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A new extragradient-viscosity algorithm for finite families of asymptotically nonexpansive mappings and variational inequality problems in Banach spaces

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Abstract

ln this paper, a new approach for finding common element of the set of solutions of the variational inequality problem for accretive mappings and the set of fixed points for asymptotically nonexpansive mappings is introduced and studied. Consequently, strong convergence results for finite families of asymptotically nonexpansive mappings and variational inequality problems are established in the setting of uniformly convex Banach space and 2-uniformly smooth Banach space. Furthermore, we prove that a slight modification of our novel scheme could be applied in finding common element of solution of variational inequality problems in Hilbert space. Our results improve, extend and generalize several recently announced results in literature.

Keywords: Asymptotically nonexpansive mapping, Banach space, Common fixed point, Extragradient-Viscosity algorithm, Strong convergence, Variational inequality
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1 Introduction

Throughout this paper, we assume, unless otherwise specified, that C is a nonempty, closed and convex subset of a Banach space E whose dual space is represented by $E^\star$, $D(T)$ and $R(T)$ are the domain and range of T, $N, R, R_+, \rightarrow$ and $\rightharpoonup$ will denote the set of natural numbers, the set of real numbers, the set of nonnegative real numbers, strong convergence and weak convergence respectively. In what follows, the mapping $J : E \longrightarrow 2^{E^\star}$ defined by

$$J(x) = \{x^\star \in E : \langle x, x^\star\rangle = \|x\|\|x^\star\|, \|x\| = \|x^\star\|\}, \tag{1.1}$$

is called normalised duality mapping, where $\langle ., .\rangle$ denotes the generalized duality pairing of elements between E and $E^\star$. It is well known that E is smooth if and only if J is single-valued, uniformly smooth if and only if each duality map J is norm-to-norm uniformly continuous on bounded subset of E. (see [3], [40] for more details on the duality mapping and its properties).

A Banach space is said to have a weakly continuous duality map, in the sense of Browder [9], if there exists a guage function $\phi : [0, \infty) \longrightarrow [0, \infty)$ with $\phi(0) = 0$ such that the duality map J with the guage function ϕ is single-valued

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and is weak-to-weak* sequentially continuous; that is, if $\{x_n\} \subset E, x_n \overset{w}{\to} x$, then $J_\phi(x_n) \overset{w^\star}{\to} J_\phi(x)$. It is known that $\ell^p(1 < p < \infty)$ has a weakly continuous duality map with guage function $\phi(t) = t^{p-1}$, see for example [3] for more details.

Let C be nonempty subset of a real Banach space E. Let $S, T : C \longrightarrow C$ be two given nonlinear mappings. The set of common fixed point of the two mappings S and T will be denoted by $\mathcal{F} = F(S) \cap F(T)$.

Definition 1.1. Recall that a nonlinear mapping T is said to be:

(a) Lipschitizian if there exists a constant L such that

$$\|Tx - Ty\| \leq L\|x - y\|, \forall x, y \in D(T), \tag{1.2}$$

where L is the Lipschitizian constant of T. If $L \in (0, 1)$ in (1.2), then T is called contraction. Note that (1.2) is equivalent to the following property: for each $n \in N$, there exists a constant $k_n > 0$ such that

$$\|Tx - Ty\| \leq k_n\|x - y\|, \forall x, y \in D(T). \tag{1.3}$$

For a Lipschitizian mapping $T : C \longrightarrow C$, we call T:

(b) uniformly Lipschitizian if $k_n = L$ and (1.3) reduces to

$$\|T^n x - T^n y\| \leq L\|x - y\|, \forall x, y \in D(T),$$

for all $n \in N$;

(c) nonexpansive if $k_n = 1$ in (1.3) for all $n \in N$; that is,

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in D(T). \tag{1.4}$$

(d) asymptotically nonexpansive [18] if for all $x, y \in D(T)$ and $n \in \mathbb{N}$, there exists a sequence $k_n \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that (1.3) is satisfied; that is, for every $x, y \in (D(T)$, there exists a constant $k_n \in [1, \infty)$ with $\lim_{n\to\infty} = 1$ such that the following inequality:

$$\|T^n x - T^n y\| \leq k_n\|x - y\| \tag{1.5}$$

holds.

Remark 1.2. If we denote the classes of mappings which are nonexpansive, asymptotically nonexpansive, uniformly Lipschitizian, Lipschitizian, uniformly continuous and continuous by $(N), (AN), (UL), (L), (UC)$ and C respectively, then the following relationship:

$$(N) \subset (AN) \subset (UL) \subset (L) \subset (UC) \subset (C) \tag{1.6}$$

holds well (see, for example, [44] for details).

Definition 1.3. A nonlinear mapping $T : C \longrightarrow C$ is called:

($a^\star$) quasi-nonexpasive if $F(T) \neq \emptyset$ and condition (c) in Definition 1.1 is satisfied; that is, for every $(x, q) \in (C \times F(T))$, the following inequality:

$$\|Tx - q\| \leq \|x - q\| \tag{1.7}$$

holds.

Example 1.4. (see [21]) Let $X = R$ be a normed linear space and $C = [0, 1]$. For each $x \in C$, we define

$$Tx = \begin{cases} \lambda x, & \text{if} \quad x \neq 0 \\ 0, & \text{if} \quad x = 0, \end{cases}$$

where $0 < \lambda < 1$. It is clear that $F(T) = \{0\}$. Now, take $q = 0$, then T is asymptotically quasi-nonexpansive mapping with constant sequence $\{1\}$.

The following examples show further relationship between nonexpansive, quasi-nonexpansive and asymptotically nonexpansive mappings.

Example 1.5. (see [13]) Let $H = R^1$ and define a mapping $T : H \longrightarrow H$ by

$$Tx = \begin{cases} \frac{x}{2} sin\frac{1}{x}, & if \quad x \neq 0 \\ 0, \quad if \quad x = 0. \end{cases}$$

Then, T is quasi-nonexpansive but not nonexpansive.

Example 1.6. (see [44], [45]) Let $X = \ell^p$, where $i < p < \infty$ and define a mapping $T : X \longrightarrow X$ by

$$Tx = (0, x_1^2, a_2x_2, a_3x_3, \cdots), \forall x = (x_1, x_2, x_3, x_4, \cdots),$$

where $\{a_n\}_{n=0}^{\infty}$ is a sequence of real numbers such that $a_2 > 0.a_n \in (0, 1)$ for $n \neq 2$ and $\sum_{n=1}^{\infty} a_n = \frac{1}{2}$. Then, $F(T) \neq \emptyset, T \in (AN) \setminus (QN)$ and $T \in (AN) \setminus (N)$.

Example 1.7. (see [46]) Let $X = R$ and $C = [0, 1]$. For each $x \in C$, define a mapping $T : C \longrightarrow C$ by

$$Tx = \begin{cases} kx, \quad if \quad 0 \leq x \leq \frac{1}{2}; \\ \frac{k}{2k-1}(k - x), \quad if \quad \frac{1}{2} \leq x \leq k; \\ 0, \quad if \quad k \leq x \leq 1, \end{cases}$$

where $\frac{1}{2} < k < 1$. Then, $T \in (AN) \setminus (N)$.

Remark 1.8. From (1.6) and the examples above, it is clear that the class of asymptotically nonexpansive mappings is larger than the classes of nonexpansive and quasi-nonexpansive mappings.

It is interesting to note that a wide range of real life problems arising from different areas of optimisation, engineering, variational inequalities, differential equations, mathematical sciences can be modeled by the equation of the form:

$$x = Tx, \tag{1.8}$$

where T is a nonexpansive mapping. The solution set of the problem defined by (1.8) coincides with the fixed point set of T. Different researchers have studied this type of operator in recent times, and more investigations are still on going to expose some of the practical implications of its inherent properties (see, for example, [5], [20], [32] for more details).

Definition 1.9. Let C be a nonempty subset of a real Banach space E. Recall that an operator $A : C \longrightarrow E$ is called :

(a) accretive if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq 0, \forall x, y \in C. \tag{1.9}$$

(b) α-inverse strongly accretive if for some $\alpha > 0$

$$\langle Ax - Ay, j(x - y)\rangle \geq \alpha\|x - y\|, \forall x, y \in C. \tag{1.10}$$

Remark 1.10. In Hilbert spaces, the normalized duality map is the identity map. Hence, in Hilbert space, accretivity and monotonicity coincide.

The following are some recent studies carried out on variational inequality problems for accretive and α-inverse strongly accretive operators:

In [1], Aoyama et al considered the following general variational inequality problem in the setting of a real Banach space: find a point $x^\star \in C$ such that, for some $j(x - x^\star) \in J(x - x^\star)$ such that

$$\langle Ax^\star, j(x - x^\star)\rangle \geq 0, \forall x \in C. \tag{1.11}$$

The solution set of (1.11) is denoted by $VI(C, A)$; that is,

$$VI(C, A) = \{x^\star \in C : \langle Ax^\star, j(x - x^\star)\rangle \geq 0, \forall x \in C\}.$$

Recently, Ceng et al [29] considered the following inequality problem: find $(x^\star, y^\star) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^\star + x^\star - y^\star, x - x^\star\rangle \geq 0, \forall x \in C, \\ \langle \mu Bx^\star + y^\star - x^\star, x - y^\star\rangle \geq 0, \forall x \in C, \end{cases} \tag{1.12}$$

which is called general system of variational inequality, where $A, B : C \longrightarrow H$ are nonlinear mappings and $\lambda, \mu > 0$ are two constants. Simultaneously, they introduced an iteration scheme for finding common element of solutions of (1.12) and the fixed point problem of nonexpansive mappings in Hilbert space; and strong convergence theorems were achieved under appropriate condition on the iteration parameters.

Very recently, Yao et al [43] studied the following general system of variational inequality problem in the setting of a real Banach space E: find $(x^\star, y^\star) \in C \times C$ such that

$$\begin{cases} \langle Ay^\star + x^\star - y^\star, j(x - x^\star)\rangle \geq 0, \forall x \in C, \\ \langle Bx^\star + y^\star - x^\star, j(x - y^\star)\rangle \geq 0, \forall x \in C, \end{cases} \tag{1.13}$$

where $A, B : C \longrightarrow H$ are nonlinear mappings.

Most recently, Cai, Shehu and Iyiola [2] introduced and studied the following general system of variational inequality problem in the framework of a real Banach space:find $(x^\star, y^\star) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^\star + x^\star - y^\star, j(x - x^\star)\rangle \geq 0, \forall x \in C, \\ \langle \mu Bx^\star + y^\star - x^\star, j(x - y^\star)\rangle \geq 0, \forall x \in C, \end{cases} \tag{1.14}$$

where $A, B : C \longrightarrow H$ are nonlinear mappings and $\lambda, \mu > 0$ are two constants. Observe that if $\lambda = 1 = \mu$, then (1.14) reduces to (1.13).

We note here that the constraints of variety of real life problems inherent in image recovery, resource allocation, signal processing, etc can be expressed as the variational inequality problem. Consequently, the problem of finding solutions of variational inequality problems is currently an interesting area of research for a good number of renown mathematicians in nonlinear operator theory.

To solve the variational inequality problem of (1.11), Aoyama et al [1] studied the following algorithm:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Q_C(I - \lambda_n A)x_n, \tag{1.15}$$

where Q_C is sunny nonexpansive retraction from E onto C and $\alpha_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset (0, \infty)$ are real number sequences. They proved the following weak convergent result:

Theorem 1.11. (Aoyama et al [1])
Let C be a nonempty closed convex subset of 2-uniformly smooth and uniformly convex Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. Let $A : C \longrightarrow E$ be an α-inverse-strongly accretive operator with $VI(C, A) \neq \emptyset$. If $\{\lambda_n\}$ and $\{\alpha_n\}$ are chosen so that $\lambda_n \in \left[a, \dfrac{\alpha}{K^2}\right]$ for some $a > 0$ and $\alpha_n \in [b, c]$ for some b, c with $0 < b < c < 1$, then the sequence $\{x_n\}$ defined by (1.15) converges weakly to z, a solution of the variational inequality (1.11), where the real number K is the 2-uniformly smoothness constant of the Banach space E.

In approximating fixed points of nonexpansive mappings, many researchers in operator theory has employed the viscosity approximation method which was introduced by Moudafi [32] in the following manner: Let C be a nonempty, closed and convex subset of a real Banach space E. Let $T : C \longrightarrow C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$ and $f : C \longrightarrow C$ be a contraction mapping. The viscosity iteration method is defined as follows:
For $x_0 \in C$, let $\{x_n\}_{n\geq 1}$ be a sequence generated by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \tag{1.16}$$

where $\{\alpha_n\}_{n\geq 1}$ is a sequence of real numbers in $(0,1)$. Under appropriate conditions, the sequence defined by (1.16) converges to a fixed point of T.

Recently, Cai, Shehu and Iyiola [2] introduced the following iterative scheme: Let C be a nonempty, closed subset of a real uniformly convex and $2-$uniformly smooth Banach space E. Let $T : C \longrightarrow C$ be an asymptotically nonexpansive mapping such that $F(T) \neq \emptyset$ and $f : C \longrightarrow C$ be a contraction mapping. Then, the extragradient-viscosity iteration method for the above mapping and problem (1.14) is defined as follows:

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n z_n; \\ z_n = Q_C(I - \lambda A)u_n; \\ u_n = Q_C(I - \mu B)x_n, \end{cases} \tag{1.17}$$

Under suitable conditions on the iteration parameters, they proved strong convergence theorem of the sequence defined by (1.17) to common element of solution of the variational inequality problem (1.14) and fixed point problem of asymptotically nonexpansive mapping. More precisely, they proved the following theorem:

Theorem 1.12. (Cai, Shehu, Iyiola [2])
Let C be a nonempty closed convex subset of 2-uniformly smooth and uniformly convex Banach space X, which admits weakly sequentially continuous duality mapping. Assume that C is a sunny nonexpansive retract of X and let Q_C be the sunny nonexpansive retraction of X onto C. Let $A, B : C \longrightarrow X$ be α- inverse-strongly accretive and β-inverse-strongly accretive mappings, respectively. Let $f : C \longrightarrow C$ be a δ-strict contraction of C into itself with coefficient $\rho \in (0,1)$. Let $T : C \longrightarrow C$ be asymptotically nonexpansive self mapping on C such that $\mathcal{F} = F(T) \cap F(G) \neq \emptyset$, where G is as defined by Lemma 2.9. For arbitrarily chosen $x_1 \in C$, let the sequence $\{x_n\}_{n\geq 1}$ be defined iteratively as follows:

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n z_n; \\ z_n = Q_C(I - \lambda A)u_n; \\ u_n = Q_C(I - \mu B)x_n, \end{cases}$$

where $0 < \lambda < \dfrac{\alpha}{K^2}, 0 < \mu < \dfrac{\beta}{K^2}$. Suppose $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0,1]$ satisfying suitable conditons. Then, the sequence $\{x_n\}$ converges strongly to $q^\star = Q_F f(q)$ and $(q, q^\star)$ is a solution of problem (1.14), where $q^\star = Q_C(q - \mu S q)$ Q_F is the sunny nonexpansive retraction of C onto F.

Moltivated and inspired by the idea in [16] and other information above, we introduce a new mapping as follows: Let C be a nonempty, closed, and convex subset of a real uniformly convex and $2-$uniformly smooth Banach space E. Let $\{A_i\}_{i=1}^N, \{B_i\}_{i=1}^N : C \longrightarrow X$ be finite families of α-inverse-strongly accretive and β-inverse-strongly accretive operators, respectively and $\{\delta_{n,i}\}_{i=1}^N$ be a real sequence in $[0,1]$. We define the mapping $Z : C \longrightarrow C$ as follows:

$$\begin{cases} w_{n,0} = I; \\ w_{n,1} = \delta_{n,1} G^1_{\lambda\mu} w_{n,0} + (1 - \delta_{n,1}) w_{n,0}; \\ w_{n,2} = \delta_{n,1} G^2_{\lambda\mu} w_{n,1} + (1 - \delta_{n,2}) w_{n,1}; \\ \vdots \\ w_{n,N-1} = \delta_{n,N-1} G^{N-1}_{\lambda\mu} w_{n,N-2} + (1 - \delta_{n,N-1}) w_{n,N-2}; \\ Z_n = w_{n,N} = \delta_{n,N} G^N_{\lambda\mu} w_{n,N-1} + (1 - \delta_{n,N}) w_{n,N-1}, \end{cases} \tag{1.18}$$

where $\{G^i_{\lambda\mu}\}_{i=1}^N = Q_C[Q_C(I - \mu_i B_i) - \lambda_i A_i(I - \mu_i B_i)], 0 < \lambda_i < \dfrac{\alpha}{K^2}, 0 < \mu_i < \dfrac{\beta}{K^2}$, for $i = 1, 2, \cdots, N$, K^2 is a uniformly smoothness constant and I is the identity mapping. The above mapping Z_n is called Z-mapping generated by $G^1_{\lambda\mu}, G^2_{\lambda\mu}, \cdots, G^N_{\lambda\mu}$ and $\delta_{n,1}, \delta_{n,2}, \cdots, \delta_{n,N}$.

Using the above definition, we introduce an iterative scheme for finding a common solution for fixed point problem of finite family of asymptotically nonexpansive mappings and finite family of variational inequality problems as follows:

$$\begin{cases} x_0 \in C; \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \sum_{i=1}^N \gamma_{n,i} S_i^n y_n; \\ y_n = Z_n x_n, \end{cases} \tag{1.19}$$

where Z_n is as defined in (1.18) and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0,1]$ with $\alpha_n + \beta_n + \sum_{n=1}^{N} \gamma_{n,i} = 1$. In addition, we obtain strong convergence theorems of the scheme defined by (1.19) under some suitable conditions on the control sequences in the setting of 2-unformly smooth and uniformly convex real Banach space, which admits weakly sequentially duality mapping.

Remark 1.13. The following remarks are evident from (1.19):

(1) If $i = 1$ and $\delta_{n,1} = \delta_n = 1, \gamma_{n,1} = \gamma_n, A_1 = A$ and $B_1 = B$, then (1.19) reduces to:

$$\begin{cases} x_0 \in C; \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S^n y_n; \\ y_n = Q_C[Q_C(x_n - \mu B x_n) - \lambda A(x_n - \mu B x_n)], \end{cases} \tag{1.20}$$

(2) If $\mu = 0 = \lambda$ in (1.20), then we get

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S^n x_n, \tag{1.21}$$

where $x_0 \in C$.

(3) If $\mu = 0 = \lambda$ in (1.19), then we get

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \sum_{i=1}^{N} \gamma_{n,i} S_i^n x_n, \tag{1.22}$$

where $x_0 \in C$.

(4) If $\delta_{n,i} = 1, B_i = I, \mu_i = 0, \alpha_n = 0 = \beta_n$ and $S_i = S = I$, where I is the identity mapping, (1.19) reduces to:

$$x_{n+1} = \gamma_{n,0} x_n + \sum_{n=1}^{N} \gamma_{n,i} Q_C(I - \lambda_i A_i) x_n, \tag{1.23}$$

where $\sum_{i=0}^{N} \gamma_{in} = 1$.

(5) If $B_i = I, \mu_i = 0, \alpha_n = 0 = \beta_n$ and $S_i = S = I$, where I is the identity mapping, (1.19) reduces to:

$$\begin{cases} x_0 \in C; \\ x_{n+1} = \gamma_{n,0} x_n + \sum_{i=1}^{N} \gamma_{n,i} y_n; \\ y_n = Z_n x_n, \end{cases} \tag{1.24}$$

where $\sum_{i=0}^{N} \gamma_{in} = 1$.

Note that (1.17) is the same as (1.20), (1.21) is more general than (1.16) since T is a subclass of S, (1.23) generalizes (1.15) while (1.21) is more general than (1.20). Consequently, the results presented in this paper extend, improve and generalize some recently announced results in the existing literature (see, for example, [1]-[10], [17], [27]-[53] and the reference therein).

2 Preliminary

For the sake of convenience, we restate the following concepts and results:
Let E be a Banach space with its dimension greater than or equal to 2. The modulus of convexity of E is a function $\delta_E(\varepsilon) : (0,2] \longrightarrow (0,2]$ defined by

$$\delta_E(\varepsilon) = \inf\{1 - \|\frac{1}{2}(x+y)\| : \|x\| = 1, \|y\| = 1, \varepsilon = \|x - y\|\}.$$

A Banach space E is uniformly convex if and if $\delta_E(\varepsilon) > 0$, for all $\varepsilon \in (0,2]$.

Let E be a normed linear space and let $S = \{x \in E : \|x\| = 1\}$. E is called smooth if

$$\lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S$. E is called uniformly smooth if it is smooth and the limit above is attained uniformly for each $x, y \in S$.

Let E be a normed space with dimension greater than or equal to 2. The modulus of smoothness of E is the function $\rho_E : [0,\infty) \longrightarrow [0,\infty)$ such that

$$\rho_E(\tau) = \sup\left\{\frac{\|x+y\|+\|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\right\}.$$

It is known that a normed linear space E is uniformly smooth if

$$\lim_{\tau\to 0}\frac{\rho_E(\tau)}{\tau} = 0.$$

Note that if there exists a constant $c > 0$ and a real number $q > 1$ such that $\rho_E(\tau) \leq c\tau$, then E is called q-uniformly smooth. Typical examples of smooth spaces are L_p, ℓ_p and W_p^m for $1 < p < \infty$, where L_p, ℓ_p or W_p^m is 2-uniformly smooth and p-uniformly convex if $2 \leq p < \infty$; 2-uniformly convex and p-uniformly smooth if $1 < p < 2$.

Let D be a subset of C and let Q be a mapping of C into D. The Q is said to sunny if

$$Q(Qx + t(x - Qx)) = Qx, \tag{2.1}$$

whenever $Qx + t(x - Qx \in C$ and $t \geq 0$. A mapping Q of C into itself is called a retraction if $Q^2 = Q$. If mapping Q into itself is a retraction, then $Qz = z$ for every $z \in R(Q)$, where $R(Q)$ is the range of Q. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D. The following three lemmas [2.1, 2.2, 2.3] are known for sunny nonexpansive retraction:

Lemma 2.1. (see [28]) Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smmoth Banach space E and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then the set $F(T)$ is a sunny nonexpansive retraction on C.

Lemma 2.2. (see [20]) Let C be a nonempty closed convex subset of a smooth Banach space E and let Q_C be a retraction from E onto C. Let D be a nonempty subset of C. Let $Q : C \longrightarrow D$ be a retraction and J be a normalised duality map on E. Then, the following are equivalent:

(i) Q_C is both sunny and nonexpansive;
(ii) $\|Qx - Qy\|^2 \leq \langle x - y.J(Ox - Qy)\rangle, \forall x, y \in C$;
(iii) $\langle x - Q_C x, J(y - Q_C x\rangle \leq 0, \forall x \in E \quad \text{and} \quad y \in CDnonumber$.

It is well known that if E is a Hilbert space, then a sunny nonexpansive retraction Q_C coincides with the metric projection P_C from E onto C. Let C be a nonempty closed and convex subset of a smooth Banach space E, $x \in E$ and $x_0 \in C$. Then, we have from Lemma 2.2 that $x_0 \in Q_C x$ if and only if $\rangle x - x_0, J(y - x_0)\langle \leq 0, \forall y \in C$, where Q_C is a sunny nonexpansive retraction from E onto C.

Lemma 2.3. (see [1]) Let C be a nonempty closed convex subset of a smooth Banach space E, Q_C be a sunny nonexpansive retraction from E onto C and A be an accretive operator of C into E. Then, for all $\lambda > 0$,

$$VI(C, A) = Fix(Q_C(I - \lambda A)).$$

Proposition 2.4. (see [2], also see [28, Theorem 4]) Let D be a closed and convex subset of a reflexive Banach space E with a uniformly Gateaux differentiable norm. If C is nonexpansive retract of D, then it in fact a sunny nonexpansive retract of D.

Lemma 2.5. (see [31]) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let β_n be a sequence in $[0, 1]$, which satisfies the following condition: $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose $x_{n+1} = \beta_n x_n + (1 - \beta_n)y_n, n \geq 0$ and $limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \leq 0$. Then, $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 2.6. (see [48]) Let $\{a_n\}$ be a sequence of nonnegative real numbers with $a_{n+1} = (1 - \alpha_n)a_n + b_n, n \geq 0$, where α_n is a sequence in $(0, 1)$ and b_n is a sequence in R such that $\sum_{n=0}^{\infty} = \infty$ and $\limsup_{n\to\infty}\dfrac{b_n}{\alpha_n} \leq 0$. Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7. (see [25], [37]) Let E be a real smooth and uniformly convex Banach space and $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \longrightarrow R$ with $g(0) = 0$ such that $g(\|x - y\| \leq \|x\|^2 - 2\langle x, j(y)\rangle + \|y\|^2$, for all $x, y \in B_r$.

Lemma 2.8. (see [9]) Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E. Let the mapping $A : C \longrightarrow E$ be an α-inverse-strongly accretive. Then, we have the following:

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \leq \|x - y\|^2 + 2\lambda(\lambda K^2 - \alpha)\|Ax - Ay\|^2,$$

where $\lambda > 0$. In particular, if $0 < \lambda \leq \dfrac{\alpha}{K^2}$, then $I - \lambda A$ is nonexpansive.

Lemma 2.9. (see [9]) Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E. Assume that C is sunny nonexpansive retract of E and let Q_C be a sunny nonexpansive retraction of E onto C. Let the mappings $A, B : C \longrightarrow E$ be α-inverse-strongly accretive and β-inverse-strongly accretive respectively. Let $G : C \longrightarrow C$ be a mapping defined by

$$G(x) = Q_C[Q_C(x - \mu Bx) - \lambda A Q_{C(x-\mu Bx)], \forall x \in C.}$$

If $0 < \lambda < \dfrac{\alpha}{K^2}$ and $0 < \mu < \dfrac{\beta}{K^2}$, then $G : C \longrightarrow C$ is nonexpansive.

Lemma 2.10. (see [14] Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E and let Q_C be a sunny nonexpansive retraction of E onto C. $A, B : C \longrightarrow E$ be two nonlinear mappings. For a given $x^\star, y^\star \in C, (x^\star, y^\star)$ is a solution to problem (1.18) if and only if $x^\star = Q_C(y^\star - \lambda Ay^\star)$, where $y^\star = Q_C(x^\star - \mu Bx^\star)$, that is $x^\star = Gx^\star$, where G is as defined by Lemma 2.9.

Lemma 2.11. (see [1],[14]) Let E be a real Banach space and $J : E \longrightarrow 2^{E^\star}$ be a normalised duality mapping, then for any $x, y \in E$, the following inequalities hold:

$$\begin{aligned} \|x + y\|^2 &\leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \forall j(x + y) \in J(x + y); \\ \|x + y\|^2 &\geq \|x\|^2 + 2\langle y, j(x)\rangle, \forall j(x) \in J(x); \end{aligned}$$

Lemma 2.12. (see [3]) Let C be a nonempty closed convex subset of a real uniformly convex Banach space E and let T a nonexpansive mapping of C into itself. If $\{x_n\}$ is a sequence of C such that $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then x is a fixed point of T.

Lemma 2.13. (see [30]) Let E be a Banach space satisfying weakly continuous duality map, K a nonempty closed convex subset of E and let $T : K \longrightarrow K$ be an asymptotically nonexpansive mapping with a fixed point. Then $I - T$ is demiclosed at zero; if $\{x_n\}$ is a sequence of K such that $x_n \rightharpoonup x$ and if $x_n - Tx_n \to 0$, then $x - Tx = 0$.

Lemma 2.14. (see [2]) Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \longrightarrow C$ be a k-strictly pseudocontractive mapping. Define $A = I - T : C \longrightarrow H$. Then, A is $\dfrac{1-k}{2}$-inverse-strongly accretive mapping; that is, for all $x, y \in C$,

$$\langle x - y, Ax - Ay\rangle \geq \frac{1-k}{2}\|Ax - Ay\|^2.$$

3 Results

Lemma 3.1. Let C be a nonempty closed convex subset of 2-uniformly smooth and uniformly convex Banach space X whose norm is strictly convex. Let $\{A_i\}_i^m, \{B_i\}_i^m : C \longrightarrow C$ be finite families of α-inverse-strongly accretive and β-inverse-strongly accretive operators, respectively such that $\cap_{i=1}^m F(G^i_{\lambda\mu}) \neq \emptyset$. where $G^i_{\lambda\mu}$ is as defined in (1.18). Let $\{\delta_{n,i}\}_{i=1}^N$ be a sequence of real numbers such that $0 \leq \delta_{n,i} \leq 1$ for $i = 1, 2, \cdots, N$. Let Z_n be the Z-mapping generated by $\{G^i_{\lambda\mu}\}_{i=1}^N$ and $\{\delta_{n,i}\}_{i=1}^N$. Then, $\{w_{n,i}\}_{i=1}^{N-1}$ and Z are nonexpansive. Moreover, $Fix(Z) = \cap_{i=1}^m F(G^i_{\lambda\mu})$.

Proof . For each $i = 1, 2, \cdots, N, G_{\lambda\mu^i}$ is nonexpansive (see lemma 2.7). Thus, from (1.18), we have the following estimates:

$$\begin{aligned}
\|w_{n,1}x - w_{n,1}y\| &= \|\delta_{n,1}G^1_{\lambda\mu}x + (1-\delta_{n,1})x - (\delta_{n,1}G^1_{\lambda\mu}y + (1-\delta_{n,1})y)\| \\
&\leq \delta_{n,1}\|G^1_{\lambda\mu}x - G^1_{\lambda\mu}y\| + (1-\delta_{n,1})\|x-y\| \\
&\leq \delta_{n,1}\|x-y\| + (1-\delta_{n,1})\|x-y\| \\
&\leq \|x-y\|,
\end{aligned}$$

$$\begin{aligned}
\|w_{n,2}x - w_{n,2}y\| &= \|\delta_{n,2}G^2_{\lambda\mu}w_{n,1}x + (1-\delta_{n,2})w_{n,1}x - (\delta_{n,2}G^2_{\lambda\mu}w_{n,1}y + (1-\delta_{n,2})w_{n,1}y)\| \\
&\leq \delta_{n,2}\|G^2_{\lambda\mu}w_{n,1}x - G^2_{\lambda\mu}w_{n,1}y\| + (1-\delta_{n,2})\|w_{n,1}x - w_{n,1}y\| \\
&\leq \delta_{n,2}\|w_{n,1}x - w_{n,1}y\| + (1-\delta_{n,2})\|w_{n,1}x - w_{n,1}y\| \\
&\leq \|w_{n,1}x - w_{n,1}y\|; \\
&\leq \|x-y\|,
\end{aligned}$$

$$\begin{aligned}
\|w_{n,3}x - w_{n,3}y\| &= \|\delta_{n,3}G^3_{\lambda\mu}w_{n,2}x + (1-\delta_{n,3})w_{n,2}x - (\delta_{n,3}G^3_{\lambda\mu}w_{n,2}y + (1-\delta_{n,3})w_{n,2}y)\| \\
&\leq \delta_{n,3}\|G^3_{\lambda\mu}w_{n,2}x - G^3_{\lambda\mu}w_{n,2}y\| + (1-\delta_{n,3})\|w_{n,1}x - w_{n,2}y\| \\
&\leq \delta_{n,3}\|w_{n,2}x - w_{n,2}y\| + (1-\delta_{n,3})\|w_{n,3}x - w_{n,1}y\| \\
&\leq \|w_{n,2}x - w_{n,2}y\|; \\
&\leq \|x-y\|
\end{aligned}$$

Continuing in this manner, we obtain that

$$\begin{aligned}
\|w_{n,N-1}x - w_{n,N-1}y\| &= \|\delta_{n,N-1}G^{N-1}_{\lambda\mu}w_{n,N-2}x + (1-\delta_{n,N-1})w_{n,N-2}x \\
&\quad -(\delta_{n,N-1}G^{N-1}_{\lambda\mu}w_{n,N-2}y + (1-\delta_{n,N-1})w_{n,N-2}y)\| \\
&\leq \delta_{n,N-1}\|G^{N-1}_{\lambda\mu}w_{n,N-2}x - G^{N-1}_{\lambda\mu}w_{n,N-2}y\| \\
&\quad +(1-\delta_{n,N-1})\|w_{n,N-2}x - w_{n,N-2}y\| \\
&\leq \delta_{n,N-1}\|w_{n,N-2}x - w_{n,N-2}y\| + (1-\delta_{n,N-1})\|w_{n,N-2}x - w_{n,N-2}y\| \\
&\leq \|w_{n,N-2}x - w_{n,N-2}y\|; \\
&\leq \|x-y\|;
\end{aligned}$$

Next, we show that $Fix(Z_n) = \cap_{i=1}^N F(G^i_{\lambda\mu})$. Firstly, we show that $\cap_{i=1}^N F(G^i_{\lambda\mu}) \subseteq Fix(Z_n)$. Let $a \in \cap_{i=1}^N F(G^i_{\lambda\mu})$, then

$$\begin{aligned}
w_{n,1}a &= \delta_{n,1}G^1_{\lambda\mu}a + (1-\delta_{n,1})a = a \\
w_{n,2}a &= \delta_{n,2}G^2_{\lambda\mu}w_{n,1}a + (1-\delta_{n,2})w_{n,1}a \\
&= \delta_{n,2}G^2_{\lambda\mu}a + (1-\delta_{n,2})a = a \\
w_{n,N-1}a &= \delta_{n,N-1}G^{N-1}_{\lambda\mu}w_{n,N-2}a + (1-\delta_{n,N-1})w_{n,N-2}a \\
&\vdots \\
&= \delta_{n,N-1}G^{N-1}_{\lambda\mu}a + (1-\delta_{n,N-1})a = a \\
w_{n,N}a &= \delta_{n,N}G^N_{\lambda\mu}w_{n,N-1}a + (1-\delta_{n,N})w_{n,N-1}a \\
&= \delta_{n,N}G^N_{\lambda\mu}a + (1-\delta_{n,N})a = a
\end{aligned}$$

Hence, $Z_na = a$; that is $a \in Fix(Z_n)$.

Again, we will show that $Fix(Z_n) \subseteq \cap_{i=1}^N F(G^i_{\lambda\mu})$. Let $b \in Fix(Z_n)$ and $a \in \cap_{i=1}^N F(G^i_{\lambda\mu})$. By the definition of Z_n, we get

$$\begin{aligned}
\|a-b\| &= \|Z_n a - b\| \\
&= \|\delta_{n,N} G^N_{\lambda\mu} w_{n,N-1} a - b + (1-\delta_{n,N}) w_{n,N-1} a - b\| \\
&= \|\delta_{n,N} (G^N_{\lambda\mu} w_{n,N-1} a - b) + (1-\delta_{n,N})(w_{n,N-1} a - b)\| \\
&\le \delta_{n,N} \|G^N_{\lambda\mu} w_{n,N-1} a - b\| + (1-\delta_{n,N})\|w_{n,N-1} a - b\| \\
&\le \delta_{n,N} \|w_{n,N-1} a - b\| + (1-\delta_{n,N})\|w_{n,N-1} a - b\| \\
&\le \|w_{n,N-1} a - b\| \\
&= \|\delta_{n,N-1} G^{N-1}_{\lambda\mu} w_{n,N-2} a - b + (1-\delta_{n,N-1}) w_{n,N-2} a - b\| \\
&= \|\delta_{n,N-1} (G^{N-1}_{\lambda\mu} w_{n,N-2} a - b) + (1-\delta_{n,N-1})(w_{n,N-2} a - b)\| \\
&\le \delta_{n,N-1} \|G^{N-1}_{\lambda\mu} w_{n,N-2} a - b\| + (1-\delta_{n,N-1})\|w_{n,N-2} a - b\| \\
&\le \delta_{n,N-1} \|w_{n,N-2} a - b\| + (1-\delta_{n,N-1})\|w_{n,N-2} a - b\| \\
&\le \|w_{n,N-2} a - b\| \\
&\vdots \\
&= \|\delta_{n,2} G^2_{\lambda\mu} w_{n,1} a - b + (1-\delta_{n,2}) w_{n,1} a - b\| \\
&= \|\delta_{n,2} (G^2_{\lambda\mu} w_{n,1} a - b) + (1-\delta_{n,2})(w_{n,1} a - b)\| \\
&\le \delta_{n,2} \|G^2_{\lambda\mu} w_{n,1} a - b\| + (1-\delta_{n,2})\|w_{n,1} a - b\| \\
&\le \delta_{n,2} \|w_{n,1} a - b\| + (1-\delta_{n,2})\|w_{n,1} a - b\| \\
&\le \|w_{n,1} a - b\| \\
&= \|\delta_{n,1} G^1_{\lambda\mu} a - b + (1-\delta_{n,1}) a - b\| \\
&= \|\delta_{n,1} (G^1_{\lambda\mu} a - b) + (1-\delta_{n,1})(a - b)\| \\
&\le \delta_{n,1} \|G^1_{\lambda\mu} a - b\| + (1-\delta_{n,1})\|a - b\| \qquad (3.1) \\
&\le \delta_{n,1} \|a - b\| + (1-\delta_{n,1})\|a - b\| \\
&\le \|a-b\| \qquad (3.2)
\end{aligned}$$

(3.2) implies that

$$\|G^1_{\lambda\mu} a - b\| = \|a-b\|$$

By the strict convexity of the norm of X, we obtain $G^1_{\lambda\mu} a = a$; that is, $a \in F(G^1_{\lambda\mu})$. This implies that $w_{n,1} a = a$. Also, from (3.2) and the fact that $w_{n,1} a = a$, we get

$$\begin{aligned}
\|a-b\| &= \|\delta_{n,2} G^2_{\lambda\mu} w_{n,1} a + (1-\delta_{n,2}) w_{n,1} a - b\| \\
&= \|\delta_{n,2} (G^2_{\lambda\mu} w_{n,1} a - b) + (1-\delta_{n,2})(w_{n,1} a - b)\| \\
&\le \delta_{n,2} \|G^2_{\lambda\mu} w_{n,1} a - b\| + (1-\delta_{n,2})\|w_{n,1} a - b\| \\
&= \delta_{n,2} \|G^2_{\lambda\mu} a - b\| + (1-\delta_{n,2})\|a - b\|, \qquad (3.3)
\end{aligned}$$

so that

$$\|G^2_{\lambda\mu} a - b\| = \|a-b\|$$

By the strict convexity of the norm of X, we obtain $G^2_{\lambda\mu} a = a$; that is, $a \in F(G^2_{\lambda\mu})$. This implies that $w_{n,2} a = a$. Continuing in this manner, we obtain

$$a = G^1_{\lambda\mu} a = G^2_{\lambda\mu} a = \cdots = G^{N-1}_{\lambda\mu} a$$

and

$$a = w_{n,1} a = w_{n,2} a = \cdots = w_{n,N-1} a.$$

Since $a \in Fix(Z_n) = Fix(w_{n,N})$ and $w_{n,N-1} a = a$, it follows that

$$a = \delta_{n,N} G^N_{\lambda\mu} a + (1-\delta_{n,N}) a.$$

This implies that $a = G^N_{\lambda\mu} a$. Consequently, $a \in F(G^N_{\lambda\mu})$. This completes the proof. □

Lemma 3.2. Let C be a nonempty closed convex subset of 2-uniformly smooth and uniformly convex Banach space X, which admits weakly sequentially continous duality mapping. Assume that C is a sunny nonexpansive retract of X and let Q_C be the sunny nonexpansive retraction of X onto C. Let $\{A_i, B_i\}_{=1}^{m} : C \longrightarrow X$ be finite families of α-inverse-strongly accretive and β-inverse-strongly accretive operators, respectively. Let $f : C \longrightarrow C$ be a ρ-strict contraction of C into itself with coefficient $\rho \in (0,1)$. Let $\{S_i\}_{i=1}^{m} : C \longrightarrow C$ be a finite families of asymptotically nonexpansive self mappings on C such that $\mathcal{F} = (\cap_{i=1}^{m} F(S_i)) \cap (\cap_{i=1}^{m} F(G^i_{\lambda\mu})) \neq \emptyset$, where $G^i_{\lambda\mu}$ is as defined in (1.18). For arbitrarily chosen $x_1 \in C$, let the sequence $\{x_n\}_{n\geq 1}$ be defined iteratively as follows:

$$\begin{cases} x_0 \in C; \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \sum_{i=1}^{N} \gamma_{n,i} S_i^n y_n; \\ y_n = Z_n x_n, \end{cases} \tag{3.4}$$

where Z_n is as defined in (1.18) and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0,1]$ satisfying the following conditions:

(a) $\alpha_n + \beta_n + \sum_{n=1}^{N} \gamma_{n,i} = 1$;
(b) $0 < \liminf \beta_n \leq \limsup \beta_n < 1$;
(c) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} \gamma_n^2 < \infty, \lim_{n\to\infty} \dfrac{k_n - 1}{\alpha_n} = 0$;
(d) S satisfy the asymptotically regularity: $\lim_{n\to\infty} \|S^{n+1}x_n - S^n x_n\| = 0$.

Then, the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Proof . Let $k_n = max_{1\leq i\leq m}\{k_n^{(i)}\}$. Firstly, we prove that $\{x_n\}$ is bounded. Let $x^\star \in \mathcal{F}$. Then, it follows from Lemma 2.9 that $x^\star = Q_C(Q_C(I - \mu B)x^\star - \lambda A Q_C(I - \mu B)x^\star))$. Let $s^\star = Q_C(I - \mu B)x^\star)$, then $x^\star = Q_C(I - \lambda A)s^\star)$. Also, from Lemma 3.1, we have

$$\|y_n - x^\star\| = \|G^i_{\lambda\mu} x_n - G^i_{\lambda\mu} x^\star\| \leq \|x_n - x^\star\| \tag{3.5}$$

By condition (c), there exists a constant ϵ with $0 < \epsilon < 1 - \delta$ and $\sum_{i=1}^{m} \gamma_{n,i}(k_n - 1) < \epsilon\alpha_n$, for each $i = 1, 2, \cdots, m$, such that following estimates hold:

$$\begin{aligned} \|x_{n+1} - x^\star\| &= \|\alpha_n f(x_n) + \beta_n x_n + \sum_{i=1}^{m} \gamma_{n,i} S_i^n y_n - x^\star\| \\ &= \|\alpha_n(f(x_n) - x^\star) + \beta_n(x_n - x^\star) + \sum_{i=1}^{m} \gamma_{n,i}(S_i^n y_n - x^\star)\| \\ &\leq \alpha_n\|f(x_n) - x^\star\| + \beta_n\|x_n - x^\star\| + \sum_{i=1}^{m} \gamma_{n,i}\|S_i^n y_n - x^\star\| \\ &\leq \alpha_n\rho\|x_n - x^\star\| + \alpha_n\|f(x^\star) - x^\star\| + (1 - \alpha_n)\|x_n - x^\star\| \\ &\quad + \sum_{i=1}^{m} \gamma_{n,i}(k_n^{(i)} - 1)\|y_n - x^\star\| \\ &\leq \alpha_n\rho\|x_n - x^\star\| + \alpha_n\|f(x^\star) - x^\star\| + (1 - \alpha_n)\|x_n - x^\star\| \\ &\quad + \sum_{i=1}^{m} \gamma_{n,i}(k_n - 1)\|x_n - x^\star\| \\ &= (1 - (1 - \rho - \epsilon)\alpha_n)\|x_n - x^\star\| + (1 - \rho - \epsilon)\alpha_n \frac{\|f(x^\star) - x^\star\|}{1 - \rho - \epsilon} \\ &\leq max\{\|x_n - x^\star\|, \frac{\|f(x^\star) - x^\star\|}{1 - \rho - \epsilon}\} \end{aligned}$$

By applying mathematical induction, we obtain

$$\|x_n - x^\star\| \leq max\{\|x_0 - x^\star\|, \frac{\|f(x^\star) - x^\star\|}{1 - \rho - \epsilon}\}, n \geq 1,$$

which implies that the sequence $\{x_n\}_{n\geq 1}$ is bounded, and so are the sequences $\{f(x_n\}_{n\geq 1}$and $\{y_n\}_{n\geq 1}$.

Next, we show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Define the sequence $\{x_n\}$ as follows:

$$x_{n+1}=(1-\alpha_n-\gamma_n)x_n+(\alpha_n+\gamma_n)\ell_n, \tag{3.6}$$

Then, it follows that

$$\begin{aligned}
\ell_{n+1}-\ell_n &= \frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{x_{n+1}-\beta_n x_n}{1-\beta_n}\\
&= \frac{\alpha_{n+1}f(x_{n+1})+\beta_{n+1}x_{n+1}+\sum_{i=1}^{m}\gamma_{n+1,i}S_i^{n+1}x_{n+1}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}\\
&\quad -\frac{\alpha_n f(x_n)+\beta_n x_n+\sum_{i=1}^{m}\gamma_{n,i}S_i^n x_n-\beta_n x_n}{1-\beta_n}\\
&= \frac{\alpha_{n+1}f(x_{n+1})+\sum_{i=1}^{m}\gamma_{n+1,i}S_i^{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n f(x_n)+\sum_{i=1}^{m}\gamma_{n,i}S_i^n x_n}{1-\beta_n}\\
&= \frac{\alpha_{n+1}f(x_{n+1})-(1-\beta_{n+1})S_i^{n+1}x_{n+1}+\sum_{i=1}^{m}\gamma_{n+1,i}S_i^{n+1}x_{n+1}}{1-\beta_{n+1}}\\
&\quad -\frac{\alpha_n f(x_n)-(1-\beta_n)S_i^n x_n+\sum_{i=1}^{m}\gamma_{n,i}S_i^n x_n}{1-\beta_n}+S_i^{n+1}x_{n+1}-S_i^n x_n\\
&= \frac{\alpha_{n+1}(f(x_{n+1})-S_i^{n+1}x_{n+1})}{1-\beta_{n+1}}-\frac{\alpha_n(f(x_n)-S_i^n x_n)}{1-\beta_n}+S_i^{n+1}x_{n+1}-S_i^n x_n,
\end{aligned}$$

for $i=1,2,\cdots,m$. The last equation implies that

$$\begin{aligned}
\|\ell_{n+1}-\ell_n\|-\|x_{n+1}-x_n\| &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-S_i^{n+1}x_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|f(x_n)-S_i^n x_n\|\\
&\quad +\|S_i^{n+1}x_{n+1}-S_i^n x_n\|-\|x_{n+1}-x_n\|\\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-S_i^{n+1}x_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|f(x_n)-S_i^n x_n\|\\
&\quad +\|S_i^{n+1}x_{n+1}-S_i^{n+1}x_n\|+\|S_i^{n+1}x_n-S_i^n x_n\|-\|x_{n+1}-x_n\|\\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-S_i^{n+1}x_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|f(x_n)-S_i^n x_n\|\\
&\quad +k_n^i\|x_{n+1}-x_n\|+\|S_i^{n+1}x_n-S_i^n x_n\|-\|x_{n+1}-x_n\|\\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-S_i^{n+1}x_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|f(x_n)-S_i^n x_n\|\\
&\quad +\|S_i^{n+1}x_n-S_i^n x_n\|+(k_n-1)\|x_{n+1}-x_n\|
\end{aligned}$$

Since $\lim_{n\to\infty}k_n=1$ and, for each $i=1,2,\cdots,m, S_i$ is asymptotically regular, it follows from conditions (c,d) that

$$\limsup_{n\to\infty}(\|\ell_{n+1}-\ell_n\|-\|x_{n+1}-x_n\|)\le 0.$$

Using Lemma 2.3, we get

$$\lim_{n\to\infty}\|\ell_n-x_n\|=0, \tag{3.7}$$

and by (3.6), we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|\ell_n-x_n\|=0. \tag{3.8}$$

□

Lemma 3.3. Let C be a nonempty closed convex subset of 2-uniformly smooth and uniformly convex Banach space X, which admits weakly sequentially continous duality mapping. Under the assumptions of Lemma 3.2, $\lim_{n\to\infty}\|x_n-S_ix_n\|=0 fori=1,2,\cdots,m$.

Proof . From

$$\begin{aligned}
\|x_{n+1}-S_i^n y_n\| &= \|\alpha_n f(x_n)+\beta_n x_n+\sum_{i=1}^{m}\gamma_{n,i}S_i^n y_n-S_i^n y_n\|\\
&= \|\alpha_n f(x_n)+\beta_n x_n+(1-\alpha_n-\beta_n)S_i^n y_n-S_i^n y_n\|\\
&= \|\alpha_n(f(x_n)-S_i^n y_n)+\beta_n(x_n-S_i^n y_n)\|\\
&\le \alpha_n\|f(x_n)-S_i^n y_n\|+\beta_n\|x_n-S_i^n y_n\|\\
&\le \alpha_n\|f(x_n)-S_i^n y_n\|+\beta_n\|x_n-x_{n+1}\|+\beta_n\|x_{n+1}-S_i^n y_n\|,
\end{aligned}$$

we get

$$\|x_{n+1}-S_i^n y_n\|\le\frac{\alpha_n}{1-\beta_n}\|f(x_n)-S_i^n y_n\|+\frac{\beta_n}{1-\beta_n}\|x_n-x_{n+1}\|\to 0\quad\text{as}\quad n\to\infty. \tag{3.9}$$

Observe that

$$\|x_n-S_i^n y_n\|\le\|x_n-x_{n+1}\|+\|x_{n+1}-S_i^n y_n\|\to 0\quad\text{as}\quad n\to\infty. \tag{3.10}$$

Using (1.18), we have

$$\begin{aligned}
\|w_{n,N}x_n-x^\star\| &= \|\delta_{n,N}G_{\lambda\mu}^N w_{n,N-1}x_n+(1-\delta_{n,N})w_{n,N-1}x_n-x^\star\|\\
&= \|\delta_{n,N}(G_{\lambda\mu}^N w_{n,N-1}x_n-x^\star)+(1-\delta_{n,N})(w_{n,N-1}x_n-x^\star)\|\\
&\le \delta_{n,N}\|G_{\lambda\mu}^N w_{n,N-1}x_n-x^\star\|+(1-\delta_{n,N})\|w_{n,N-1}x_n-x^\star\|\\
&\le \delta_{n,N}\|w_{n,N-1}x_n-x^\star\|+(1-\delta_{n,N})\|w_{n,N-1}x_n-x^\star\|\\
&\le \|w_{n,N-1}x_n-x^\star\|\\
&= \|\delta_{n,N-1}G_{\lambda\mu}^{N-1}w_{n,N-2}x_n+(1-\delta_{n,N-1})w_{n,N-2}x_n-x^\star\|\\
&= \|\delta_{n,N-1}(G_{\lambda\mu}^{N-1}w_{n,N-2}x_n-x^\star)+(1-\delta_{n,N-1})(w_{n,N-2}x_n-x^\star)\|\\
&\le \delta_{n,N-1}\|G_{\lambda\mu}^{N-1}w_{n,N-2}x_n-x^\star\|+(1-\delta_{n,N-1})\|w_{n,N-2}x_n-x^\star\|\\
&\le \delta_{n,N-1}\|w_{n,N-2}x_n-x^\star\|+(1-\delta_{n,N-1})\|w_{n,N-2}x_n-x^\star\|\\
&\le \|w_{n,N-2}x_n-x^\star\|\\
&\vdots\\
&\le \|w_{n,2}x_n-x^\star\|\\
&= \|\delta_{n,2}G_{\lambda\mu}^2 w_{n,1}x_n+(1-\delta_{n,2})w_{n,1}x_n-x^\star\|\\
&= \|\delta_{n,2}(G_{\lambda\mu}^2 w_{n,1}x_n-x^\star)+(1-\delta_{n,2})(w_{n,1}x_n-x^\star)\|\\
&\le \delta_{n,2}\|G_{\lambda\mu}^2 w_{n,1}x_n-x^\star\|+(1-\delta_{n,2})\|w_{n,1}x_n-x^\star\|\\
&\le \delta_{n,2}\|w_{n,1}x_n-x^\star\|+(1-\delta_{n,2})\|w_{n,1}x_n-x^\star\|\\
&\le \|w_{n,1}x_n-x^\star\|
\end{aligned} \tag{3.11}$$

(3.11) implies that

$$\begin{aligned}
\|w_{n,N}x_n-x^\star\|^2 &\le \|\delta_{n,1}G_{\lambda\mu}^1 x_n+(1-\delta_{n,1})x_n-x^\star\|^2\\
&= \|\delta_{n,1}(G_{\lambda\mu}^1 x_n-x^\star)+(1-\delta_{n,1})(x_n-x^\star)\|^2\\
&\le \delta_{n,1}^2\|G_{\lambda\mu}^1 x_n-x^\star\|^2+(1-\delta_{n,1})^2\|x_n-x^\star\|^2\\
&\quad +2\delta_{n,1}(1-\delta_{n,1})\|G_{\lambda\mu}^1 x_n-x^\star\|\|x_n-x^\star\|\\
&\le \delta_{n,1}^2\|G_{\lambda\mu}^1 x_n-x^\star\|^2+(1-\delta_{n,1})^2\|x_n-x^\star\|^2\\
&\quad +\delta_{n,1}(1-\delta_{n,1})[\|G_{\lambda\mu}^1 x_n-x^\star\|^2+\|x_n-x^\star\|^2]\\
&\le [\delta_{n,1}^2+\delta_{n,1}(1-\delta_{n,1})]\|G_{\lambda\mu}^1 x_n-x^\star\|^2+[(1-\delta_{n,1})^2+\delta_{n,1}(1-\delta_{n,1})]\|x_n-x^\star\|^2\\
&= \delta_{n,1}\|G_{\lambda\mu}^1 x_n-x^\star\|^2+(1-\delta_{n,1})\|x_n-x^\star\|^2\\
&= \delta_{n,1}\|Q_C((I-\mu_1 B_1)x_n-\lambda A_1(I-\mu_1 B_1)x_n)-x^\star\|^2\\
&\quad +(1-\delta_{n,1})\|x_n-x^\star\|^2
\end{aligned} \tag{3.12}$$

By setting $u_{n,1} = Q_C((I - \mu_1 B_1)x_n)$ and $v_{n,1} = Q_C((I - \lambda_1 A_1)u_{n,1})$, (3.12) becomes

$$\|w_{n,N}x_n - x^\star\|^2 \leq \delta_{n,1}\|v_{n,1} - x^\star\|^2 + (1 - \delta_{n,1})\|x_n - x^\star\|^2 \tag{3.13}$$

From Lemma 2.8, we obtain

$$\begin{aligned} \|u_{n,1} - s^\star\|^2 &= \|Q_C(I - \mu_1 B_1)x_n) - Q_C(I - \mu_1 B_1)x^\star)\|^2 \\ &\leq \|(I - \mu_1 B_1)x_n - (I - \mu_1 B_1)x^\star\|^2 \\ &= \|x_n - x^\star - \mu_1(B_1 x_n - B_1 x^\star)\|^2 \\ &\leq \|x_n - x^\star\|^2 - 2\mu_1(\beta - K^2\beta)\|B_1 x_n - B_1 x^\star\|^2 \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} \|v_{n,1} - x^\star\|^2 &= \|Q_C(I - \lambda_1 A_1)u_{n,1}) - Q_C(I - \lambda_1 A_1)s^\star)\|^2 \\ &\leq \|(I - \lambda_1 A_1)u_{n,1} - (I - \lambda_1 A_1)s^\star\|^2 \\ &= \|u_{n,1} - s^\star - \lambda_1(A_1 u_{n,1} - A_1 s^\star)\|^2 \\ &\leq \|u_{n,1} - s^\star\|^2 - 2\lambda_1(\alpha - K^2\alpha)\|A_1 u_{n,1} - A_1 s^\star\|^2 \end{aligned} \tag{3.15}$$

(3.14) and (3.15) imply

$$\begin{aligned} \|v_{n,1} - x^\star\|^2 &\leq \|x_n - x^\star\|^2 - 2\mu_1(\beta - K^2\beta)\|B_1 x_n - B_1 x^\star\|^2 \\ &\quad -2\lambda_1(\alpha - K^2\alpha)\|A_1 u_{n,1} - A_1 s^\star\|^2 \end{aligned} \tag{3.16}$$

(3.13) and (3.16) imply that

$$\begin{aligned} \|w_{n,N}x_n - x^\star\|^2 &\leq \delta_{n,1}[\|x_n - x^\star\|^2 - 2\mu_1(\beta - K^2\beta)\|B_1 x_n - B_1 x^\star\|^2 \\ &\quad -2\lambda_1(\alpha - K^2\alpha)\|A_1 u_{n,1} - A_1 s^\star\|^2] + (1 - \delta_{n,1})\|x_n - x^\star\|^2 \\ &\leq \|x_n - x^\star\|^2 - 2\delta_{n,1}\mu_1(\beta - K^2\beta)\|B_1 x_n - B_1 x^\star\|^2 \\ &\quad -2\delta_{n,1}\lambda_1(\alpha - K^2\alpha)\|A_1 u_{n,1} - A_1 s^\star\|^2 \end{aligned} \tag{3.17}$$

Also, from (3.11), we obtain

$$\begin{aligned} \|w_{n,N}x_n - x^\star\|^2 &\leq \|w_{n,2}x_n - x^\star\|^2 \\ &\leq \|\delta_{n,2}G^2_{\lambda\mu}w_{n,1}x_n + (1 - \delta_{n,2})w_{n,1}x_n - x^\star\|^2 \\ &= \|\delta_{n,2}(G^2_{\lambda\mu}w_{n,1}x_n - x^\star) + (1 - \delta_{n,2})(w_{n,1}x_n - x^\star)\|^2 \\ &\leq \delta^2_{n,2}\|G^2_{\lambda\mu}w_{n,1}x_n - x^\star\|^2 + (1 - \delta_{n,2})^2\|w_{n,1}x_n - x^\star\|^2 \\ &\quad +2\delta_{n,2}(1 - \delta_{n,2})\|G^2_{\lambda\mu}w_{n,1}x_n - x^\star\|\|w_{n,1}x_n - x^\star\| \\ &\leq \delta^2_{n,2}\|G^2_{\lambda\mu}w_{n,1}x_n - x^\star\|^2 + (1 - \delta_{n,2})^2\|w_{n,1}x_n - x^\star\|^2 \\ &\quad +\delta_{n,2}(1 - \delta_{n,2})[\|G^2_{\lambda\mu}w_{n,1}x_n - x^\star\|^2 + \|w_{n,1}x_n - x^\star\|^2] \\ &\leq [\delta^2_{n,2} + \delta_{n,2}(1 - \delta_{n,2})]\|G^2_{\lambda\mu}w_{n,1}x_n - x^\star\|^2 + [(1 - \delta_{n,2})^2 \\ &\quad +\delta_{n,2}(1 - \delta_{n,2})]\|w_{n,1}x_n - x^\star\|^2 \\ &= \delta_{n,2}\|G^2_{\lambda\mu}w_{n,1}x_n - x^\star\|^2 + (1 - \delta_{n,2})\|x_n - x^\star\|^2 \\ &= \delta_{n,2}\|Q_C((I - \mu_2 B_2)w_{n,1}x_n - \lambda_2 A_2(I - \mu_2 B_2)w_{n,1}x_n) - x^\star\|^2 \\ &\quad +(1 - \delta_{n,2})\|x_n - x^\star\|^2 \end{aligned} \tag{3.18}$$

Setting $u_{n,2} = Q_C((I - \mu_2 B_1)w_{n,1}x_n)$ and $v_{n,2} = Q_C((I - \lambda_2 A_2)u_{n,2})$, (3.18) becomes

$$\|w_{n,N}x_n - x^\star\|^2 \leq \delta_{n,2}\|v_{n,2} - x^\star\|^2 + (1 - \delta_{n,2})\|x_n - x^\star\|^2 \tag{3.19}$$

Using Lemma 2.8, we obtain

$$\begin{aligned}\|u_{n,2}-s^\star\|^2 &= \|Q_C(I-\mu_2B_2)w_{n,1}x_n)-Q_C(I-\mu_2B_2)w_{n,1}x^\star)\|^2\\ &\le \|(I-\mu_2B_2)w_{n,1}x_n-(I-\mu_2B_2)w_{n,1}x^\star\|^2\\ &= \|w_{n,1}x_n-x^\star-\mu_2(B_2w_{n,1}x_n-B_2w_{n,1}x^\star)\|^2\\ &\le \|w_{n,1}x_n-x^\star\|^2-2\mu_2(\beta-K^2\beta)\|B_2w_{n,1}x_n-B_2w_{n,1}x^\star\|^2\\ &\le \|x_n-x^\star\|^2-2\mu_2(\beta-K^2\beta)\|B_2w_{n,1}x_n-B_2w_{n,1}x^\star\|^2 \qquad (3.20)\end{aligned}$$

and

$$\begin{aligned}\|v_{n,2}-x^\star\|^2 &= \|Q_C(I-\lambda_2A_2)u_{n,2})-Q_C(I-\lambda_2A_2)s^\star)\|^2\\ &\le \|(I-\lambda_2A_2)u_{n,2}-(I-\lambda_2A_2)s^\star\|^2\\ &= \|u_{n,2}-s^\star-\lambda_2(A_2u_{n,2}-A_2s^\star)\|^2\\ &\le \|u_{n,2}-s^\star\|^2-2\lambda_2(\alpha-K^2\alpha)\|A_2u_{n,2}-A_2s^\star\|^2 \qquad (3.21)\end{aligned}$$

(3.20) and (3.21) imply

$$\begin{aligned}\|v_{n,2}-x^\star\|^2 &\le \|x_n-x^\star\|^2-2\mu_2(\beta-K^2\beta)\|B_2w_{n,1}x_n-B_2w_{n,1}x^\star\|^2\\ &\quad -2\lambda_2(\alpha-K^2\alpha)\|A_2u_{n,2}-A_2s^\star\|^2 \qquad (3.22)\end{aligned}$$

(3.19) and (3.22) imply that

$$\begin{aligned}\|w_{n,N}x_n-x^\star\|^2 &\le \delta_{n,2}[\|x_n-x^\star\|^2-2\mu_2(\beta-K^2\beta)\|B_2w_{n,1}x_n-B_2w_{n,1}x^\star\|^2\\ &\quad -2\lambda_2(\alpha-K^2\alpha)\|A_2u_{n,2}-A_2s^\star\|^2]+(1-\delta_{n,2})\|x_n-x^\star\|^2\\ &\le \|x_n-x^\star\|^2-2\delta_{n,2}\mu_1(\beta-K^2\beta)\|B_2w_{n,1}x_n-B_2w_{n,1}x^\star\|^2\\ &\quad -2\delta_{n,2}\lambda_2(\alpha-K^2\alpha)\|A_2u_{n,2}-A_2s^\star\|^2 \qquad (3.23)\end{aligned}$$

Again, using (3.11) and continuing in the manner as above, with $u_{n,N}=Q_C((I-\mu_NB_N)w_{n,N-1}x_n)$ and $v_{n,N}=Q_C((I-\lambda_NA_N)u_{n,N})$, we have

$$\begin{aligned}\|v_{n,N}-x^\star\|^2 &\le \|x_n-x^\star\|^2-2\mu_N(\beta-K^2\beta)\|B_Nw_{n,N}x_n-B_2w_{n,N-1}x^\star\|^2\\ &\quad -2\lambda_N(\alpha-K^2\alpha)\|A_Nu_{n,N}-A_Ns^\star\|^2\end{aligned}$$

and

$$\begin{aligned}\|w_{n,N}x_n-x^\star\|^2 &\le \|x_n-x^\star\|^2-2\delta_{n,N}\mu_N(\beta-K^2\beta)\|B_Nw_{n,N-1}x_n-B_Nw_{n,N-1}x^\star\|^2\\ &\quad -2\delta_{n,N}\lambda_N(\alpha-K^2\alpha)\|A_Nu_{n,N}-A_Ns^\star\|^2\end{aligned}$$

In general, for $i=1,2,\cdots,N$, we have

$$\begin{aligned}\|v_{n,i}-x^\star\|^2 &\le \|x_n-x^\star\|^2-2\mu_i(\beta-K^2\beta)\|B_iw_{n,i-1}x_n-B_iw_{n,i-1}x^\star\|^2\\ &\quad -2\lambda_i(\alpha-K^2\alpha)\|A_iu_{n,i}-A_is^\star\|^2, \qquad (3.24)\end{aligned}$$

$$\|w_{n,N}x_n-x^\star\|^2 \le \delta_{n,i}\|v_{n,i}-x^\star\|^2+(1-\delta_{n,i})\|x_n-x^\star\|^2 \qquad (3.25)$$

and

$$\begin{aligned}\|w_{n,N}x_n-x^\star\|^2 &\le \|x_n-x^\star\|^2-2\delta_{n,i}\mu_i(\beta-K^2\beta)\|B_iw_{n,i-1}x_n-B_Nw_{n,i-1}x^\star\|^2\\ &\quad -2\delta_{n,i}\lambda_i(\alpha-K^2\alpha)\|A_iu_{n,i}-A_is^\star\|^2 \qquad (3.26)\end{aligned}$$

From (3.4) and the convexity of $\|,.,\|^2$, we get

$$\begin{aligned}
\|x_{n+1}-x^\star\|^2 &= \|\alpha_n f(x_n)+\beta_n x_n+\sum_{i=1}^{m}\gamma_{n,i}S_i^n y_n - x^\star\|^2 \\
&= \|\alpha_n(f(x_n)-x^\star)+\beta_n(x_n-x^\star)+\sum_{i=1}^{m}\gamma_{n,i}(S_i^n y_n - x^\star)\|^2 \\
&\leq \alpha_n\|f(x_n)-x^\star\|^2+\beta_n\|x_n-x^\star\|^2+\sum_{i=1}^{m}\gamma_{n,i}\|S_i^n y_n - x^\star\|^2 \\
&\leq \alpha_n M^\star+\beta_n\|x_n-x^\star\|^2+\sum_{i=1}^{m}\gamma_{n,i}k_n^2\|y_n - x^\star\|^2, \qquad (3.27)
\end{aligned}$$

where $M^\star = \sup_{n\geq 1} = \|f(x_n)-x^\star\|^2$.

(3.26) and (3.27) imply that

$$\begin{aligned}
\|x_{n+1}-x^\star\|^2 &\leq \alpha_n M^\star+\beta_n\|x_n-x^\star\|^2+\sum_{i=1}^{m}\gamma_{n,i}k_n^2[\|x_n-x^\star\|^2 \\
&\quad -2\delta_{n,i}\mu_i(\beta-K^2\beta)\|B_i w_{n,i-1}x_n - B_i w_{n,i-1}x^\star\|^2 \\
&\quad -2\delta_{n,i}\lambda_i(\alpha-K^2\alpha)\|A_i u_{n,i}-A_i s^\star\|^2] \\
&= \alpha_n M^\star+(1-\alpha_n)\|x_n-x^\star\|^2+\sum_{i=1}^{m}\gamma_{n,i}(k_n^2-1)\|x_n-x^\star\|^2 \\
&\quad -2\sum_{i=1}^{m}\delta_{n,1}\gamma_{n,i}\mu_i(\beta-K^2\beta)\|B_i w_{n,i-1}x_n - B_i w_{n,i-1}x^\star\|^2 \\
&\quad -2\sum_{i=1}^{m}\delta_{n,i}\gamma_{n,i}\lambda_i(\alpha-K^2\alpha)\|A_i u_{n,i}-A_i s^\star\|^2
\end{aligned} \qquad (3.28)$$

Set $D = 2\sum_{i=1}^{m}\delta_{n,1}\gamma_{n,i}\mu_i(\beta-K^2\beta)\|B_i w_{n,i-1}x_n - B_i w_{n,i-1}x^\star\|^2 + 2\sum_{i=1}^{m}\delta_{n,i}\gamma_{n,i}\lambda_i(\alpha-K^2\alpha)\|A_i u_{n,i}-A_i s^\star\|^2$. Then, it follows from the last inequality that

$$\begin{aligned}
D &\leq \alpha_n M^\star+(1-\alpha_n)\|x_n-x^\star\|^2-\|x_{n+1}-x^\star\|^2+\sum_{i=1}^{m}\gamma_{n,i}(k_n^2-1)\|x_n-x^\star\|^2 \\
&= \alpha_n M^\star+\|-(x_{n+1}-x_n)+x_{n+1}-x^\star\|^2-\|x_{n+1}-x^\star\|^2 \\
&\quad +\sum_{i=1}^{m}\gamma_{n,i}(k_n^2-1)\|x_n-x^\star\|^2 \\
&\leq \alpha_n M^\star+\|x_{n+1}-x_n\|^2+\sum_{i=1}^{m}\gamma_{n,i}(k_n^2-1)\|x_n-x^\star\|^2 \qquad (3.29)
\end{aligned}$$

Using (3.8), conditions (c), Lemma 3.1, the fact that $0<\lambda<\dfrac{\xi}{K^2}, 0<\mu<\dfrac{\eta}{K^2}$ and $\lim_{n\longrightarrow\infty}k_n = 1$, we get

$$\lim_{n\longrightarrow\infty} D = 0$$

Thus, for $i=1,2,\cdots,N$, we have

$$\lim_{n\longrightarrow\infty}\|B_i w_{n,i-1}x_n - B_i w_{n,i-1}x^\star\| = 0 = \lim_{n\longrightarrow\infty}\|A_i u_{n,i}-A_i t^\star\| \qquad (3.30)$$

Since, using Lemma 2.7 and Lemma 2.2,

$$\begin{aligned}\|u_{n,i}-s^\star\|^2 &= \|Q_C(I-\mu_iB_i)w_{n,i-1}x_n)-Q_C(I-\mu_iB_i)w_{n,i-1}x^\star)\|^2\\ &\le \langle w_{n,i-1}x_n-\mu_iB_iw_{n,i-1}x_n-(w_{n,i-1}x^\star-\mu_iB_iw_{n,i-1}x^\star),j(u_{n,i}-s^\star)\rangle\\ &= \langle w_{n,i-1}x_n-w_{n,i-1}x^\star,j(u_{n,i}-s^\star)\rangle+\mu_i\langle B_iw_{n,i-1}x^\star-B_iw_{n,i-1}x_n,j(u_{n,i}-s^\star)\rangle\\ &\le \frac{1}{2}[\|w_{n,i-1}x_n-w_{n,i-1}x^\star\|^2+\|u_{n,i}-s^\star\|^2-g^\star(\|w_{n,i-1}x_n-u_{n,i}-(x^\star-s^\star)\|)]\\ &\quad+\mu_i\|B_iw_{n,i-1}x_n-B_iw_{n,i-1}x^\star\|\|u_{n,i}-s^\star\|,\end{aligned}$$

it follows that

$$\begin{aligned}\|u_{n,i}-s^\star\|^2 &\le \|x_n-x^\star\|^2-g^\star(\|w_{n,i-1}x_n-u_{n,i}-(x^\star-s^\star)\|)\\ &\quad+2\mu_i\|B_iw_{n,i-1}x_n-B_iw_{n,i-1}x^\star\|\|u_{n,i}-s^\star\| \qquad (3.31)\end{aligned}$$

Applying the same method used for (3.31), we get

$$\begin{aligned}\|v_{n,i}-x^\star\|^2 &= \|Q_C(I-\lambda_iA_i)u_{n,i})-Q_C(I-\lambda_iA_i)s^\star)\|^2\\ &\le \langle u_{n,i}-\lambda_iA_iu_{n,i}-(s^\star-\lambda_iA_is^\star),j(v_{n,i}-x^\star)\rangle\\ &= \langle u_{n,i}-s^\star,j(v_{n,i}-x^\star)\rangle+\lambda_i\langle A_is^\star-A_iu_{n,i},j(v_{n,i}-x^\star)\rangle\\ &\le \frac{1}{2}[\|u_{n,i}-t^\star\|^2+\|v_{n,i}-x^\star\|^2-g^{\star\star}(\|u_{n,i}-v_{n,i}+(x^\star-s^\star)\|)]\\ &\quad+\lambda_i\|A_is^\star-A_iu_{n,i}\|\|v_{n,i}-x^\star\|,\end{aligned}$$

so that

$$\begin{aligned}\|v_{n,i}-x^\star\|^2 &\le \|u_{n,i}-s^\star\|^2-g^{\star\star}(\|u_{n,i}-v_{n,i}+(x^\star-s^\star)\|)\\ &\quad+2\lambda_i\|A_is^\star-A_iu_{n,i}\|\|v_{n,i}-x^\star\| \qquad (3.32)\end{aligned}$$

(3.31) and (3.32) imply that

$$\begin{aligned}\|v_{n,i}-x^\star\|^2 &\le \|x_n-x^\star\|^2-g^\star(\|w_{n,i-1}x_n-u_{n,i}-(x^\star-s^\star)\|)\\ &\quad+2\mu_i\|B_iw_{n,i-1}x_n-B_iw_{n,i-1}x^\star\|\|u_{n,i}-s^\star\|\\ &\quad-g^{\star\star}(\|u_{n,i}-v_{n,i}+(x^\star-s^\star)\|)\\ &\quad+2\lambda_i\|A_is^\star-A_iu_{n,1}\|\|v_{n,i}-x^\star\| \qquad (3.33)\end{aligned}$$

From (3.25) and (3.33), we have

$$\begin{aligned}\|w_{n,N}-x^\star\|^2 &\le \delta_{n,i}[\|x_n-x^\star\|^2-g^\star(\|w_{n,i-1}x_n-u_{n,i}-(x^\star-s^\star)\|)\\ &\quad+2\mu_i\|B_iw_{n,i-1}x_n-B_iw_{n,i-1}x^\star\|\|u_{n,i}-s^\star\|-g^{\star\star}(\|u_{n,i}-v_{n,i}+(x^\star-s^\star)\|)\\ &\quad+2\lambda_i\|A_is^\star-A_iu_{n,i}\|\|v_{n,i}-x^\star\|]+(1-\delta_{n,i})\|x_n-x^\star\|^2\\ &= \|x_n-x^\star\|^2-g^\star(\|w_{n,i-1}x_n-u_{n,i}-(x^\star-s^\star)\|)\\ &\quad+2\mu_i\|B_iw_{n,i-1}x_n-B_iw_{n,i-1}x^\star\|\|u_{n,i}-s^\star\|\\ &\quad-g^{\star\star}(\|u_{n,i}-v_{n,i}+(x^\star-s^\star)\|)+2\lambda_i\|A_is^\star-A_iu_{n,i}\|\|v_{n,i}-x^\star\| \qquad (3.34)\end{aligned}$$

Thus, from (3.27), (3.34) and the inequality:

$$\begin{aligned}\|x_{n+1}-x^\star\|^2 &\le \alpha_nM^\star+\beta_n\|x_n-x^\star\|^2+\sum_{i=1}^m\gamma_{n,i}k_n^2[\|x_n-x^\star\|^2\\ &\quad-g^\star(\|w_{n,i-1}x_n-u_{n,i}-(x^\star-s^\star)\|)+2\mu_i\|B_iw_{n,i-1}x_n-B_iw_{n,i-1}x^\star\|\|u_{n,i}-s^\star\|\\ &\quad-g^{\star\star}(\|u_{n,i}-v_{n,i}+(x^\star-s^\star)\|)+2\lambda_i\|A_is^\star-A_iu_{n,i}\|\|v_{n,i}-x^\star\|]\end{aligned}$$

$$\begin{aligned}
&= \alpha_n M^\star + (1-\alpha_n)\|x_n - x^\star\|^2 + \sum_{i=1}^m \gamma_{n,i}(k_n^2-1)\|x_n - x^\star\|^2 \\
&\quad - \sum_{i=1}^m \gamma_{n,i}k_n^2 g^\star(\|w_{n,i-1}x_n - u_{n,i} - (x^\star - s^\star)\|) \\
&\quad +2\sum_{i=1}^m \gamma_{n,i}k_n^2\mu_i\|B_i w_{n,i-1}x_n - B_i w_{n,i-1}x^\star\|\|u_{n,i} - s^\star\| \\
&\quad - \sum_{i=1}^m \gamma_{n,i}k_n^2 g^{\star\star}(\|u_{n,i} - v_{n,i} + (x^\star - s^\star)\|) \\
&\quad +2\sum_{i=1}^m \gamma_{n,i}k_n^2\lambda_i\|A_i s^\star - A_i u_{n,1}\|\|v_{n,i} - x^\star\|] \qquad (3.35)
\end{aligned}$$

we obtain, with $D^\star = \sum_{i=1}^m \gamma_{n,i}k_n^2 g^\star(\|w_{n,i-1}x_n - u_{n,i} - (x^\star - s^\star)\|) + \sum_{i=1}^m \gamma_{n,i}k_n^2 g^{\star\star}(\|u_{n,i} - v_{n,i} + (x^\star - s^\star)\|)$, the following estimation:

$$\begin{aligned}
D^\star &\le \alpha_n M^\star + (1-\alpha_n)\|x_n - x^\star\|^2 + \|x_{n+1} - x^\star\|^2 + \sum_{i=1}^m \gamma_{n,i}(k_n^2-1)\|x_n - x^\star\|^2 \\
&\quad +2\sum_{i=1}^m \gamma_{n,i}k_n^2\mu_i\|B_i w_{n,i-1}x_n - B_i w_{n,i-1}x^\star\|\|u_{n,i} - s^\star\| \\
&\quad +2\sum_{i=1}^m \gamma_{n,i}k_n^2\lambda_1\|A_i s^\star - A_i u_{n,i}\|\|v_{n,i} - x^\star\|] \\
&\le \alpha_n M^\star + \|-(x_{n+1} - x_n) + (x_{n+1} - x^\star)\|^2 - \|x_{n+1} - x^\star\|^2 + \sum_{i=1}^m \gamma_{n,i}(k_n^2-1)\|x_n - x^\star\|^2 \\
&\quad +2\sum_{i=1}^m \gamma_{n,i}k_n^2\mu_i\|B_i w_{n,i-1}x_n - B_i w_{n,i-1}x^\star\|\|u_{n,i} - s^\star\| \\
&\quad +2\sum_{i=1}^m \gamma_{n,i}k_n^2\lambda_i\|A_i s^\star - A_i u_{n,i}\|\|v_{n,i} - x^\star\|] \\
&\le \alpha_n M^\star + \|-(x_{n+1} - x_n)\|^2 + \|x_{n+1} - x^\star\|^2 - \|x_{n+1} - x^\star\|^2 + \sum_{i=1}^m \gamma_{n,i}(k_n^2-1)\|x_n - x^\star\|^2 \\
&\quad +2\sum_{i=1}^m \gamma_{n,i}k_n^2\mu_i\|B_i w_{n,i-1}x_n - B_i w_{n,i-1}x^\star\|\|u_{n,i} - s^\star\| \\
&\quad +2\sum_{i=1}^m \gamma_{n,i}k_n^2\lambda_i\|A_i s^\star - A_i u_{n,i}\|\|v_{n,i} - x^\star\|] \\
&\le \alpha_n M^\star + \|x_{n+1} - x_n\|^2 + \sum_{i=1}^m \gamma_{n,i}(k_n^2-1)\|x_n - x^\star\|^2 + 2\sum_{i=1}^m \gamma_{n,i}k_n^2\mu_i\|B_i w_{n,i-1}x_n - B_i w_{n,i-1}x^\star\|\|u_{n,i} - s^\star\| \\
&\quad +2\sum_{i=1}^m \gamma_{n,i}k_n^2\lambda_i\|A_i s^\star - A_i u_{n,i}\|\|v_{n,i} - x^\star\|]
\end{aligned}$$

Using (3.8), conditions (c), Lemma 3.1, the fact that $0 < \lambda < \dfrac{\xi}{K^2}$ and $0 < \mu < \dfrac{\eta}{K^2}$ and $\lim_{n\to\infty} k_n = 1$, we get from the last inequality that

$$\lim_{n\longrightarrow\infty} D^\star = 0$$

Thus,

$$\lim_{n\longrightarrow\infty} g^\star(\|w_{n,i-1}x_n - u_{n,i} - (x^\star - s^\star)\|) = 0 = \lim_{n\longrightarrow\infty} g^{\star\star}(\|u_{n,i} - v_{n,i} + (x^\star - s^\star)\|) \qquad (3.36)$$

Hence, from the properties of $g^\star$ and $g^{\star\star}$, we get

$$\lim_{n\to\infty} \|w_{n,i-1}x_n - u_{n,i} - (x^\star - s^\star)\| = 0 = \lim_{n\to\infty} \|u_{n,i} - v_{n,i} + (x^\star - s^\star)\| \qquad (3.37)$$

Using (3.37) and the inequality:

$$\|w_{n,i-1}x_n - v_{n,i}\| \leq \|w_{n,i-1}x_n - u_{n,i} - (x^\star - s^\star)\| + \|u_{n,i} - v_{n,i} + (x^\star - s^\star)\|, \tag{3.38}$$

we obtain

$$\lim_{n\to\infty} \|w_{n,i-1}x_n - v_{n,i}\| = 0, i = 1, 2, \cdots, N. \tag{3.39}$$

Recall that

$$w_{n,N}x_n = \delta_{n,i}v_{n,i}x_n + (1 - \delta_{n,i})w_{n,i-1}x_n$$

Consequently, for $i = 1, 2, \cdots, N$, we have

$$\|w_{n,N}x_n - w_{n,i-1}x_n\| \leq \|v_{n,i}x_n - w_{n.i-1}x_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.40}$$

(3.40) implies that

$$\|w_{n,N}x_n - w_{n,i-1}x_n\| = \|w_{n,N}x_n - w_{n,i-2}x_n\| = \cdots = \|w_{n,N}x_n - x_n\| = 0 \tag{3.41}$$

Since $\{w_{n.N}x_n\}$ is bounded, by Lemma 3.2, it follows that the limit exists. Define the mapping $w : C \longrightarrow C$ by

$$wx_n = \lim_{n\to\infty} w_{n,N}x_n,$$

so that

$$\lim_{n\to\infty} \|w_{n,N}x_n - wx_n\| = 0, \forall n \geq 1. \tag{3.42}$$

Observe that

$$\|x_n - wx_n\| \leq \|x_n - w_{n,N}x_n\| + \|w_{n,N}x_n - wx_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.43}$$

From (3.10) and (3.41), we obtain

$$\|y_n - S_i^n y_n\| \leq \|y_n - x_n\| + \|x_n - S_i^n y_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.44}$$

Again, from (3.41) and (3.44), we have

$$\begin{aligned} \|x_n - S_i^n x_n\| &\leq \|x_n - y_n\| + \|y_n - S_i^n y_n\| + \|S_i^n y_n - S_i^n x_n\| \\ &\leq (1 + k_n)\|x_n - y_n\| + \|y_n - S_i^n y_n\| \to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.45}$$

Furthermore, observe that

$$\begin{aligned} \|x_n - S_i x_n\| &\leq \|x_n - S_i^n x_n\| + \|S_i^n x_n - S_i^{n+1} x_n\| + \|S_i^{n+1} x_n - S_i x_n\| \\ &\leq (1 + k_n)\|x_n - S_i^n x_n\| + \|S_i^{n+1} x_n - S_i x_n\| \to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.46}$$

□

Theorem 3.4. Let C be a nonempty closed convex subset of 2-uniformly smooth and uniformly convex Banach space X, which admits weakly sequentially continous duality mapping and for which the norm is strictly convex. Under the assumptions of Lemma 3.2, the sequence $\{x_n\}$ converges strongly to $q^\star = Q_F f(q)$ and $(q, q^\star)$ is a solution of problem 1.18, where $q^\star = Q_C(q - \mu B_i q)$ and Q_F is the sunny nonexpansive retraction of C onto F.

Proof . Since $Q_\mathcal{F} f$ is a contraction mapping (very easy to verify), it follows from Banach contraction principle that there exists a unique $q \in C$ such that $Q_\mathcal{F} f(q) = q$. By the definition of sunny nonexpansive retraction $Q_\mathcal{F}$, we have $q \in \mathcal{F}$.

Next, we show that

$$\limsup_{n\to} \langle f(q) - q, j(x_n - q)\rangle \leq 0, \tag{3.47}$$

where $Q_\mathcal{F} f(q) = q$. Boundedness of $\{x_n\}$ guarantees the existence of a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup z$ as $k \to \infty$ and

$$\limsup_{n\to\infty} \langle f(q) - q, j(x_n - q)\rangle = \limsup_{k\to\infty} \langle f(q) - q, j(x_{n_k} - q)\rangle \tag{3.48}$$

(3.43) and Lemma 2.12 imply that $z \in \cap_{i=1}^m F(G_{\lambda\mu}^i)$. Again, (3.46) and Lemma 2.11 imply that $z \in (\cap_{i=1}^m F(S_i)) \cap (\cap_{i=1}^m (F(G_{\lambda\mu}^i)$.

Since j is weakly sequentially continous, Lemma 2.2 and (3.48) imply that

$$\begin{aligned}
\limsup_{n\to\infty}\langle f(q)-q, j(x_n-q)\rangle &= \limsup_{k\to\infty}\langle f(q)-q, j(x_{n_k}-q)\rangle \\
&= \langle f(q)-q, j(z-q)\rangle \\
&\leq 0,
\end{aligned}$$

which is as desired. Next, we show that $x_n \to q = Q_F f(q)$ as $n \to \infty$. Now, using (3.4) and Lemma 2.11, we have

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|\beta_n(x_n-q)+\sum_{i=1}^m \gamma_{n,i}(S_i^n y_n - q) + \alpha_n(f(x_n)-q)\|^2 \\
&\leq \|\beta_n(x_n-q)+\sum_{i=1}^m \gamma_{n,i}(S_i^n y_n - q)\|^2 + 2\alpha_n\langle f(x_n)-q, j(x_{n+1}-q\rangle \\
&\leq (\beta_n\|x_n-q\|+\sum_{i=1}^m \gamma_{n,i}\|S_i^n y_n - q\|)^2 + 2\alpha_n\rho\|x_n-q\|\|x_{n+1}-q\| \\
&\quad +2\alpha_n\langle f(q)-q, j(x_{n+1}-q\rangle \\
&\leq ((1-\alpha_n)\|x_n-q\|+\sum_{i=1}^m \gamma_{n,i}(k_n-1)\|y_n-q\|)^2 + 2\alpha_n\rho\|x_n-q\|\|x_{n+1}-q\| \\
&\quad +2\alpha_n\langle f(q)-q, j(x_{n+1}-q\rangle \\
&\leq (1-2\alpha_n+\alpha^2+2(1-\alpha_n)\sum_{i=1}^m \gamma_{n,i}(k_n-1)+\sum_{i=1}^m \gamma_{n,i}^2(k_n-1)^2)\|x_n-q\|^2 \\
&\quad +\alpha_n\rho(\|x_n-q\|^2+\|x_{n+1}-q\|^2)+2\alpha_n\langle f(q)-q, j(x_{n+1}-q\rangle \\
&\leq (1+\alpha_n\rho-2\alpha_n+2(1-\alpha_n)\sum_{i=1}^m \gamma_{n,i}(k_n-1))\|x_n-q\|^2 \\
&\quad +[\alpha_n^2+\sum_{i=1}^m \gamma_{n,i}^2(k_n-1)^2]\|x_n-q\|^2+\alpha_n\rho\|x_{n+1}-q\|^2+2\alpha_n\langle f(q)-q, j(x_{n+1}-q\rangle \\
&\leq [1-(2-\rho)\alpha_n]\|x_n-q\|^2+\{\alpha_n^2+(k_n-1)[\sum_{i=1}^m \gamma_{n,i}^2(k_n-1) \\
&\quad +2(1-\alpha_n)\sum_{i=1}^m \gamma_{n,i}]\}M_2+\alpha_n\rho\|x_{n+1}-q\|^2+2\alpha_n\langle f(q)-q, j(x_{n+1}-q\rangle
\end{aligned}$$

where $M_2 = \sup_{n\geq 1}\|x_n-q\|^2$. Put

$$\sigma_n = \frac{2(1-\rho)\alpha_n}{1-\alpha_n\rho},$$

$$\begin{aligned}
\tau_n &= \frac{\alpha_n}{1-\alpha_n\rho}\Big\{\{\alpha_n+\frac{(k_n-1)}{\alpha_n}[\sum_{i=1}^m \gamma_{n,i}^2(k_n-1)+2(1-\alpha_n)\sum_{i=1}^m \gamma_{n,i}]\}M_2 \\
&\quad +2\langle f(q)-q, j(x_{n+1}-q\rangle\Big\}.
\end{aligned}$$

and

$$\begin{aligned}
\omega_n &= \frac{1}{1-\rho}\Big\{\{\alpha_n+\frac{(k_n-1)}{\alpha_n}[\sum_{i=1}^m \gamma_{n,i}^2(k_n-1)+2(1-\alpha_n)\sum_{i=1}^m \gamma_{n,i}]\}M_2 \\
&\quad +2\langle f(q)-q, j(x_{n+1}-q\rangle\Big\}.
\end{aligned}$$

Then the last inequality becomes

$$a_{n+1} \leq (1-\sigma_n)a_n + \omega_n, \tag{3.49}$$

where $a_n = \|x_n - q\|^2$ and $\omega_n = \dfrac{\tau_n}{\sigma_n}$. Hence, from condition (c) and (3.49), we get

$$\sigma_n \to 0 \quad \text{as} \quad n \to \infty \quad , \sum_{n=1}^{\infty} \sigma_n = \infty \quad \text{and} \quad \limsup_{n\to\infty} \omega_n \leq 0.$$

Thus, from Lemma 2.6, the result follows as required (i.e., $x_n \to q$ as $n \to \infty$) and this completes the proof. □

Remark 3.5. In Hilbert space, the sunny nonexpansive operator Q_C automatically becomes the projection operator P_C.

Base on the remark above, (1.18) becomes: Let K be a nonempty subset of a Hilbert space H. The mapping $Z : K \longrightarrow K$ is defined as follows:

$$\begin{cases} w_{n,0} = I; \\ w_{n,1} = \delta_{n,1} G^1_{\lambda\mu} w_{n,0} + (1-\delta_{n,1}) w_{n,0}; \\ w_{n,2} = \delta_{n,1} G^2_{\lambda\mu} w_{n,1} + (1-\delta_{n,2}) w_{n,1}; \\ \vdots \\ w_{n,N-1} = \delta_{n,N-1} G^{N-1}_{\lambda\mu} w_{n,N-2} + (1-\delta_{n,N-1}) w_{n,N-2}; \\ Z_n = w_{n,N} = \delta_{n,N} G^N_{\lambda\mu} w_{n,N-1} + (1-\delta_{n,N}) w_{n,N-1}, \end{cases} \tag{3.50}$$

where $\{G^i_{\lambda\mu}\}_{i=1}^N = P_C[P_C(I-\mu_i B_i) - \lambda_i A_i(I-\mu_i B_i)], 0 < \lambda_i < \dfrac{\alpha}{K^2}, 0 < \mu_i < \dfrac{\beta}{K^2}$, for $i = 1, 2, \cdots, N$, K^2 is a uniformly smoothness constant and I is the identity mapping. The above mapping Z_n is called Z-mapping generated by $G^1_{\lambda\mu}, G^2_{\lambda\mu}, \cdots, G^N_{\lambda\mu}$ and $\delta_{n,1}, \delta_{n,2}, \cdots, \delta_{n,N}$.

Corollary 3.6. Let K be a nonempty closed convex subset of of a Hilbert space H. Let P_C be a metric projection from H onto C. Let $\{A_i, B_i\}_{=1}^m : K \longrightarrow X$ be finite families of α-inverse-strongly accretive and β-inverse-strongly accretive operators, respectively. Let $f : K \longrightarrow K$ be a ρ-strict contraction of C into itself with coefficient $\rho \in (0,1)$. Let $\{S_i\}_{i=1}^m : K \longrightarrow K$ be a finite families of asymptotically nonexpansive self mappings on C such that $\mathcal{F} = (\cap_{i=1}^m F(S_i)) \cap (\cap_{i=1}^m F(G^i_{\lambda\mu})) \neq \emptyset$, where $G^i_{\lambda\mu}$ is as defined in (3.50). For arbitrarily chosen $x_1 \in C$, let the sequence $\{x_n\}_{n\geq 1}$ be defined iteratively as follows:

$$\begin{cases} x_0 \in K; \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \sum_{i=1}^N \gamma_{n,i} S_i^n y_n; \\ y_n = Z_n x_n, \end{cases} \tag{3.51}$$

where Z_n is as defined in (3.50) and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0,1]$ satisfying the following conditions:

(a) $\alpha_n + \beta_n + \sum_{n=1}^N \gamma_{n,i} = 1$;
(b) $0 < \liminf \beta_n \leq \limsup \beta_n < 1$;
(c) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^\infty \gamma_n^2 < \infty, \lim_{n\to\infty} \dfrac{k_n - 1}{\alpha_n} = 0$;
(d) S satisfy the asymptotically regularity: $\lim_{n\to\infty} \|S^{n+1}x_n - S^n x_n\| = 0$.

Then, the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Proof . Since H is a Hilbert space, then 2-uniformly smooth constant $K = \dfrac{\sqrt{2}}{2}$. The result follows from Theorem 3.4. □

Remark 3.7. Since every asymptotically nonexpansive mapping is a superclass of the classes of nonexpansive and quasi-nonexpqnsive mappings (see Example 1.3 above), the above results remain valid when S is either nonexpansive or quasi-nonexpansive mapping.

Application

As a direct consequence of Theorem 3.4, we have the following result:

Let K be a nonempty closed convex subset of a real Hilbert space H. A mapping $T : K \longrightarrow K$ is said to be k-strictly pseudocontractive mapping if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\| \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \forall x, y \in K. \tag{3.52}$$

It is well known that if T is k-strictly pseudocontractive, then $A = I - T : K \longrightarrow H$ is $\dfrac{1-k}{2}$-inverse-strongly accretive (see [5] for details).

Theorem 3.8. Let K be a nonempty closed convex subset of of a Hilbert space H. Let P_C be a metric projection from H onto K. Let the mappings $\{T_i, T_i^\star\}_{i=1}^m : K \longrightarrow K$ be two finite families of $\{k_i, k_i^\star\}_{i=1}^m$-strictly pseudocontractive, respectively. Let $\{A_i, B_i\}_{=1}^m : K \longrightarrow X$ be finite families of α-inverse-strongly accretive and β-inverse-strongly accretive operators, respectively. Let $f : K \longrightarrow K$ be a ρ-strict contraction of C into itself with coefficient $\rho \in (0, 1)$. Let $\{S_i\}_{i=1}^m : K \longrightarrow K$ be a finite families of asymptotically nonexpansive self mappings on C such that $\mathcal{F} = (\cap_{i=1}^m F(S_i)) \cap (\cap_{i=1}^m F(G^i_{\lambda\mu})) \neq \emptyset$, where $G^i_{\lambda\mu}$ is as defined in (3.50). For arbitrarily chosen $x_1 \in C$, let the sequence $\{x_n\}_{n\geq 1}$ be defined iteratively as follows:

$$\begin{cases} x_0 \in K; \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \sum_{i=1}^N \gamma_{n,i} S_i^n y_n; \\ y_n = \delta_{n,i}[(1-\lambda_i)I + \lambda_i T_i]u_{n,i} + (1-\delta_{n,i})w_{n,i-1}; \\ u_{n,i} = [(1-\mu_i)I + \mu_i T_i^\star]x_n, i = 2, 3, \cdots, N, \end{cases} \tag{3.53}$$

where $w_{n,1} = \delta_{n,1}[(1-\lambda_1)I + \lambda_1 T_1]u_{n,1}, u_{n,1} = \delta_{n,1}[(1-\mu_1)I + \mu_1 T_1^\star]x_n$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_{n,i}\}$ are sequences in $[0, 1]$ satisfying the following conditions:

(a) $\alpha_n + \beta_n + \sum_{n=1}^N \gamma_{n,i} = 1$;
(b) $0 < \liminf \beta_n \leq \limsup \beta_n < 1$;
(c) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^\infty \gamma_n^2 < \infty, \lim_{n\to\infty} \dfrac{k_n - 1}{\alpha_n} = 0$;
(d) S satisfy the asymptotically regularity: $\lim_{n\to\infty} \|S^{n+1}x_n - S^n x_n\| = 0$.

Then, the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Proof . Taking $A = I - T_i : K \longrightarrow H$ and $B = I - T_i^\star : K \longrightarrow H$, it follows from Lemma 2.14 that $A =: K \longrightarrow H$ is α-inverse-strongly accretive with $\alpha = \dfrac{1-k_i}{2}$ and $B =: K \longrightarrow H$ is β-inverse-strongly accretive with $\beta = \dfrac{1-k_i^\star}{2}$. Observe that

$$\begin{aligned} P_C[P_C(I - \mu_i B_i) - \lambda_i A_i(I - \mu_i B_i)] &= P_C[P_C(I - \mu_i(I - T_i^\star) - \lambda_i(I - T_i)(I - \mu_i(I - T_i^\star)))] \\ &= P_C[P_C((I - \mu_i)I + \mu_i T_i^\star) - \lambda_i(I - T_i)((I - \mu_i)I + \mu_i T_i^\star)))] \\ &= P_C((I - \mu_i)I + \mu_i T_i^\star) - \lambda_i(I - T_i)((I - \mu_i)I + \mu_i T_i^\star))) \\ &= ((I - \mu_i)I + \mu_i T_i^\star) - \lambda_i(I - T_i)((I - \mu_i)I + \mu_i T_i^\star)) \\ &= [(I - \lambda_i)I + \lambda_i T_i]((I - \mu_i)I + \mu_i T_i) \end{aligned}$$

For $i = 2, \cdots, N$, let $\eta_n = P_C[P_C(w_{n,i-1}x_n - \mu_i B_i w_{n,i-1}x_n) - \lambda_i A_i(w_{n,i-1}x_n - \mu_i B_i w_{n,i-1}x_n)]$ so that

$$\begin{aligned} \delta_{n,i}\eta_n + (1-\delta_{n,i})w_{n,1}x_n &= \delta_{n,i}(I - \lambda_i)I + \lambda_i T_i]((I - \mu_i)w_{n,i-1}x_n + \mu_i T_i w_{n,i-1}x_n) \\ &\quad + (1-\delta_{n,i})w_{n,i}x_n \end{aligned} \tag{3.54}$$

Conclusion

Fixed point problem has so many practical applications, especially in convex feasibility problems and set theoretic signal estimations (see [2] and the reference therein). Conversely, numerous problems in physics, optimization, economics and engineering reduce to finding a solution of a particular variational inequality. Using modified extragradient method, we study an iteration algorithm for finding a common element of a set of solutions of a finite family of variational inequality problems for inverse strongly acccretive mappings and the set of fixed points for a finite family of asymptotically nonexpansive mappings in the setting of a real 2-uniformly smooth and uniformly convex Banach space. Strong convergence results, which improve, extend and generalize most of the currently existing results in literature, were obtained. However, it still remains an open question on whether the result of Theorem 3.4 could be obtained if the mapping S is asymptotically quasi-nonexpansive (or mappings larger than asymptotically quasi-nonexpansive). □

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