$ and b is continuous function on I. We study the existence of solutions in the class of continuous functions x. For this aim, we consider firstly the general state-dependent integral equation via Chandrasekhar kernel

$$x(t) = b(t) + \lambda\, x\Big(\int_0^1 \frac{t}{t+s}\, g(s, x(s))\, ds\Big), \quad t \in I, \tag{1.3}$$

under some conditions on the functions b and g. Furthermore, we establish the existence of the solution for the equation (1.2) and study the continuous dependence of this solution on the function a.

2 Existence of solutions

Denote by $C = C(I, \mathbb{R})$ the class of continuous functions defined on the interval I and with values in $\mathbb{R}$. In this section, we shall study the existence of at least one solution for the integral equation (1.3). Consider the following assumptions:

(i) $g : I \times I \to \mathbb{R}^+$ satisfies Carathéodory condition. i.e. is measurable in $t \in I$, $\forall x \in I$ and continuous in $x \in I$, $\forall t \in I$ and there exists a function $a : I \to I$, $a \in L^1(I)$ and $a(t) \le t$ such that:

$$|g(t, x(t))| \le a(t)\, x(t).$$

(ii) $b : I \to I$ is continuous, with $b(0) = 0$, $\sup_{\forall t \in I} |b(t)| = \mathfrak{b}$ and $\lambda \in (0, 1)$.

Remark:1
Some examples for the function b :

- When $b(t) = t$, such that $b(0) = 0$ and $\mathfrak{b} = 1$.
- When $b(t) = \sin t$, such that $b(0) = 0$ and $\mathfrak{b} = 1$.

Remark:2

- From assumptions (i),(ii) and using equation (1.3) we deduce that

$$x(0) = \lambda x(0) \;\Rightarrow (1 - \lambda)x(0) = 0 \;\Rightarrow\; x(0) = 0.$$

- $a(t) \le t$, $t \in I$ implies that $\sup_{\forall t \in I} |a(t)| = 1$.

Theorem 2.1. Let the assumptions (i)-(ii) be satisfied. Then (1.3) has a solution $x \in C$.

Proof . Let $\lambda \in (0,1)$ and define the set $S_L \subset C$ by

$$S_L \ = \ \{ \ x \in I : |x(t) - x(s)| \leq L|t-s|, \quad L \ = \ \frac{\mathfrak{b}}{1-\lambda} \ \}$$

and define the map F by

$$Fx(t) \ = \ b(t) \ + \ \lambda \ x\bigg(\ \int_0^1 \ \frac{t}{t+s} \ g(s,x(s)) \ ds\bigg), \qquad t \in I,$$

then for $x \in S_L$ we have

$$\begin{aligned}
|Fx(t)| \quad & \leq \ \mathfrak{b} \ + \ \lambda \ \bigg|x\bigg(\ \int_0^1 \ \frac{t}{t+s} \ g(s,x(s)) \ ds\bigg) - x(0)\bigg| \\
& \leq \ \mathfrak{b} \ + \ \lambda \ L \int_0^1 \ \frac{t}{t+s} \ |g(s,x(s))| \ ds \\
& \leq \ \mathfrak{b} \ + \ \lambda \ L \int_0^1 \ |g(s,x(s))| \ ds \\
& \leq \ \mathfrak{b} \ + \ \lambda \ L \int_0^1 a(s)x(s) \ ds \\
& \leq \ \mathfrak{b} \ + \ \lambda \ L \ = \ L.
\end{aligned}$$

This implies $F : S_L \to S_L$ and the class $\{Fx\}$ is uniformly bounded on S_L.

Next, $x \in S_L$ and $t_1, \ t_2 \in I, \ t_1 < t_2, \ \ |t_2 - t_1| \leq \delta,$ then

$$\begin{aligned}
|Fx(t_2) - Fx(t_1)| \quad & \leq \quad \lambda\bigg|x\bigg(\int_0^1 \ \frac{t_2}{t_2+s}g(s,x(s))ds\bigg) - x\bigg(\int_0^1 \ \frac{t_1}{t_1+s}g(s,x(s))\bigg)ds\bigg| \\
& \leq \quad \lambda \ L\bigg|\int_0^1 \ \bigg(\frac{t_2}{t_2+s} - \frac{t_1}{t_1+s}\bigg)g(s,x(s))ds\bigg| \\
& \leq \quad \lambda \ L \ \int_0^1 \ (\frac{(t_2-t_1) \ s}{(t_2+s)(t_1+s)}g(s,x(s))ds
\end{aligned}$$

$$\begin{aligned}
& \leq \quad \lambda \ L \ |t_2 - t_1| \ \int_0^1 \ \frac{t_2}{(t_2+s)} \frac{s}{(t_1+s)} \ a(s) \ x(s)ds \\
& \leq \quad \lambda \ L \ |t_2 - t_1| \ \int_0^1 \ x(s)ds \\
& \leq \quad \lambda \ L \ |t_2 - t_1| \ \int_0^1 \ ds \\
& \leq \quad \lambda \ L \ |t_2 - t_1| \ \leq \ L \ |t_2 - t_1|.
\end{aligned}$$

Then $\{Fx\}$ is equicontinuous on S_L.

Now, let $\{x_n(t)\} \in S_L$, $x_n(t) \to x_o(t)$, then we get

$$
\begin{aligned}
& \left| x_n\Big(\int_0^1 \frac{t}{t+s}\, g(s,x_n(s))\, ds \Big) - x_o\Big(\int_0^1 \frac{t}{t+s}\, g(s,x_o(s))\, ds \Big) \right| \\
= & \left| x_n\Big(\int_0^1 \frac{t}{t+s}\, g(s,x_n(s))\, ds \Big) - x_n\Big(\int_0^1 \frac{t}{t+s}\, g(s,x_o(s))\, ds \Big) \right. \\
+ & \left. x_n\Big(\int_0^1 \frac{t}{t+s}\, g(s,x_o(s))\, ds \Big) - x_o\Big(\int_0^1 \frac{t}{t+s}\, g(s,x_o(s))\, ds \Big) \right| \\
\leq & \left| x_n\Big(\int_0^1 \frac{t}{t+s}\, g(s,x_n(s))\, ds \Big) - x_n\Big(\int_0^1 \frac{t}{t+s}\, g(s,x_o(s))\, ds \Big) \right| \\
+ & \left| x_n\Big(\int_0^1 \frac{t}{t+s}\, g(s,x_o(s))\, ds \Big) - x_o\Big(\int_0^1 \frac{t}{t+s}\, g(s,x_o(s))\, ds \Big) \right| \\
\leq & L \left| \int_0^1 \frac{t}{t+s}\, g(s,x_n(s))\, ds - \int_0^1 \frac{t}{t+s}\, g(s,x_o(s))\, ds \right| \\
+ & \left| x_n\Big(\int_0^1 \frac{t}{t+s}\, g(s,x_o(s))\, ds \Big) - x_o\Big(\int_0^1 \frac{t}{t+s}\, g(s,x_o(s))\, ds \Big) \right| \\
\leq & L \int_0^1 \frac{t}{t+s}\, |g(s,x_n(s)) - g(s,x_o(s))|\, ds \\
+ & \left| x_n\Big(\int_0^1 \frac{t}{t+s}\, g(s,x_o(s))\, ds \Big) - x_o\Big(\int_0^1 \frac{t}{t+s}\, g(s,x_o(s))\, ds \Big) \right| \\
\leq & L \int_0^1 |g(s,x_n(s))\, ds - g(s,x_o(s))\, ds \; + \; \epsilon_2 \\
\leq & L\, \epsilon_1 \; + \; \epsilon_2 \; = \; \epsilon.
\end{aligned}
$$

Applying Lebesgue dominated Theorem [9], we have

$$
\begin{aligned}
\lim_{n\to\infty} Fx_n(t) \;&=\; b(t) \;+\; \lambda \lim_{n\to\infty} x_n\Big(\int_0^1 \frac{t}{t+s}\, g(s,x_n(s))\, ds \Big) \\
&=\; b(t) \;+\; \lambda \lim_{n\to\infty} x_n\Big(\int_0^1 \frac{t}{t+s}\, g(s,\lim_{n\to\infty} x_n(s))\, ds \Big) \\
&=\; b(t) \;+\; \lambda\, x_o\Big(\int_0^1 \frac{t}{t+s}\, g(s,x_o(s))\, ds \Big) \;=\; Fx_o(t).
\end{aligned}
$$

This proves that F is continuous and by Schauder fixed point theorem in[9], thus the state-dependent integral equation (1.3) has a solution $x \in S_L$. □

Corollary 2.2. Let the assumptions of Theorem 2.1 be satisfied with $g(t,x(t)) = a(t)\; x(t)$. Then the state-dependent integral equation (1.2) has a solution $x \in S_L$.

Example 1
Consider the state-dependent integral equation via Chandrasekhar kernel

$$x(t) \;=\; t\big(1 - sin(\frac{\pi}{2}t)\big) \;+\; \lambda x\Big(\int_0^1 \frac{t}{t+s}\; cos(\frac{\pi}{2}s)\; x(s)ds \Big), \tag{2.1}$$

where $a(t) = cos(\frac{t\pi}{2})$ and $b(t) = t\big(1 - sin(\frac{\pi}{2}t)\big)$. We can easily, verify assumptions of Theorem 2.1, then equation has a continuous solution.

Example 2
Consider the state-dependent integral equation via Chandrasekhar kernel

$$x(t) \;=\; t(1 - e^{t^2}) \;+\; \lambda x\Big(\int_0^1 \frac{t}{t+s}\; e^{-(1-t^3)} cos(\frac{\pi}{2}s)\; x(s)ds \Big), \tag{2.2}$$

where $a(t) = e^{-(1-t^3)} cos(\frac{t\pi}{2})$ and $b(t) = t(1-e^{t^2})$. We can easily, verify assumptions of Theorem 2.1, then equation has a continuous solution.

3 The uniqueness and the data dependency of the solution

Here, we discuss the existence of a unique solution of the integral equation (1.2)and its continuous dependence on the function a.

Theorem 3.1. Let the assumptions of Corollary 2.2 be satisfied. If $\lambda\ (1+\ L)\ <\ 1,$ then the solution of the state-dependent integral equation (1.2) is unique.

Proof . Let x, y be two solutions of (1.3), then

$$\begin{aligned}
|x(t)-y(t)| &= \lambda \left| x\left(\int_0^1 \frac{t}{t+s}\ a(s)x(s)\ ds \right) - y\left(\int_0^1 \frac{t}{t+s}\ a(s)y(s)\ ds \right) \right| \\
&= \lambda \left| x\left(\int_0^1 \frac{t}{t+s}\ a(s)x(s)\ ds \right) - x\left(\int_0^1 \frac{t}{t+s}\ a(s)y(s)\ ds \right) \right. \\
&+ \left. x\left(\int_0^1 \frac{t}{t+s}\ a(s)y(s)\ ds \right) - y\left(\int_0^1 \frac{t}{t+s}\ a(s)y(s)\ ds \right) \right| \\
&\leq \lambda\ L \left| \int_0^1 \frac{t}{t+s}\ a(s)x(s)\ ds - \int_0^1 \frac{t}{t+s}\ a(s)y(s)\ ds \right| \\
&+ \lambda \left| x\left(\int_0^1 \frac{t}{t+s}\ a(s)y(s)\ ds \right) - y\left(\int_0^1 \frac{t}{t+s}\ a(s)y(s)\ ds \right) \right| \\
&\leq \lambda\ L \int_0^1 \frac{t}{t+s}\ a(s)|x(s)-y(s)|\ ds\ +\ \lambda\ ||x-y|| \\
&\leq \lambda\ L\ ||x-y||\ +\ \lambda\ ||x-y|| \\
&\leq \lambda(1+L)||x-y||, \\
&\Rightarrow\ (1-\lambda(1+L))||x-y||\ \leq\ 0, \qquad \Rightarrow\ x(t)\ =\ y(t).
\end{aligned}$$

Then the solution of equation (1.2) is unique. □

Definition 3.2. The solution of the state-dependent integral equation (1.2) depends continuously on the function a, if $\forall\epsilon > 0$, there exist $\delta > 0$ such that

$$||a-a^*|| \leq \delta \quad \Rightarrow \quad ||x-x^*|| \leq \epsilon,$$

$$x^*(t)\ =\ b(t)\ +\ \lambda\ x^*\left(\int_0^1 \frac{t}{t+s}\ a^*(s)x^*(s)\ ds \right), \qquad t \in [0,1]. \tag{3.1}$$

Theorem 3.3. Let the assumptions of Theorem 3.1 be satisfied for the two functions a and a^*. Then the solution of (1.2) depends continuously on the function a.

Proof . Let x and x^* be the two solution of (1.2) and (3.1), then

$$\begin{aligned}
|x^*(t)-x(t)| &= \lambda \left| x\Big(\int_0^1 \frac{t}{t+s}\ a(s)x(s)\ ds\Big) - x^*\Big(\int_0^1 \frac{t}{t+s}\ a^*(s)x^*(s)\ ds\Big)\right| \\
&= \lambda \left| x\Big(\int_0^1 \frac{t}{t+s}\ a(s)x(s)\ ds\Big) - x^*\Big(\int_0^1 \frac{t}{t+s}\ a(s)x(s)\ ds\Big)\right. \\
&+ x^*\Big(\int_0^1 \frac{t}{t+s}\ a(s)x(s)\ ds\Big) - x^*\Big(\int_0^1 \frac{t}{t+s}\ a(s)x^*(s)\ ds\Big) \\
&+ \left. x^*\Big(\int_0^1 \frac{t}{t+s}\ a(s)x^*(s)\ ds\Big) - x^*\Big(\int_0^1 \frac{t}{t+s}\ a^*(s)x^*(s)\ ds\Big)\right| \\
&\le \lambda \left| x\Big(\int_0^1 \frac{t}{t+s}\ a(s)x(s)\ ds\Big) - x^*\Big(\int_0^1 \frac{t}{t+s}\ a(s)x(s)\ ds\Big)\right| \\
&+ \lambda \left| x^*\Big(\int_0^1 \frac{t}{t+s}\ a(s)x(s)\ ds\Big) - x^*\Big(\int_0^1 \frac{t}{t+s}\ a(s)x^*(s)\ ds\Big)\right| \\
&+ \lambda \left| x^*\Big(\int_0^1 \frac{t}{t+s}\ a(s)x^*(s)\ ds\Big) - x^*\Big(\int_0^1 \frac{t}{t+s}\ a^*(s)x^*(s)\ ds\Big)\right| \\
&\le \lambda \left| x\Big(\int_0^1 \frac{t}{t+s}\ a(s)x(s)\ ds\Big) - x^*\Big(\int_0^1 \frac{t}{t+s}\ a(s)x(s)\ ds\Big)\right| \\
&+ \lambda\ L \left| \int_0^1 \frac{t}{t+s}\ a(s)x(s)\ ds - \int_0^1 \frac{t}{t+s}\ a(s)x^*(s)\ ds\right| \\
&+ \lambda\ L \left| \int_0^1 \frac{t}{t+s}\ a(s)x^*(s)\ ds - \int_0^1 \frac{t}{t+s}\ a^*(s)x^*(s)\ ds\right| \\
&\le \lambda\ ||x-x^*|| + \lambda\ L \int_0^1 \frac{t}{t+s}\ a(s)|x(s)-x^*(s)|\ ds \\
&+ \lambda\ L \int_0^1 \frac{t}{t+s}\ |a(s)-a^*(s)|x^*(s)\ ds \\
&\le \lambda\ (||x-x^*|| \ + \ L\ ||x-x^*|| \ + \ L \int_0^1 \frac{t}{t+s} x^*(s)\ |a(s)-a^*(s)|ds) \\
&\le \lambda\ (||x-x^*|| \ + \ L\ ||x-x^*|| \ + \ L\ \delta). \\
&\Rightarrow \quad ||x-x^*|| \ \le \ \frac{1}{1-\lambda(1+L)} L\ \delta \ = \ \epsilon.
\end{aligned}$$

□

Conclusion

Many problems of radiative transfer have been reduced by S. Chandrasekhar [7], to the problem of solving the well known Chandrasekhar integral equation [8], [30]. The problems of radiative transfer give rise to interesting integral equations that may be analysed qualitatively (see [8], [6]and [20]) or qualitatively (see [30], [12] and [22]).

This work is concerned with the continuous solution of a new state-dependent integral equation via Chandrasekhar kernel by a direct application of Schauder fixed point theorem. Firstly, we have proved the existence of at least one solution for equation (1.3) and as a particular case, we deduce the existence of solutions for equation (1.2). Next, the existence of a unique solution has been proved. Furthermore, we established the continuous dependence of solution on some data.

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