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Efficient analytical method for the solution of some fractional -order nonlinear differential equations

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Abstract

A novel technique called the Variational Adomian decomposition method (VIADM) is used to approximate an analytical solution for several types of non-linear fractional differential equations. Some examples are presented to back up our findings. The solution procedure and results indicated that the proposed method is very effective, reliable, and straightforward. The results show how effective and precise the present technology is at resolving various nonlinear problems in applied science. The MATLAB software carried out all the computations and graphics. Fractional derivatives are mentioned in Caputo Sense. Moreover, a graphical representation was made for the solution of some examples. For integer and fractional order problems, solution graphs are shown.

Keywords: Caputo operator, Approximate Solution, Partial differential equations, Adomian Decomposition Method, Fractional Calculus.
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1 Introduction

Mathematical modelling of several phenomena problems refers to fractional non-linear differential equations in various areas of engineering, fluid mechanics and other applied sciences. Fractional Calculus (FC) has been applied and may be described effectively using fractional calculus mathematical techniques, in recent decades, [5, 6, 10, 11]. Because at most fractional non-linear differential equations (FDEs) don't have accurate analytic solutions, the numerical and approximation techniques should be tested. Adomian decomposition (ADM) [1, 8], Variational Iteration method [3, 4, 7, 13], Homotopy perturbation (HAM) [2, 9, 14] and others are examples of modern analytic approaches The VIM and the ADM are the most obvious ways of the solutions FDEs for providing the analytic solutions, in addition, numerical-approximate solutions alone linearisation (discretization) for non-linear equations.

In this work, we will provide a novel approach of (VIADM) for the solution of (FDEs) along with the initial and boundary conditions [12].

2 Variational Iteration Adomian Decomposition Method for Solving PDEs with Fractional Order

Consider the fractional differential equation as follows:

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$$^{C}D_t^{\alpha}y(x,t) + Ly(x,t) + Ny(x,t) - f(x,t) = 0, \tag{2.1}$$

where, $^{C}D_t^{\alpha}$ means the Caputo Fractional order derivative, N, L are the Nonlinear terms and linear differential Operator respectively, and f refers the source term.

Therefore, the construct correction functional using VIM for Eq. (2.1) is provided by:

$$y_{n+1}(x,t) = y_n(x,t) + \int_0^t \lambda(x,s)[^{C}D_t^{\alpha}y_n(x,s) + L(\tilde{y}_n(x,s)) + N(\tilde{y}_n(x,s)) - f(x,s)]ds, \tag{2.2}$$

where, $\tilde{y}(x,t)$ is a restricted variation. To solve Eq. (2.2) by VIM, λ (Lagrange Multiplier)is determined via integration by parts. To choose y_0 may be selected by a function which satisfies the initial and boundary conditions. Then the related Variational Iteration formula is given by:

$$y_{n+1}(x,t) = y_n(x,t) + J_t^{\alpha}[\lambda(x,s)[^{C}D_t^{\alpha}(y_n(x,s)) + L(\tilde{y}_n(x,s)) + N(\tilde{y}_n(x,s)) - f(x,s)]]. \tag{2.3}$$

The continuous functions $y(x,s)$ and $f(x,s)$ and $m-1 < \alpha \leq m$. In the Caputo Sense, α means a parameter explaining the order of the fractional derivative and J_t^{α} means Riemann-Liouvile integral operator of fractional order $\alpha = 1 + \beta - m$. Then, Eq. (2.3) becomes:

$$y_{n+1}(x,t) = y_n(x,t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\lambda(x,s)\left[\frac{\partial^{\beta}}{\partial t^{\beta}}y_n(x,s) + L(\tilde{y}_n(x,s)) + N(\tilde{y}_n(x,s)) - f(x,s)\right]ds$$
$$0 < \beta \leq 1 \tag{2.4}$$

by approximate $^{C}D_t^{\alpha}y$ by $\left(\frac{\partial^{\beta}}{\partial t^{\beta}}y\right)$. To find $\lambda(x,s)$, suppose $\alpha = 1,\ \beta = m$, the correction functional of Eq. (2.4) may be written as follows approximately:

$$y_{n+1}(x,t) = y_n(x,t) + \int_0^t \lambda(x,s)\left[L(\tilde{y}_n(x,s)) + N(\tilde{y}_n(x,s)) - f(x,s) + \frac{\partial^m}{\partial t^m}y_n(x,s)\right]ds,$$

where, y_n is considered as a restricted variation, and $\delta\tilde{y}_n = 0$

$$\delta y_{n+1}(x,t) = \delta y_n(x,t) + \delta\int_0^t \lambda(x,s)\left[\frac{\partial^m}{\partial t^m}y_n(x,s) + L(\tilde{y}_n(x,s)) + N(\tilde{y}_n(x,s)) - f(x,s)\right]ds \tag{2.5}$$

Then, consequently Eq. (2.5) with $m = 1$ will be reduced to:

$$\delta y_{n+1}(x,t) = \delta y_n(x,t) + \delta\int_0^t \lambda(x,s)\left(\frac{\partial}{\partial t}y_n(x,s) + L(\tilde{y}_n(x,s)) + N(\tilde{y}_n(x,s)) - f(x,s)\right)ds. \tag{2.6}$$

By using the method of integration by parts on Eq. (2.6) will give the following formula:

$$\delta y_{n+1}(x,t) = \delta y_n + \lambda((x,s))\delta y_n|_{s=t} + \int_0^t \lambda'(x,s)\delta y_n ds$$

and then

$$\delta y_{n+1} = (1 + \lambda(x,s))\delta y_n + \int_0^t \lambda'(x,s)\delta y_n(x,s)ds = 0.$$

As a result, the stationary conditions are obtained:

$$\lambda'(x,s) = 0 \text{ and } 1 + \lambda(x,s)|_{s=t} = 0.$$

Thus, the general Lagrange multiplier may be defined as $\lambda(x,s) = -1$.

So, the following iteration formula for Eq. (2.2) is obtained by:

$$y_{n+1}(x,t) = y_n(x,t) - J_t^{\alpha}\left[\frac{\partial}{\partial t}y_n(x,s) + Ly_n(x,s) + Ny_n(x,s) - f(x,s)\right].$$

Now, for $m = 2$, after substituting the value of m, we get:

$$\alpha = \beta - 1,\ \beta = m \quad 1 < \alpha \leq 2.$$

The Eq. (2.3) becomes:

$$y_{n+1}(x,t) = y_n(x,t) + \frac{1}{\Gamma(\beta-1)} \int_0^t (t-s)^{\beta-2} \lambda(x,s) \left[\frac{\partial^m}{\partial t^m} y_s(x,s) + L(\tilde{y}_n(x,s)) + N(\tilde{y}_n(x,s)) - f(x,s) \right] ds \tag{2.7}$$

By taking the first variation of Eq. (2.7) with respect to y_n and taking thus $\delta y_n = 0$ and $\delta y'_n = 0$, yields to:

$$\delta y_{n+1}(x,t) = \delta y_n(x,t) + \delta \left\{ \frac{1}{\Gamma(\beta-1)} \int_0^t (t-s)^{\beta-2} \lambda(x,s) \left[\frac{\partial^m}{\partial t^m} y_n(x,s) \ + L(\tilde{y}_n(x,s)) \right.\right.$$
$$\left. + N(\tilde{y}_n(x,s)) - f(x,s) \right] ds \}$$

where, y_n is considered as a restricted variation and hence:

$$\delta y_{n+1}(x,t) = \delta y_n(x,t) + \frac{1}{\Gamma(\beta-1)} \int_0^t (t-s)^{\beta-2} \lambda(t,s) \delta \left(\frac{\partial^2}{\partial t^2} y_n(t,s) \right) ds \tag{2.8}$$

Carrying the integration by parts twice, then Eq. (2.8) will be:

$$\delta y_{n+1}(x,t) = \delta y_n(x,t) + \lambda(t,s) \delta y'_n(t,s)|_0^t - \lambda' \delta y_n(t,s)|_0^t + \int_0^t \lambda''(t,s) \delta y_n(t,s) ds$$

and

$$\delta y_{n+1}(x,t) = (1-\lambda') \delta y_n(t,s)|_{t=s} + \lambda \delta y'_n(t,s)|_{t=s} + \int_0^t \lambda''(t,s) \delta y_n(t,s) ds.$$

Since δy_n is arbitrary, and from the theory of Calculus of variation, and the following Euler equation is obtained:

$$\lambda''(t,s) = 0 \tag{2.9}$$

with the stationary conditions:

$$1 - \lambda'(t,s)|_{t=s} = 0, \qquad \lambda(t,s)|_{t=s} = 0. \tag{2.10}$$

Solving Eqs. (2.9) - (2.10), will give the following solution:

$$\lambda(t,s) = s - t.$$

So, the following iteration formula (the correction functional for Eq. (2.7) will reads as follows:

$$y_{n+1}(x,t) = y_n(x,t) + \frac{1}{\Gamma(\beta-1)} \int_0^t (t-s)^{\beta-2} (s-t) \left[\frac{\partial^\beta}{\partial t^\beta} y_n(x,s) + L(y_n(x,s)) + N(y_n(x,s)) - f(x,s) \right] ds. \tag{2.11}$$

Thus,

$$y_{n+1}(x,t) = y_n(x,t) + \frac{\beta-1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} (s-t) \left[\frac{\partial^\beta}{\partial t^\beta} y_n(x,s) + L(y_n(x,s)) + N(y_n(x,s)) - f(x,s) \right] ds.$$

So, the following iteration formula becomes:

$$y_{n+1}(x,t) = y_n(x,t) - (\beta-1) J_t^\alpha \left[\frac{\partial^\beta}{\partial t^\beta} y_n(x,s) + L(y_n(x,s)) + N(y_n(x,s)) - f(x,s) \right]$$

3 Applications and Results

In this part, certain numerical examples will be investigated in order to verify the accuracy for the suggested approach given in the previous section.

Example 3.1. Consider

$$ {}^{C}D^{\alpha}_{0,t}y - \frac{\partial^2 y}{\partial x^2} - y = 0 \tag{3.1}$$

with the initial conditions:

$$y(x,0) = 1 + \sin x,\ \ y_t(x,0) = 0.$$

First, the correction functional will be found by utilizing VIM to get the solution of Eq. (3.1)

$$y_{n+1}(x,t) = y_n(x,t) + \int_0^t \lambda(x,s)\left[\frac{\partial^\alpha}{\partial t^\alpha} y_n(x,s) - \frac{\partial^2 \tilde{y}_n(x,s)}{\partial x^2} - \tilde{y}_n(x,s)\right] ds.$$

Taking the correction functional stationary and $\delta\tilde{y}_n = 0$ and $\delta y'_n = 0$, we get:

$$\delta y_{n+1}(x,t) = \delta y_n(x,t) + \int_0^t \delta\lambda(x,s)\left[\frac{\partial^\alpha}{\partial t^\alpha} y_n(x,s) - \frac{\partial^2 \tilde{y}_n(x,s)}{\partial x^2} - \tilde{y}_n(x,s)\right] ds. \tag{3.2}$$

Second, on the iteration of Eq. (3.2), apply Adomian polynomials for nonlinear terms, Eq. (3.2) becomes:

$$\delta y_{n+1}(x,t) = \delta y_n(x,t) + \int_0^t \delta\lambda(x,s)\left[\frac{\partial^\alpha}{\partial t^\alpha} y_n(x,s) - \frac{\partial^2 (\tilde{y}_n(x,s))}{\partial x^2} - \sum_{i=0}^{\infty} A_i\right] ds \tag{3.3}$$

where, $\sum_{i=0}^{\infty} A_i$ represents Adomian polynomials and $y = \sum_{i=0}^{\infty} A_i$ to find λ, approximate $\left(\frac{\partial^\alpha}{\partial t^\alpha}\right)$ by y'' and using the method of integration by parts twice, then Eq. (3.3)

$$\begin{aligned}\delta y_{n+1}(x,t) &= \delta y_n(x,t) + \delta\lambda y'_n(x,s)|_{s=t} - \delta\lambda' y_n(x,s)|_{s=t} + \int_0^t \delta\lambda'' y_n(x,s) ds = 0\\ &= (1-\lambda')|_{s=t}\delta y_n(x,t) + \lambda(x,s)|_{s=t}\delta y'_n(t,s) + \int_0^t \lambda''\delta y_n(x,s) ds = 0.\end{aligned}$$

Since δy_n is arbitrary, and using theory of calculus of variation, get

$$\lambda''(x,s)|_{s=t} = 0$$

and

$$\lambda(x,s)|_{s=t} = 0, \quad 1 - \lambda'(x,s)|_{s=t} = 0.$$

Simplifying the equations, and solving the above equations will give the following solution $\lambda = s - t$ and can be get the variational iteration formula from Eq. (3.3) as follows:

$$y_{n+1}(x,t) = y_n(x,t) + \int_0^t (s - t^\alpha)\left[\frac{\partial^\alpha y_n(x,s)}{\partial t^\alpha} - \frac{\partial^2 y_n(x,s)}{\partial x^2} - \sum_{i=0}^{\infty} A_i\right] ds.$$

Start with the initial approximation:

$$y_0 = 1 + \sin x.$$

By using Eq. (3.3), we have the following iteration formula

$$y_{n+1}(x,t) = y_n(x,t) + \int_0^t (s-t)\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}\left[\left(\frac{\partial^\alpha y_n}{\partial t^\alpha}\right) - \frac{\partial^2 y_n}{\partial x^2} - \sum_{i=0}^{\infty} A_i\right] ds.$$

Then

$$
\begin{aligned}
y_1(x,t) &= y_0(x,t) - \frac{(\alpha-1)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha}\left[\left(\frac{\partial^{\alpha} y_n}{\partial t^{\alpha}}\right) - \frac{\partial^2 y_n}{\partial x^2} - \sum_{i=0}^{\infty} A_i\right] ds \\
&= y_0(x,t) + \frac{t^{\alpha}}{\Gamma(\alpha+1)}\left[\left(\frac{\partial^{\alpha} y_0}{\partial t^{\alpha}}\right) - \frac{\partial^2 y_0}{\partial x^2}\right] \\
&= 1 + \sin x + \frac{1}{\Gamma(\alpha+1)} t^{\alpha} \\
y_2(x,t) &= y_1(x,t) + \frac{t^{\alpha}}{\Gamma(\alpha+1)}\left[\left(\frac{\partial^{\alpha} y_1}{\partial t^{\alpha}}\right) - \frac{\partial^2 y_1}{\partial x^2} - y_1\right] \\
&= \sin x + \frac{1}{\Gamma(\alpha+1)} t^{\alpha} + \frac{1}{\Gamma(2\alpha+1)} t^{2\alpha} + 1.
\end{aligned}
$$

Thus,

$$
y_{n+1}(x,t) = \sin x + \frac{1}{\Gamma(\alpha+1)} t^{\alpha} + \frac{1}{\Gamma(2\alpha+1)} t^{2\alpha} + ... + \frac{1}{\Gamma(n\alpha+1)} t^{n\alpha} + 1.
$$

To find the validity of the approximated solution, when $\alpha = 2$, the exact solution is:

$$
y(x,t) = \sin x + \cosh t.
$$

Therefore,

$$
y_{n+1}(x,t) = 1 + \sin x + \frac{t^2}{\Gamma(3)} + \frac{t^4}{\Gamma(5)} + \frac{t^6}{\Gamma(7)} + ... = \sin x\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \frac{t^8}{8!} + ...\right) = \sin x + \cosh t.
$$

The following Figures show the exact and obtained results of solutions (approximate solutions) at $\alpha = 1.6,\ 1.8,\ 2$, here we use four terms to approximate the exact solution, the proposed method VIADM has a high convergence order and higher accuracy we get. Similarly, in figures the 3D exact and obtained results are plotted at $\alpha = 1.6,\ 1.8,\ 2$. All the exact and approximate results on the Graphs have shown are very closed and explain the reliability of the present technique.

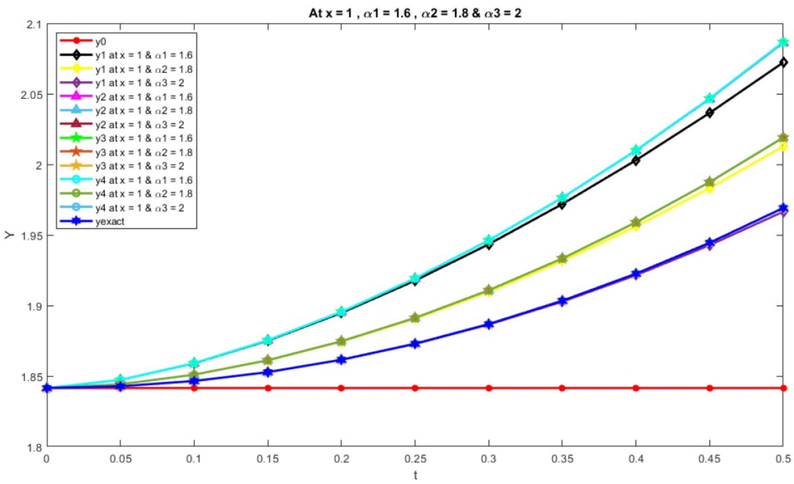

Figure 1: Show the Exact and approximate solution for $\alpha = 1.6,\ 1.8,\ 2$

Example 3.2. Consider the nonhomogeneous linear Klein-Gordon equation with fractional order:

$$
\frac{\partial^{\alpha} y}{\partial t^{\alpha}} - \frac{\partial^2 y}{\partial x^2} + y = x^3 t^3 + 6x^3 t - 6xt^3 \quad t > 0,\ 1 < x \leq 2 \tag{3.4}
$$

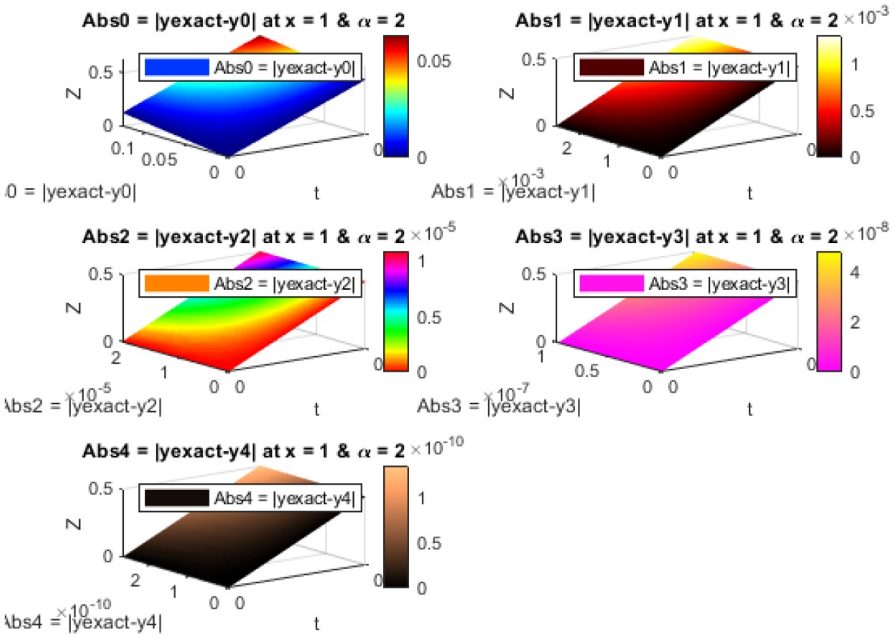

Figure 2: Show the 3D Absolute solution plots at $\alpha = 2$

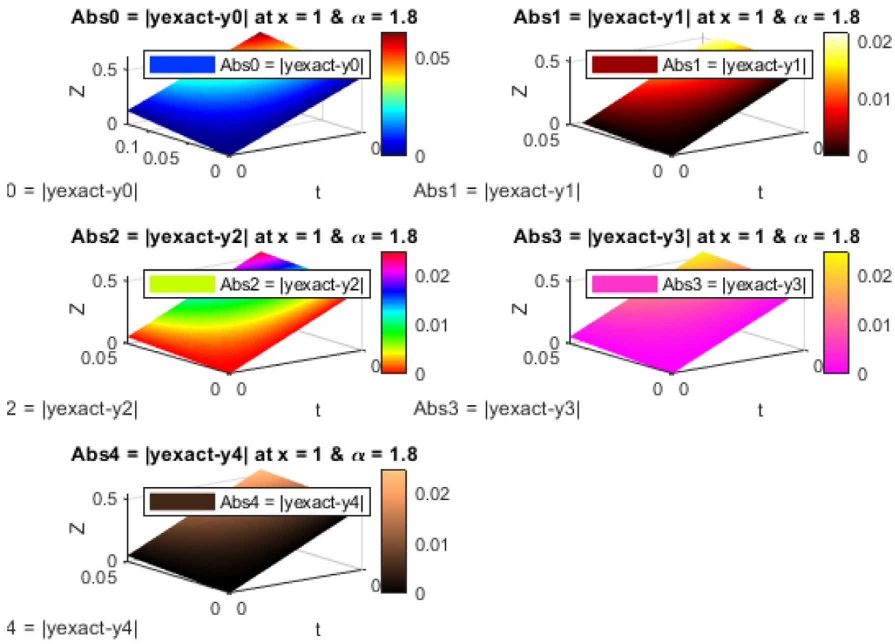

Figure 3: Show the 3D Absolute solution plots at $\alpha = 1.8$

with

$$y(x,0) = 0, \;\; y_t(x,0) = 0.$$

According to the VIADM, to the above equation, the iteration formula for Eq. (3.4) is given by:

$$y_{n+1}(x,t) = y_n(x,t) - (\alpha - 1)J_t^{\alpha}\left[\frac{\partial^{\alpha} y}{\partial t^{\alpha}} - \frac{\partial^2 y}{\partial x^2} + y - 6x^3t - x^3t^3 + 6xt^3\right]$$

and applying the Adomian polynomials to the nonlinear terms, we get:

$$y_{n+1}(x,t) = y_n(x,t) - (\alpha-1)J_t^{\alpha}\left[\frac{\partial^\alpha y}{\partial t^\alpha} - \frac{\partial^2 y}{\partial x^2} + y - \sum_{i=0}^{\infty} A_i - \sum_{i=0}^{\infty} B_i + \sum_{i=0}^{\infty} C_i\right],$$

where,

$$\sum_{i=0}^{\infty} A_i = 6x^3 t$$

$$\sum_{i=0}^{\infty} B_i = x^3 t^3$$

$$\sum_{i=0}^{\infty} C_i = 6xt^3.$$

If we begin with:

$$y_0(x,t) = y_0(x,0) + ty_t(x,0) = 0,$$

we can obtain

$$y_1(x,t) = (\alpha-1)\left[6x^3\frac{t^{1+\alpha}}{\Gamma(2+\alpha)} + x^3\frac{6t^{\alpha+3}}{\Gamma(\alpha+4)} - 36x\frac{t^{\alpha+3}}{\Gamma(\alpha+4)}\right]$$

$$y_2(x,t) = 6x^3\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 6x^3\frac{t^{\alpha+3}}{\Gamma(\alpha+4)} - 36x\frac{t^{\alpha+3}}{\Gamma(\alpha+4)},$$

and

$$(\alpha-1)^2\left[6x^3\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - 36x\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 6x^3\frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)} - 72\frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)}\right]$$

Then, the general solution when $\alpha = 2$ is given by:

$$y(x,t) = x^3t^3 + x^3\frac{6t^5}{\Gamma(6)} - 36\frac{xt^5}{\Gamma(6)} + 36\frac{xt^5}{\Gamma(6)} - 36x\frac{6t^7}{\Gamma(8)} - 6x^3\frac{t^5}{\Gamma(6)} - 6x^3\frac{t^7}{\Gamma(8)} + 36\frac{xt^7}{\Gamma(8)} + ...$$

By canceling some noise terms, yields the true solution of Eq. (3.4)

$$y(x,t) = x^3t^3$$

4 Conclusions

Fractional nonlinear differential equations with initial and boundary conditions are investigated analytically. Fractional derivatives are given in the Caputo Sense for every case. The solutions graphs are provided to demonstrate the best relevance of the suggested method. The graphs show that the proposed approach works well for solving problems of both integer and fractional order. To check the validity and efficiency of the available method, certain instances of the analytical solution are measured. The findings show that when the issues converge towards integer order, the accuracy of the suggested approach improves.

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