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Stochastic maximum principle for a Markov regime switching jump-diffusion in infinite horizon

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Abstract

In this paper, we study a stochastic optimal control problem for a Markov regime switching jump-diffusion model. Sufficient and necessary maximum principles for optimal control under partial information in infinite horizon are derived. We illustrate our results by a problem of optimal consumption problem from a cash flow with regime.

Keywords: Stochastic maximum principle, Optimal control, Partial information, Markov regime switching jump-diffusion model
2020 MSC: 93E20, 60H10

1 Introduction

The maximum principle is one of the most important methods used to solve optimal control problem, and due to its applications in several fields such as economics, biology and finance, it attracted a large number of researchers. Kushner wa the first who studied the stochastique case [7] , Bensoussan [1] used the convexe perturbation method to derive the stochastic maximum principle in local form. In the continuous case Peng [12] proved the general maximum principle for the stochastic control system by using a second order variational equation and second order adjoint equation to overcome the difficulty appearing along with the nonconvex control domain and control entering the diffusion term, this works was extended in the jumps case by Tang [15]. There are many results for other stochastic control systems; we refer the reader to Young and Zhou [16], Hafayed et al [4], Hafayed and Syed [5], Hafayed et al [6] , Meherrem and Hafayed [9]. The optimal control problem for a Markov regime-switching model has recently received much attention, e.g., see Donnelly [2], Elliot et al [3] , Menoukeu [8], Sun et al[14], Zhang et al [17]. In infinite horizon the stochastic maximum principle has been studied by many authors. For example see Hadam et al [13], Agram et al[10, 11]. Our contribution in this paper is to extend the result of Hadam et al [13] to the diffusion-jumps with regime switching, that is we establish a necessary and sufficient stochastic maximum principle for optimal control within a regime-switching diffusion-jumps model on infinite horizon.

The paper is organized as follows. In section2, we present the optimal control problem for our Markov regime switching jump-diffusion model and the main assumptions. In section 3, we prove the existence–uniqueness theorem for BSDE with jumps and regimes. In Sections 4 and 5 sufficient and necessary maximum principles are developed under partial information. An optimal portfolio and consumption in a switching diffusion market is studied in Section 6.

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2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}\}_{t\geq 0}, P)$ be complete filtered probability space. The filtration $\{\mathcal{F}\}_{t\geq 0}$ is right-continuous, P-completed and all of the processes defined below including the Markov chain, the Brownian motions and the Poisson random measures are adapted to it. We consider a continuous-time, finie-state Markov chain $\{\alpha(t)/t\geq 0\}$ with a finite state space $\mathcal{S} = \{e_1, ..., e_D\}$, where $D\in\mathbb{N}$, $e_i\in\mathbb{R}^D$, and jthe component of e_i is the Kronecker delta δ_{ij} for each $i,j = 1,2,...,D$. the state space $\mathcal{S}$ is called a canonical state space and its use faciliates the mathematics. We suppose that the chain is homogeneous and irreducible. To specify statistical or probabilistic properties of the chain α. we define the generator $\Lambda = \{\lambda_{ij}\ 1\leq i\leq j\leq D\}$ of the chain under P. this is also called the rate matrix, or the Q-matrix. Here, for each $i,j = 1,2,..,D$, λ_{ij} is the constant transition intensity of the chain from state e_i to state e_j at time t. Note that $\lambda_{ij}\geq 0$ for $i\neq j$ and $\sum_{j=1}^{D}\lambda_{ij} = 0$, so $\lambda_{ii}\leq 0$. In what follows for each $i,j = 1,2,..,D$ which $i\neq j$, we suppose that $\lambda_{ij} > 0$, so $\lambda_{ii} < 0$.

Elliott et al. [3] obtained the following semimartingale dynamics for the chain α :

$$\alpha(t) = \alpha(0) + \int_0^t \Lambda^T\alpha(u)\,du + \mathcal{M}(t)$$

where $\{\mathcal{M}(t)\setminus t\geq 0\}$ is an $\mathbb{R}^D$-valued, $\left(\{\mathcal{F}\}_{t\geq 0}, P\right)$-martingale and y^T denotes the transpose of a matrix (or, in particular, a victor).

To model the controlled state process, we first need to introduce a set of Markov jump martingales associated with the chain α. Here we follow the results of Elliott et al. [3].

For each $i,j = 1,2,..,D$, wich $i\neq j$, and $t\in[0,\infty[$ let $J^{ij}(t)$ be the number of jumps from state e_i to state e_j up to time t. Then

$$\begin{aligned} J^{ij}(t) &= \sum_{0\leq s\leq t}\langle\alpha(s-), e_i\rangle\langle\alpha(s), e_j\rangle \\ &= \sum_{0\leq s\leq t}\langle\alpha(s-), e_i\rangle\langle\alpha(s) - \alpha(s-), e_j\rangle \\ &= \int_0^t\langle\alpha(s-), e_i\rangle\langle d\alpha(s), e_j\rangle \\ &= \int_0^t\langle\alpha(s-), e_i\rangle\langle\Lambda^T\alpha(s), e_j\rangle ds + \int_0^t\langle\alpha(s-), e_i\rangle\langle d\mathcal{M}(s), e_j\rangle ds \\ &= \lambda_{ij}\int_0^t\langle\alpha(s-), e_i\rangle ds + m_{ij}(t), \end{aligned}$$

where $m_{ij} = \{m_{ij}(t)\setminus t\in\tau\}$ with $m_{ij}(t) = \int_0^t\langle\alpha(s-), e_i\rangle\langle d\mathcal{M}(s), e_j\rangle$ is an $\left(\{\mathcal{F}\}_{t\geq 0}, P\right)$-martingale, the m_{ij}'s are called the basie martingales associated with the chain α.

Now, for each fined $j = 1,2,..,D$, let $\Phi_j(t)$ be the number of jumps into state e_j up to time t.

Then

$$\begin{aligned} \Phi_j(t) &= \sum_{i=1,\ i\neq j}^{D} J^{ij}(t) \\ &= \sum_{i=1,\ i\neq j}\lambda_{ij}\int_0^t\langle\alpha(s), e_i\rangle ds + \widetilde{\Phi}_j(t), \end{aligned}$$

where $\widetilde{\Phi}_j(t) = \sum_{i=1,\ i\neq j}^{D} m_{ij}(t)$ and, for each $j = 1,2,..,D$, $\widetilde{\Phi}_j(t) = \left\{\widetilde{\Phi}_j(t)\setminus t\in\tau\right\}$ is a an $\left(\{\mathcal{F}_t\}_{t\geq 0}, P\right)$-martingale.

Write for each $j = 1,2,...,D$

$$\lambda_j(t) = \sum_{i=1,\ i\neq j}^{D}\lambda_{ij}\int_0^t\langle\alpha(s), e_i\rangle ds. \tag{2.1}$$

Then for each $j = 1, 2, .., D$,

$$\widetilde{\Phi}_j(t) = \Phi_j(t) - \lambda_j(t), \tag{2.2}$$

is an $\left(\{\mathcal{F}\}_{t\geq 0}, P\right)$-martingale.

We now introduce a Markov regime-switching Poisson random measures. Let $\mathbb{R}^+ = [0, +\infty[$ be the time index set and $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ be a measurable space. Where $\mathcal{B}(\mathbb{R}^+)$ is the Borel σ-field generated by the open subsets of $\mathbb{R}^+$.

Let $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$ and $\mathcal{B}_0$ the Borel σ-field generated by open subset O of $\mathbb{R}_0$ whose closure $\overline{O}$ does not contain the point 0. In what follows, suppose that $\mathcal{N}^i(dz, dt)$, $\int = 1, ..., M$, are independent Poisson random measure on $(\mathbb{R}^+ \times \mathbb{R}_0, \mathcal{B}(\mathbb{R}^+) \times \mathcal{B}_0)$ where $M \in \mathbb{N}$. Assume that the Poisson random measures $\mathcal{N}^i(dz, dt)$ has the following compensator :

$$\eta^i_\alpha(dt, dz) = \nu^i_{\alpha(t-)}(dz)\,dt = \langle\alpha(t-), \nu^i(dz)\rangle dt, \tag{2.3}$$

where

$$\nu^i(dz) = \left(\nu^i_{e_1}(dz), \nu^i_{e_2}(dz), .., \nu^i_{e_D}(dz)\right)^T \in \mathbb{R}^D$$

For each $i = 1, 2, ..., M$, $j = 1, 2, .., D$, $\nu^i_{e_j}$ is assumed to be $\sigma-$finite measure on $\mathbb{R}_0$ satisfying $\nu^i_{e_j}(O) < \infty$, $\forall O \in \mathcal{B}_0$ and $\int_{\mathbb{R}_0} \min\left(1, \ z^2\right) \nu^i_{e_j}(dz) < \infty$. Here we use the subscript α in η^i_α to indicate the dependence of the probability law of the Poisson random measures on the Markov chain. Indeed, $\nu^i_{e_j}(dz)$ is the conditional Lévy density of jump sizes of the random measure $\mathcal{N}^i(dz, dt)$ when $\alpha(t-) = e_j$. Moreover, denote the compensated Poisson random measuren $\widetilde{\mathcal{N}_\alpha}(dz, dt)$ by

$$\widetilde{\mathcal{N}_\alpha}(dz, dt) := \left(\mathcal{N}^1_\alpha(dz, dt) - v^1_\alpha(dz)\,dt, ..., \mathcal{N}^M_\alpha(dz, dt) - v^M_\alpha(dz)\,dt\right)^T. \tag{2.4}$$

We now introduce the state process $X = \{X(t) \backslash t \in [0, \infty[\}$. Suppose that we are given a set $U \subset \mathbb{R}^K$ and a control process $u(t) = u(t, w) : [0, \infty[\times \Omega \to U$. We also require that $\{u(t, w) \backslash t \in [0, \infty[\}$ is $\mathcal{F}_t$-predictable and has right limits. Let $X(t) = X^{(u)}(t)$ be a controlled Markov regime-switching jumps-diffusion in $\mathbb{R}^L$ described by the stochastic differential equation

$$\begin{cases} dX(t) = & b(t, X(t), u(t), \alpha(t))\,dt + \sigma(t, X(t), u(t), \alpha(t))\,dB(t) \\ & +\displaystyle\int_{\mathbb{R}_0} \eta(t, X(t), u(t), \alpha(t), z)\,\widetilde{\mathcal{N}_\alpha}(dz, dt) \\ & +\gamma(t, X(t), u(t), \alpha(t))\,d\widetilde{\Phi}(t) \qquad 0 \leq t \leq \infty, \\ X(0) = & x_0. \end{cases} \tag{2.5}$$

Here $b : [0, \infty[\times \mathbb{R}^L \times U \times \mathcal{S} \to \mathbb{R}^L$, $\sigma : [0, \infty[\times \mathbb{R}^L \times U \times \mathcal{S} \to \mathbb{R}^{L\times N}$, $\eta : [0, \infty[\times \mathbb{R}^L \times U \times \mathcal{S} \times \mathbb{R}_0 \to \mathbb{R}^{L\times M}$ and $\gamma : [0, \infty[\times \mathbb{R}^L \times U \times \mathcal{S} \to \mathbb{R}^{L\times D}$, are given continuous functions , $B(t) := (B_1(t), ..., B_N(t))^T$ is an $N-$dimensional standard Brownian motion, $\widetilde{\mathcal{N}_\alpha}(dz, dt)$ is M-dimentional Markov regime-switching random measures definied by (2.4) $\widetilde{\Phi}(t) = \left(\widetilde{\Phi}_1, .., \widetilde{\Phi}_D\right)$ whith $\widetilde{\Phi}_j(t)$, $j = 1, 2, .., D$, defined by (2.2).

Let $\varepsilon_t \subset \mathcal{F}_t$ be a given subfiltration, representing the information avialable to the controller at time t, $t \geq 0$. The control process $u(t)$ assumed to be $\{\varepsilon_t\}_{t\geq 0}$ predictable and with value in a convexe set $U \subset \mathbb{R}^K$. Let $\mathcal{A}_\varepsilon$ be our family of ε_t-predictable controls.

Consider a performance criterion defined for each $x \in \mathbb{R}^L, e_i \in \mathcal{S}$ as

$$J(x, e_i, u) = E_{x,e_i}\left[\int_0^\infty f(t, X(t), u(t), \alpha(t))\,dt\right].$$

Here E_{x,e_i} is the conditional expectation given $X(0) = 0$ and $\alpha(0) = e_i$ under P, and

$$E\left[\int_0^\infty \left\{|f(t, X(t), u(t), \alpha(t))| + \left|\frac{\partial f}{\partial x}((t, X(t), u(t), \alpha(t)))\right|^2\right\} dt\right] < \infty,$$

for all $u \in \mathcal{A}_{\varepsilon}$, we study the problem to find $u^* \in \mathcal{A}_{\varepsilon}$ such that

$$J\left(x^*, e_i, u^*\right) = \sup_{u \in \mathcal{A}_{\varepsilon}} J\left(x, e_i, u\right). \tag{2.6}$$

Denote by $\mathcal{R}$ the set of functions $r : [0, \infty[\times \mathbb{R}_0^L \to \mathbb{R}^{L\times M}$ such that

$$\int_{\mathbb{R}_0} \left|\eta_{nm}\left(t, x, u, e_i, z\right) r_{nm}\left(t, z\right)\right| \nu_{e_i}^m\left(dz\right) < \infty, \text{ forl all } n, m, x, t,$$

and $\mathcal{M}^2$ the set of functions $s\left(\cdot\right) : [0, \infty[\to \mathbb{R}^{L\times D}$ such that

$$\sum_{m=1}^{D} \sum_{n=1}^{L} \gamma_{nm}\left(t, x, u, e_i\right) s_{nm}\left(t\right) \lambda_{im}\left(t\right) < \infty, \text{ forl all } n, m, x, t,$$

and define the Hamiltonian $H : [0, \infty[\times \mathbb{R}^L \times U \times \mathcal{S} \times \mathbb{R}^L \times \mathbb{R}^{L\times N} \times \mathcal{R} \times \mathbb{R}^{L\times D} \to \mathbb{R}$ by

$$\begin{aligned} H\left(t, x, u, e_i, p, q, r, s\right) &= f\left(t, x, u, e_i\right) + b^T\left(t, x, u, e_i\right) p + tr\left(\sigma^T\left(t, x, u, e_i\right) q\right) \\ &\quad + \int_{\mathbb{R}_0} \sum_{n=1}^{L} \sum_{m=1}^{M} \eta_{nm}\left(t, x, u, e_i, z\right) r_{nm}\left(t, z\right) \nu_{e_i}^m\left(dz\right) \\ &\quad + \sum_{m=1}^{D} \sum_{n=1}^{L} \gamma_{nm}\left(t, x, u, e_i\right) s_{nm}\left(t\right) \lambda_{im}. \end{aligned} \tag{2.7}$$

The adjoint equation in the unknown $\mathcal{F}_t$-predictable processes $\left(p\left(t\right), q\left(t\right), r\left(t, z\right), s\left(t\right)\right)$ where $p\left(t\right) \in \mathbb{R}^L, q\left(t\right) \in \mathbb{R}^{L\times N}, r\left(t, z\right) \in \mathbb{R}^{L\times M}, s\left(t\right) \in \mathbb{R}^{L\times D}$ is the following backward stochastic differential equation (BSDE)

$$\begin{aligned} dp\left(t\right) = &\ -\frac{\partial H}{\partial x}\left(t, X\left(t\right), u\left(t\right), \alpha\left(t\right), p\left(t\right), q\left(t\right), r\left(t, .\right), s\left(t\right)\right) dt \\ &+ q\left(t\right) dB\left(t\right) + \int_{\mathbb{R}_0} r\left(t, z\right) \widetilde{\mathcal{N}}_{\alpha}\left(dz, dt\right) + s\left(t\right) d\widetilde{\Phi}\left(t\right), \quad t \geq 0. \end{aligned} \tag{2.8}$$

3 Existence and uniqueness

In this section, we prove the existence and uniqueness of the solution $\left(Y\left(t\right), Z\left(t\right), K\left(t, \varsigma\right), V\left(t\right)\right)$ of infinite horizon BSDEs of the form:

$$\begin{cases} dY\left(t\right) = & -g\left(t, \alpha\left(t\right), Y\left(t\right), Z\left(t\right), K\left(t, .\right), V\left(t\right)\right) dt + Z\left(t\right) dB\left(t\right) \\ & + \int_{\mathbb{R}_0} -K\left(t, \varsigma\right) \widetilde{\mathcal{N}}_{\alpha}\left(d\varsigma, dt\right) + V\left(t\right) d\widetilde{\Phi}\left(t\right), \quad 0 \leq t \leq \tau, \\ \lim\limits_{t\to\tau} Y\left(t\right) = & \xi\left(\tau\right) 1_{[0,\infty[}\left(\tau\right), \end{cases} \tag{3.1}$$

where $\tau \leq \infty$ is a given $\mathcal{F}_t$-stopping time, possibly infinite. We assume the following.

(H1) The function $g : \Omega \times \mathbb{R}_+ \times \mathcal{S} \times \mathbb{R}^L \times \mathbb{R}^{L\times N} \times \mathcal{R} \times \mathbb{R}^{L\times D} \to \mathbb{R}^L$, is such that there exist real numbers μ, λ, K_1, K_2 and K_3 such that K_1, K_2 and $K_3 > 0$, and $\lambda > 2\mu + K_1^2 + K_2^2 + K_3^2$.

We assume that the function g satisfies the following requirement:

(a) $g\left(.e_i, y, z, k, v\right)$ is progressively measurable for all y, z, k, v and

$$\left|g\left(t, e_i, y, z, k, v\right) - g\left(t, e_i, y, z', k', v'\right)\right| \leq K_1 \left\|z - z'\right\| + K_2 \left\|k - k'\right\|_{\mathcal{R}} + K_3 \left\|v - v'\right\|_{\mathcal{M}^2},$$

where

$$\begin{aligned} \left\|z\right\|^2 &= trace\left(zz^*\right), \\ \left\|k\left(.\right)\right\|_{\mathcal{R}}^2 &= \textstyle\sum_{l=1}^{L} \sum_{m=1}^{M} \int \left|k_{lm}\left(z\right)\right|^2 v_{e_i}^m\left(dz\right), \\ \left\|v\right\|_{\mathcal{M}^2}^2 &= \textstyle\sum_{l=1}^{L} \displaystyle\sum_{j=1}^{D} \left|\nu_{lj}\left(t\right)\right|^2 \lambda_j\left(t\right). \end{aligned}$$

(b)

$$\langle y-y', g(t,e_i,y,z,k,v)-g(t,e_i,y',z,k,v)\rangle \leq \mu |y-y'|, \text{ for all } y,y',z,k,v \ P-a.s.$$

(c)

$$E\int_0^\tau e^{\lambda t}|g(t,e_i,0,0,0,0)|^2 dt<\infty,$$

(d) $y\mapsto g(t,e_i,y,z,k,v)$ is continuous for all t, e_i, z,k, v. $P-a.s.$

(H2) A final condition ξ which is a an $\mathcal{F}_\tau-$mesurable and m-dimensional random variablen such that

$$E\left[e^{\lambda\tau}|\xi|^2\right]<\infty,$$
$$E\int_0^\tau e^{\lambda t}|g(t,e_i,\xi_t,\eta_t,\psi_t,\varphi_t)|^2 dt<\infty,$$

where τ is an $\mathcal{F}_t$-stopping time , $\xi_t=E(\xi/\mathcal{F}_t)$, $\eta\in L^2_{\mathcal{F},p}, \psi\in F^2_p$ and $\varphi\in M^2_p$ such that:

$$\xi=E(\xi)+\int_0^\infty \eta(s)\,dB_s+\int_0^\infty\int_{\mathbb{R}_0}\psi(s,\varsigma)\widetilde{\mathcal{N}}_\alpha(d\varsigma,ds)+\int_0^\infty\varphi(s)\,d\widetilde{\Phi}(s),$$

where

$$\begin{aligned}
L^2_{\mathcal{F},p} &=\left\{f:\mathbb{R}^{L\times N}\text{-valued } \mathcal{F}_t-\text{predictable process, s,t. } E\left[\int_0^\infty |f(t)|^2 dt\right]<\infty\right\}.\\
F^2_p &=\left\{f:\mathbb{R}^{L\times M}\text{-valued } \mathcal{F}_t-\text{predictable process, s,t. } E\left[\int_0^\infty \|f(t,.)\|^2_{\mathcal{R}} dt\right]<\infty\right\}.\\
M^2_p &=\left\{f:\mathbb{R}^{L\times D}\text{-valued } \mathcal{F}_t-\text{predictable process, s,t. } E\left[\int_0^\infty \|f(t)\|^2_{\mathcal{M}^2} dt\right]<\infty\right\}.
\end{aligned}$$

A solution of the BSDE (3.1),is a quadreplet (Y,Z,K,V) of progressively measurable processes with values in $\mathbb{R}^L\times\mathbb{R}^{L\times N}\times\mathbb{R}^{L\times M}\times\ \mathbb{R}^{L\times D}$ s.t $Z_t,K_t,V_t=0$, when $t>\tau$, and

$$\begin{cases}
E\left(\sup_{t\geq 0}e^{\lambda t}|Y(t)|^2+\int_0^\tau e^{\lambda t}\|Z(t)\|^2 dt+\int_0^\tau e^{\lambda t}\|K(t)\|^2_{\mathcal{R}} dt+\int_0^\tau e^{\lambda t}\|V(t)\|^2_{\mathcal{M}^2} dt\right)<\infty,\\
Y(t)=Y(T)+\int_{t\wedge\tau}^{T\wedge\tau} g(s,\alpha(s),Y(s),Z(s),K(t,.),V(s))\,ds-\int_{t\wedge\tau}^{T\wedge\tau} Z(s)\,dB(s)\\
\quad -\int_{t\wedge\tau}^{T\wedge\tau}\int_{\mathbb{R}_0} K(s,\varsigma)\widetilde{\mathcal{N}}_\alpha(d\varsigma,ds)-\int_{t\wedge\tau}^{T\wedge\tau} V(s)\,d\widetilde{\Phi}(s)\,;\ \text{for all deterministic } T<\infty.\\
Y_t=\xi \text{ on the set } \{t\geq\tau\}.
\end{cases}$$

Theorem 3.1.(Existence and Uniqueness) Under the above conditions there exists a unique solution (Y_t,Z_t,K_t,V_t) of the BSDE (3.1), which satisfies moreover, for any $\lambda>2\mu+K_1^2+K_2^2+K_3^2$,

$$\begin{aligned}
&E\left(\sup_{0\leq t\leq\tau}e^{\lambda t}|Y(t)|^2+\textstyle\int_0^\tau e^{\lambda t}\|Z(t)\|^2 dt+\int_0^\tau e^{\lambda t}\|K(t)\|^2_{\mathcal{R}} dt+\int_0^\tau e^{\lambda t}\|V(t)\|^2_{\mathcal{M}^2} dt\right)\\
&<cE\left(e^{\lambda\tau}|\xi|^2+\int_0^\tau e^{\lambda t}|g(t,e_i,0,0,0,0)|^2 dt\right).
\end{aligned} \tag{3.2}$$

Proof of uniqueness. Let (Y,Z,K,V) and (Y',Z',K',V') be two solutions, which satisfy (3.1) and let $\left(\bar{Y},\bar{Z},\bar{K},\bar{V}\right)=$

$(Y-Y', Z-Z', K-K', V-V')$. It follows from Itô's formula, and the above assumption that

$$\begin{aligned}
& e^{\lambda(T\wedge\tau)}\left|\bar{Y}(T)\right|^2 - e^{\lambda(t\wedge\tau)}\left|\bar{Y}(t)\right|^2 \\
&= -\int_{(t\wedge\tau)}^{(T\wedge\tau)} e^{\lambda s}\langle g(s,\alpha(s),Y(s),Z(s),K(s,.),V(s)) \\
&\quad - g(s,\alpha(s),Y'(s),Z'(s),K'(s,.),V'(s)), Y(s)-Y'(s)\rangle\, ds \\
&\quad + \int_{(t\wedge\tau)}^{(T\wedge\tau)} (e^{\lambda s}\left\|\bar{Z}(s)\right\|^2 + \lambda e^{\lambda s}\left|\bar{Y}(s)\right|^2) ds \\
&\quad + \int_{(t\wedge\tau)}^{(T\wedge\tau)} e^{\lambda s}\left\|\bar{K}(s,\zeta)\right\|_{\mathcal{R}}^2 ds + \int_{(t\wedge\tau)}^{(T\wedge\tau)} e^{\lambda s}\left\|V(s)\right\|_{\mathcal{M}^2}^2 ds \\
&\quad +2\int_{(t\wedge\tau)}^{(T\wedge\tau)} e^{\lambda s}\left\langle \bar{Y}(s), \bar{Z}(s)\,dB(s)\right\rangle + \int_{(t\wedge\tau)}^{(T\wedge\tau)}\int_{\mathbb{R}_0} e^{\lambda s}\left(\bar{K}^2(s,\zeta) - 2\left\langle \bar{Y}(s), \bar{K}(s,\zeta)\right\rangle\right)\widetilde{\mathcal{N}}_\alpha(d\varsigma, ds) \\
&\quad + \int_{(t\wedge\tau)}^{(T\wedge\tau)} e^{\lambda s}\left(\bar{V}^2(s) - 2\left\langle \bar{Y}(s), \bar{V}(s)\right\rangle\right) d\widetilde{\Phi}(s).
\end{aligned}$$

so

$$\begin{aligned}
& e^{\lambda(t\wedge\tau)}\left|\bar{Y}(t)\right|^2 + \int_{(t\wedge\tau)}^{(T\wedge\tau)} e^{\lambda s}\left(\lambda\left|\bar{Y}(s)\right|^2 + \left\|\bar{Z}(s)\right\|^2\right) ds + \int_{(t\wedge\tau)}^{(T\wedge\tau)} e^{\lambda s}\left(\left\|\bar{K}(s,\zeta)\right\|_{\mathcal{R}}^2 + \left\|\bar{V}(s)\right\|_{\mathcal{M}^2}^2\right) ds \\
& \le e^{\lambda(T\wedge\tau)}\left|\bar{Y}(T)\right|^2 \\
&\quad +2\int_{(t\wedge\tau)}^{(T\wedge\tau)} e^{\lambda s}\left(\mu\left|\bar{Y}(s)\right|^2 + K_1\left|\bar{Y}(s)\right|\left\|\bar{Z}(s)\right\| + K_2\left|\bar{Y}(s)\right|\left\|\bar{K}(s,\zeta)\right\|_{\mathcal{R}} + K_3\left|\bar{Y}(s)\right|\left\|\bar{V}(s)\right\|_{\mathcal{M}^2}\right) ds \\
&\quad -2\int_{(t\wedge\tau)}^{(T\wedge\tau)} e^{\lambda s}\left\langle \bar{Y}(s), \bar{Z}(s)\,dB(s)\right\rangle - \int_{(t\wedge\tau)}^{(T\wedge\tau)}\int_{\mathbb{R}_0} e^{\lambda s}\left(\bar{K}^2(s,\zeta) - 2\left\langle \bar{Y}(s), \bar{K}(s,\zeta)\right\rangle\right)\widetilde{\mathcal{N}}_\alpha(d\varsigma, ds) \\
&\quad - \int_{(t\wedge\tau)}^{(T\wedge\tau)} e^{\lambda s}\left(\bar{V}^2(s) - 2\left\langle \bar{Y}(s), \bar{V}(s)\right\rangle\right) d\widetilde{\Phi}(s).
\end{aligned}$$

By the fact that

$$\begin{aligned}
2K_1\left|\bar{Y}(s)\right|\left\|\bar{Z}(s)\right\| &\le \left\|\bar{Z}(s)\right\|^2 + K_1^2\left|\bar{Y}(s)\right|^2, \\
2K_2\left|\bar{Y}(s)\right|\left\|\bar{K}(s,\zeta)\right\|_{\mathcal{R}} &\le \left\|\bar{K}(s,\zeta)\right\|_{\mathcal{R}}^2 + K_2^2\left|\bar{Y}(s)\right|^2, \\
2K_3\left|\bar{Y}(s)\right|\left\|\bar{V}(s)\right\|_{\mathcal{M}^2} &\le \left\|\bar{V}(s)\right\|_{\mathcal{M}^2}^2 + K_3^2\left|\bar{Y}(s)\right|^2,
\end{aligned}$$

and since $\lambda > 2\mu + K_1^2 + K_2^2 + K_3^2$,we deduce that for $t < T$,

$$E\left(e^{\lambda(t\wedge\tau)}\left|\bar{Y}(t)\right|^2\right) \le E\left(e^{\lambda(T\wedge\tau)}\left|\bar{Y}(T)\right|^2\right).$$

The same result holds with λ remplaced by λ', with

$$2\mu + K_1^2 + K_2^2 + K_3^2 < \lambda' < \lambda.$$

Hence

$$E\left(e^{\lambda'(t\wedge\tau)}\left|\bar{Y}(t)\right|^2\right) \le e^{\left(\lambda-\lambda'\right)T} E\left(e^{\lambda(T\wedge\tau)}\left|\bar{Y}(T)\right|^2 \mathbf{I}_{\{T<\tau\}}\right).$$

With our conditions the second factor of the right hand side remains bounded as $T \to \infty$, while the first factor tend to 0 as $T \to \infty$. Uniquenessis is proved.

Proof of existence. For each n, we construct a solution $\{(Y^n(t), Z^n(t), K^n(t), V^n(t))\ ; t \ge 0\}$ of the BSDE

$$\begin{cases}
Y^n(t) &= \xi + \int_{t\wedge\tau}^{n\wedge\tau} g(s,\alpha(s),Y^n(s),Z^n(s),K^n(t,.),V^n(s))\,ds - \int_{t\wedge\tau}^{n\wedge\tau} Z^n(s)\,dB(s) \\
& \quad - \int_{t\wedge\tau}^{n\wedge\tau}\int_{\mathbb{R}_0} K^n(s,\varsigma)\widetilde{\mathcal{N}}_\alpha(d\varsigma, ds) - \int_{t\wedge\tau}^{n\wedge\tau} V^n(s)\,d\widetilde{\Phi}(s),\ t \ge 0,
\end{cases}$$

as follows. $\{(Y^n(t), Z^n(t), K^n(t), V^n(t)); 0 \leq t \leq n\}$ is defined as the solution of the following BSDE on the fixed intervall $[0, n]$:

$$\begin{cases} Y^n(t) & = E(\xi/\mathcal{F}_n) + \int_t^n \mathbf{I}_{[0,\tau]} g(s, \alpha(s), Y^n(s), Z^n(s), K^n(t,.), V^n(s))\, ds \\ & - \int_t^n Z^n(s)\, dB(s) - \int_t^n \int_{\mathbb{R}_0} K^n(s,\varsigma) \widetilde{\mathcal{N}}_\alpha(d\varsigma, ds) \\ & - \int_t^n V^n(s)\, d\widetilde{\Phi}(s), \quad 0 \leq t \leq n, \end{cases}$$

$\{(Y^n(t), Z^n(t), K^n(t), V^n(t))\ ; t \geq n\}$ is defined by

$$Y^n(t) = \xi_t, Z^n(t) = \eta(s), K^n(t) = \psi(s,\varsigma), V^n(t) = \varphi(s).$$

For any $\varepsilon > 0$, $0 < \rho < 1, 0 < \alpha < 1$ $, 0 < \beta < 1$, we have for all $t \geq 0, y \in \mathbb{R}^L, e_i \in D, z \in \mathbb{R}^{L\times N}, k \in \mathbb{R}^{L\times M}, v \in \mathbb{R}^{L\times D}$ if $c = \frac{1}{\epsilon}$,

$$\begin{aligned} 2\langle y, g(t,e_i,y,z,k,v)\rangle & = 2\langle y, g(t,e_i,y,z,k,v) - g(t,e_i,0,z,k,v)\rangle \\ & +2\langle y, g(t,e_i,0,z,k,v) - g(t,e_i,0,0,k,v)\rangle \\ & +2\langle y, g(t,e_i,0,0,k,v) - g(t,e_i,0,0,0,v)\rangle \\ & +2\langle y, g(t,e_i,0,0,0,v) - g(t,e_i,0,0,0,0)\rangle \\ & +2\langle y, g(t,e_i,0,0,0,0)\rangle \\ & \leq \left(2\mu + \tfrac{1}{\rho}K_1^2 + \tfrac{1}{\alpha}K_2^2 + \tfrac{1}{\beta}K_3^2 + \varepsilon\right)|y|^2 \\ & +\rho\|z\| + \alpha\|k(.)\|_{\mathcal{R}}^2 + \beta\|v\|_{\mathcal{M}^2}^2 \\ & +c|g(t,e_i,0,0,0,0)|^2. \end{aligned}$$

From these and Itô′s formula, we deduce that

$$\begin{aligned} & e^{\lambda(t\wedge\tau)}|Y^n(t\wedge\tau)|^2 + \int_{(t\wedge\tau)}^{\tau} e^{\lambda s}\left(\bar{\lambda}|Y^n(s)|^2 + \bar{\rho}\|Z^n(s)\|^2\right) ds \\ & + \int_{(t\wedge\tau)}^{\tau} \bar{\alpha} e^{\lambda s}\|K^n(s,\zeta)\|_{\mathcal{R}}^2\, ds + \int_{(t\wedge\tau)}^{\tau} \bar{\beta} e^{\lambda s}\|V^n(s)\|_{\mathcal{M}^2}^2\, ds \\ & \leq e^{\lambda\tau}|\xi|^2 + c\int_{(t\wedge\tau)}^{\tau} e^{\lambda s}|g(s,e_i,0,0,0,0)|^2\, ds \\ & -2\int_{(t\wedge\tau)}^{\tau} e^{\lambda s}\langle Y^n(s), Z^n(s)\, dB(s)\rangle \\ & - \int_{(t\wedge\tau)}^{\tau}\int_{\mathbb{R}_0} e^{\lambda s}\left((K^n)^2(s,\zeta) + 2\langle Y^n(s), K^n(s,\zeta)\rangle\right)\widetilde{\mathcal{N}}_\alpha(d\varsigma, ds) \\ & - \int_{(t\wedge\tau)}^{\tau} e^{\lambda s}\left((V^n)^2(s) + 2\langle Y^n(s), V^n(s)\rangle\right) d\widetilde{\Phi}(s), \end{aligned}$$

with $\bar{\lambda} = \lambda - 2\mu - \frac{1}{\rho}K_1^2 - \frac{1}{\alpha}K_2^2 - \frac{1}{\beta}K_3^2 - \varepsilon > 0$, $\bar{\rho} = 1 - \rho > 0, \bar{\alpha} = 1 - \alpha$ and $\bar{\beta} = 1 - \beta$. It then follows from Burkholder's inequality

$$\begin{aligned} & E\Bigg[\sup_{t\geq s} e^{\lambda(t\wedge\tau)}|Y^n(t\wedge\tau)|^2 + \int_{(t\wedge\tau)}^{\tau} e^{\lambda r}\left(|Y^n(r)|^2 + \|Z^n(r)\|^2\right) dr \\ & + \int_{(t\wedge\tau)}^{\tau} e^{\lambda r}\left(\|K^n(r,\zeta)\|_{\mathcal{R}}^2 + \|V^n(r)\|_{\mathcal{M}^2}^2\right) dr\Bigg] \\ & \leq CE\left[e^{\lambda\tau}|\xi|^2 + \int_{(t\wedge\tau)}^{\tau} e^{\lambda r}|g(r,e_i,0,0,0,0)|^2\, dr\right]. \end{aligned}$$

Let now $m > n$, and define

$$\begin{aligned} \triangle Y(t) &= Y^m(t) - Y^n(t), \ \triangle Z(t) = Z^m(t) - Z^n(t), \\ \triangle K(t) &= K^m(t) - K^n(t), \ \triangle V(t) = V^m(t) - V^n(t). \end{aligned}$$

We first have that for $n \leq t \leq m$,

$$\begin{aligned} \triangle Y(t) &= \int_{t\wedge\tau}^{m\wedge\tau} g\left(s, \alpha(s), Y^m(s), Z^m(s), K^m(t,.), V^m(s)\right) ds \\ &- \int_{t\wedge\tau}^{m\wedge\tau} \triangle Z^m(s)\, dB(s) - \int_{t\wedge\tau}^{m\wedge\tau} \int_{\mathbb{R}_0} \triangle K^m(s,\varsigma)\, \widetilde{\mathcal{N}}_\alpha(d\varsigma, ds) \\ &- \int_{t\wedge\tau}^{m\wedge\tau} \triangle V^m(s)\, d\widetilde{\Phi}(s). \end{aligned}$$

Consequently, again for $n \leq t \leq m$,

$$\begin{aligned} & e^{\lambda(t\wedge\tau)} \left|\triangle Y(t)\right|^2 + \int_{(t\wedge\tau)}^{(m\wedge\tau)} e^{\lambda s} \left(\lambda \left|\triangle Y(s)\right|^2 + \left\|\triangle Z(s)\right\|^2\right) ds \\ & + \int_{(t\wedge\tau)}^{(m\wedge\tau)} e^{\lambda s} \left(\left\|\triangle K(s,\zeta)\right\|_{\mathcal{R}}^2 + \left\|\triangle V(s)\right\|_{\mathcal{M}^2}^2\right) ds \\ & = 2 \int_{(t\wedge\tau)}^{(m\wedge\tau)} \left(e^{\lambda s} \left\langle g\left(s, \alpha(s), Y^m(s), Z^m(s), K^m(s,.), V^m(s)\right), \triangle Y(s)\right\rangle\right) ds \\ & -2 \int_{(t\wedge\tau)}^{(m\wedge\tau)} e^{\lambda s} \left\langle \triangle Y(s), \triangle Z(s)\, dB(s)\right\rangle - \int_{(t\wedge\tau)}^{(m\wedge\tau)} \int_{\mathbb{R}_0} e^{\lambda s} \left((\triangle K)^2(s,\zeta) + 2\left\langle \triangle Y(s), \triangle K(s,\zeta)\right\rangle\right) \widetilde{\mathcal{N}}_\alpha(d\varsigma, ds) \\ & - \int_{(t\wedge\tau)}^{(m\wedge\tau)} e^{\lambda s} \left((\triangle V)^2(s) + 2\left\langle \triangle Y(s), \triangle V(s)\right\rangle\right) d\widetilde{\Phi}(s). \\ & \leq 2 \int_{(t\wedge\tau)}^{(m\wedge\tau)} e^{\lambda s} \left\{\mu \left|\triangle Y(s)\right|^2 + K_1 \left|\triangle Y(s)\right| \left\|\triangle Z(s)\right\| + K_2 \left|\triangle Y(s)\right| \left\|\triangle K(s,\zeta)\right\|_{\mathcal{R}} \right. \\ & \left. + K_3 \left|\triangle Y(s)\right| \left\|\triangle V(s)\right\|_{\mathcal{M}^2}\right\} ds \\ & -2 \int_{(t\wedge\tau)}^{(m\wedge\tau)} e^{\lambda s} \left|\triangle Y(s)\right| \left|g\left(s, e_i, \xi_s, \eta_s, \psi_s, \varphi_s\right)\right|^2 ds - 2 \int_{(t\wedge\tau)}^{(m\wedge\tau)} e^{\lambda s} \left\langle \triangle Y(s), \triangle Z(s)\, dB(s)\right\rangle \\ & - \int_{(t\wedge\tau)}^{(m\wedge\tau)} \int_{\mathbb{R}_0} e^{\lambda s} \left((\triangle K)^2(s,\zeta) + 2\left\langle \triangle Y(s), \triangle K(s,\zeta)\right\rangle\right) \widetilde{\mathcal{N}}_\alpha(d\varsigma, ds) \\ & - \int_{(t\wedge\tau)}^{(m\wedge\tau)} e^{\lambda s} \left((\triangle V)^2(s) + 2\left\langle \triangle Y(s), \triangle V(s)\right\rangle\right) d\widetilde{\Phi}(s). \end{aligned}$$

We then deduce, by an argument that already used , that

$$\begin{aligned} & E\left[\sup_{n\leq t\leq m} e^{\lambda(t\wedge\tau)} \left|Y(t\wedge\tau)\right|^2 + \int_{n\wedge\tau}^{m\wedge\tau} e^{\lambda s} \left(\left|\triangle Y(s)\right|^2 + \left\|\triangle Z(s)\right\|^2\right.\right. \\ & \left.\left. + \left\|\triangle K(s,\zeta)\right\|_{\mathcal{R}}^2 + \left\|\triangle V(s)\right\|_{\mathcal{M}^2}^2\right) ds\right] \\ & \leq C \int_{(n\wedge\tau)}^{\tau} e^{\lambda s} \left|g\left(s, e_i, \xi_s, \eta_s, \psi_s, \varphi_s\right)\right|^2 ds, \end{aligned}$$

and this last term tends to zero, as $n \to \infty$. Next , for $t \leq n$,

$$\begin{aligned} \triangle Y(t) &= \triangle Y(n) + \int_{(t\wedge\tau)}^{(n\wedge\tau)} \left\{g\left(s, \alpha(s), Y^m(s), Z^m(s), K^m(s,.), V^m(s)\right)\right. \\ & \left. - g\left(s, \alpha(s), Y^n(s), Z^n(s), K^n(s,.), V^n(s)\right)\right\} ds^2 \\ & - \int_{t\wedge\tau}^{n\wedge\tau} \triangle Z(s)\, dB(s) - \int_{t\wedge\tau}^{n\wedge\tau} \int_{\mathbb{R}_0} \triangle K(s,\varsigma)\, \widetilde{\mathcal{N}}_\alpha(d\varsigma, ds) - \int_{t\wedge\tau}^{n\wedge\tau} \triangle V(s)\, d\widetilde{\Phi}(s). \end{aligned}$$

It follows from the same argument as in the proof of uniqueness that

$$\begin{aligned} E\left(e^{\lambda(t\wedge\tau)}\left|\triangle Y(t)\right|^2\right) &\leq E\left(e^{\lambda(n\wedge\tau)}\left|\triangle Y(n)\right|^2\right) \\ &\leq C\int_{(n\wedge\tau)}^{\tau} e^{\lambda s}\left|g\left(s,e_i,\xi_s,\eta_s,\psi_s,\varphi_s\right)\right|^2 ds. \end{aligned}$$

It now follows that the sequence (Y^n, Z^n, K^n, V^n) is Cauchy with the norm

$$\|(Y^n,Z^n,K^n,V^n)\|^2 = E\left[\sup_{0\leq t\leq\tau} e^{\lambda t}\left|Y(t)\right|^2 + \int_0^{\tau} e^{\lambda t}\left(\left|Y(t)\right|^2 + \left\|Z(t)\right\|^2 + \left\|K(t)\right\|^2_{\mathcal{R}} + \left\|V(t)\right\|^2_{\mathcal{M}^2}\right)dt\right],$$

and that the limit $(Y, Z; K, V)$ is a solution of the BSDE (3.1) The proof is complete. ■

4 Optimal control with partial information and infinite horizon

In the following we assume that $L = M = N = 1$.

Now, let us get back to the problem of maximizing the performance functional

$$J(x,e_i,u) = E_{x,e_i}\left[\int_0^{\infty} f\left(t, X(t), u(t), \alpha(t)\right)dt\right],$$

where $X(t)$ is of the from (2.5). Our goal is to find a $u^* \in \mathcal{A}_\varepsilon$ such that

$$J(x^*, e_i, u^*) = \sup_{u\in\mathcal{A}_\varepsilon} J(x, e_i, u),$$

where $u(t)$ is a control which adapted to subfiltration $\varepsilon_t \subset \mathcal{F}_t$, with value in a set $U \subset \mathbb{R}$.

Let H be the Hamiltonian defined by (2.7) and (p, q, r, s) the solution to the adjoint equation (2.8). Then we have the following maximum principle.

Theorem 4.1. (Sufficient Infinite Horizon Maximum Principle) Let $u^* \in \mathcal{A}_\varepsilon$ and let

$(p^*(t), q^*(t), r^*(t,z), s^*(t))$ be an associated solution to Eq (2.8). Assume that for all $u \in \mathcal{A}_\varepsilon$ the following terminal condition holds :

$$0 \leq E\left[\overline{\lim_{t\to\infty}}\left[p^*(t)\left(X(t) - X^*(t)\right)\right]\right] < \infty. \tag{4.1}$$

Moreover, assume that $H(t, x, u, e_i, p^*(t), q^*(t), r^*(t,\cdot), s^*(t))$ is concave in x and u and

$$\begin{aligned} &E\left[H\left(t, X^*(t), u^*(t), \alpha(t), p^*(t), q^*(t), r^*(t,.), s^*(t)\right)/\varepsilon_t\right] \\ &= \max_{u\in U} E\left[H\left(t, X^*(t), u, \alpha(t), p^*(t), q^*(t), r^*(t,.), s^*(t)\right)/\varepsilon_t\right]. \end{aligned} \tag{4.2}$$

In addition we assume that for all $T < \infty$,

$$E\left[\int_0^T \left(X^*(t) - X^u(t)\right)^2 \left\{\left(q^*\right)^2(t) + \int_{\mathbb{R}_0}\left(r^*\right)^2(t,z)\,\nu_\alpha(dz) + \sum_{j=1}^{D}\left(s_j^*\right)^2(t)\,\lambda_j(t)\right\}dt\right] < \infty, \tag{4.3}$$

and

$$E\left[\int_0^T \left(p^*\right)^2(t)\left\{\left(\sigma(t)\right)^2 + \int_{\mathbb{R}_0}\left(\eta(t,z)\right)^2\nu_\alpha(dz) + \sum_{j=1}^{D}\left(\gamma^j\right)^2\lambda_j(t)\right\}dt\right] < \infty \tag{4.4}$$

$$E\left[\left|\frac{\partial}{\partial u}H\left(t, X^*(t), u^*(t), \alpha(t), p^*(t), q^*(t), r^*(t,.), s^*(t)\right)\right|^2\right] < \infty, \tag{4.5}$$

and that

$$E\left[\int_0^{\infty}\left|H\left(t, X(t), u(t), \alpha(t), p^*(t), q^*(t), r^*(t,.), s^*(t)\right)\right|\right] < \infty, \tag{4.6}$$

for all u. Then we have that $u^*(t)$ is optimal.

Proof. Let

$$\begin{aligned} I^{\infty} &:= E\left[\int_0^{\infty}\left\{f\left(t,X\left(t\right),u\left(t\right),\alpha\left(t\right)\right)-f\left(t,X^*\left(t\right),u^*\left(t\right),\alpha\left(t\right)\right)\right\}dt\right] \\ &= J\left(x,e_i,u\right)-J\left(x^*,e_i,u^*\right). \end{aligned}$$

Then $I^{\infty}=I_1^{\infty}-I_2^{\infty}-I_3^{\infty}-I_4^{\infty}-I_5^{\infty}$, where

$$\begin{aligned} I_1^{\infty} := E\Big[\int_0^{\infty}&\left(H\left(s,X\left(s\right),u\left(s\right),\alpha\left(s\right),p^*\left(s\right),q^*\left(s\right),r^*\left(s,.\right),s^*\left(s\right)\right)\right. \\ &-\left.H\left(s,X^*\left(s\right),u^*\left(s\right),\alpha\left(s\right),p^*\left(s\right),q^*\left(s\right),r^*\left(s,.\right),s^*\left(s\right)\right)\right)ds\Big], \end{aligned}$$

$$I_2^{\infty}:=E\left[\int_0^{\infty}p^*\left(s\right)\left(b\left(s,X\left(s\right),u\left(s\right),\alpha\left(s\right)\right)-b^*\left(s,X^*\left(s\right),u^*\left(s\right),\alpha\left(s\right)\right)\right)ds\right],$$

$$I_3^{\infty}:=E\left[\int_0^{\infty}q^*\left(s\right)\left(\sigma\left(s,X\left(s\right),u\left(s\right),\alpha\left(s\right)\right)-\sigma^*\left(s,X^*\left(s\right),u^*\left(s\right),\alpha\left(s\right)\right)\right)ds\right],$$

$$I_4^{\infty}:=E\left[\int_0^{\infty}\int_{\mathbb{R}_0}\left(\eta\left(s,X\left(s\right),u\left(s\right),\alpha\left(s\right),z\right)-\eta^*\left(s,X^*\left(s\right),u^*\left(s\right),\alpha\left(s\right),z\right)\right)r^*\left(s,z\right)\nu_{\alpha(s)}\left(dz\right)ds\right],$$

$$I_5^{\infty}:=E\left[\int_0^{\infty}\sum_{j=1}^{D}\left(\gamma^j\left(s,X\left(s\right),u\left(s\right),\alpha\left(s\right)\right)-\gamma^{*j}\left(s,X^*\left(s\right),u^*\left(s\right),\alpha\left(s\right)\right)\right)s_j^*\left(s\right)\lambda_j\left(s\right)ds\right].$$

For the simplification we put

$$H_{t,x,u,\alpha,p^*,q^*,r^*,s^*}:=H\left(t,x,u,\alpha\left(t\right),p^*\left(t\right),q^*\left(t\right),r^*\left(t,\cdot\right),s^*\left(t\right)\right),$$

and the same for the other expressions. We have from concavity that

$$\begin{aligned} &H_{t,X,u,\alpha,p^*,q^*,r^*,s^*}-H_{t,X^*,u^*,\alpha,p^*,q^*,r^*,s^*} \\ &\leq \frac{\partial}{\partial x}H\left(t,X^*\left(t\right),u^*\left(t\right),\alpha\left(t\right),p^*\left(t\right),q^*\left(t\right),r^*\left(t,.\right),s^*\left(t\right)\right)\left(X\left(t\right)-X^*\left(t\right)\right) \\ &+\frac{\partial}{\partial u}H\left(t,X^*\left(t\right),u^*\left(t\right),\alpha\left(t\right),p^*\left(t\right),q^*\left(t\right),r^*\left(t,.\right),s^*\left(t\right)\right)\left(u\left(t\right)-u^*\left(t\right)\right) \end{aligned} \tag{4.7}$$

Then we have from (4.2),(4.5) and that $u(t)$ is adapted to ε_t,

$$\begin{aligned} 0 &\geq \frac{\partial}{\partial u}E\left[H_{t,X^*,u,\alpha,p^*,q^*,r^*,s^*}/\varepsilon_t\right]_{u=u^*(t)}\left(u\left(t\right)-u^*\left(t\right)\right) \\ &= \frac{\partial}{\partial u}E\left[H_{t,X^*,u^*,\alpha,p^*,q^*,r^*,s^*}\left(u\left(t\right)-u^*\left(t\right)\right)/\varepsilon_t\right]. \end{aligned} \tag{4.8}$$

Combining (2.8), (4.3), (4.7) and (4.8), we get

$$\begin{aligned} I_1^{\infty} &\leq E\left[\int_0^{\infty}\frac{\partial}{\partial x}H_{t,X^*,u^*,\alpha,p^*,q^*,r^*,s^*}\left(X\left(s\right)-X^*\left(s\right)\right)ds\right]=E\left[\int_0^{\infty}dp^*\left(s\right)\left(X\left(s\right)-X^*\left(s\right)\right)\right] \\ &:= -J_1. \end{aligned}$$

From (4.3) , (4.4), and Ito's formula , we have that

$$
\begin{aligned}
0 &\leq E\left[\overline{\lim_{t\to\infty}}\left[p^*(t)\left(X(t)-X^*(t)\right)\right]\right] \\
&= E\left[\overline{\lim_{t\to\infty}}\int_0^t p^*(s)\left(b(s,X(s),u(s),\alpha(s))-b(s,X^*(s),u^*(s),\alpha(s))\right)ds\right. \\
&+\int_0^t p^*(s)\left(\sigma(s,X(s),u(s),\alpha(s))-\sigma^*(s,X^*(s),u^*(s),\alpha(s))\right)dB(s) \\
&+\int_0^t\int_{\mathbb{R}_0} p^*(s)\left(\eta(s,X(s),u(s),\alpha(s),z)-\eta^*(s,X^*(s),u^*(s),\alpha(s),z)\right)\widetilde{\mathcal{N}}_\alpha(ds,dz) \\
&+\int_0^t p^*(s)\left(\gamma(s,X(s),u(s),\alpha(s))-\gamma^*(s,X^*(s),u^*(s),\alpha(s))\right)d\widetilde{\Phi}(t)+\int_0^\infty\left(X(s)-X^*(s)\right) \\
&\times\left(-\frac{\partial}{\partial x}H^*(s,X^*(s),u^*(s),\alpha(s),p^*(s),q^*(s),r^*(s,.),s^*(s))\right)ds \\
&+\int_0^t q^*(s)\left(X(s)-X^*(s)\right)dB(s)+\int_0^\infty\int_{\mathbb{R}_0} r^*(s,z)\left(X(s)-X^*(s)\right)\widetilde{\mathcal{N}}_\alpha(ds,dz) \\
&+\int_0^t s^*(s)\left(X(s)-X^*(s)\right)d\widetilde{\Phi}(t) \\
&+\int_0^t q^*(s)\left(\sigma(s,X(s),u(s),\alpha(s))-\sigma^*(s,X^*(s),u^*(s),\alpha(s))\right)ds \\
&+\int_0^t\int_{\mathbb{R}_0} r^*(s,z)\left(\eta(s,X(s),u(s),\alpha(s),z)-\eta^*(s,X^*(s),u^*(s),\alpha(s),z)\right)v_{\alpha(s)}(dz)\,ds \\
&+\int_0^t\int_{\mathbb{R}_0} r^*(s,z)\left(\eta(s,X(s),u(s),\alpha(s),z)-\eta^*(s,X^*(s),u^*(s),\alpha(s),z)\right)\widetilde{\mathcal{N}}_\alpha(ds,dz) \\
&+\int_0^t s^*(s)\left(\gamma(s,X(s),u(s),\alpha(s))-\gamma^*(s,X^*(s),u^*(s),\alpha(s))\right)d\widetilde{\Phi}(t) \\
&\left.+\int_0^t\sum_{j=1}^D s_j^*(s)\left(\gamma^j(s,X(s),u(s),\alpha(s))-\gamma^{*j}(s,X^*(s),u^*(s),\alpha(s))\right)\lambda_j(s)\,ds\right].
\end{aligned}
$$

From (4.3) and (4.4) , we have that

$$
\begin{aligned}
&E\left[\int_0^\infty p^*(s)\left(b(s,X(s),u(s),\alpha(s))-b(s,X^*(s),u^*(s),\alpha(s))\right)ds+\int_0^\infty\left(X(s)-X^*(s)\right)\right. \\
&\times\left(-\frac{\partial}{\partial x}H^*(s,X^*(s),u^*(s),\alpha(s),p^*(s),q^*(s),r^*(s,.),s^*(s))\right)ds \\
&+\int_0^\infty q^*(s)\left(\sigma(s,X(s),u(s),\alpha(s))-\sigma^*(s,X^*(s),u^*(s),\alpha(s))\right)ds \\
&+\int_0^\infty\int_{\mathbb{R}_0} r^*(s,z)\left(\eta(s,X(s),u(s),\alpha(s),z)-\eta^*(s,X^*(s),u^*(s),\alpha(s),z)\right)v_{\alpha(s)}(dz)\,ds \\
&\left.+\int_0^\infty\sum_{j=1}^D s_j^*(s)\left(\gamma^j(s,X(s),u(s),\alpha(s))-\gamma^{*j}(s,X^*(s),u^*(s),\alpha(s))\right)\lambda_j(s)\,ds\right] \\
&= I_2^\infty+J_1^\infty+I_3^\infty+I_4^\infty+I_5^\infty.
\end{aligned}
$$

Finally, combining the above we get

$$
\begin{aligned}
J(x,e_i,u)-J(x^*,e_i,u^*) &\leq I_1^\infty-I_2^\infty-I_3^\infty-I_4^\infty-I_5^\infty \\
&\leq -J_1^\infty-I_2^\infty-I_3^\infty-I_4^\infty-I_5^\infty \\
&\leq 0.
\end{aligned}
$$

This holds for all $u\in\mathcal{A}_\varepsilon$, so the proof is complete. ■

5 Necessary Maximum Principle

In this section, we establish optimality necessary conditions for our control problem. We will to prove : if u^* is optimal does it satisfy

$$\begin{aligned} &E\left[H\left(t,\ X^*(t),\ u^*(t),\ \alpha(t),\ p^*(t),\ q^*(t),\ r^*(t,.),\ s^*(t)\right)/\varepsilon_t\right] \\ &= \max_{u\in U} E\left[H\left(t,\ X^*(t),\ u,\ \alpha(t),\ p^*(t),\ q^*(t),\ r^*(t,.),\ s^*(t)\right)/\varepsilon_t\right]. \end{aligned} \tag{5.1}$$

We assume the following:

(A1) For all t, h such that $0 \leq t \leq t+h \leq \infty$ and for all bounded ε_t-measurable random variables $\theta = \theta(\omega)$, the control process $\beta(s)$ defined by

$$\beta(s) = \theta 1_{[t,t+h]}(s),$$

belongs to $\mathcal{A}_\varepsilon$. Here

$$1_{[t,t+h]}(s) = \begin{cases} 1 & \text{if } t \in [t,\ t+h], \\ 0 & \text{otherwise.} \end{cases}$$

(A2) For all $u \in \mathcal{A}_\varepsilon$ and all $\beta \in \mathcal{A}_\varepsilon$ bounded, there exists $\epsilon > 0$ such that

$$u + \epsilon\beta \in \mathcal{A}_\varepsilon \text{ for all } \epsilon \in [-\delta, \delta].$$

(A3) The derivative process

$$\xi(t) := \frac{d}{d\epsilon} X^{u+\epsilon\beta}(t)\Big|_{\epsilon=0},$$

exists and belongs to $L^2(m \times P)$, where m denotes the Lebesgue measure on $\mathbb{R}$.

$$\begin{aligned} d\xi(t) &= \left\{\frac{\partial b}{\partial x}(t)\xi(t) + \frac{\partial b}{\partial u}(t)\beta(t)\right\}dt + \left\{\frac{\partial \sigma}{\partial x}(t)\xi(t) + \frac{\partial \sigma}{\partial u}(t)\beta(t)\right\}dB(t) \\ &+ \int_{\mathbb{R}_0}\left\{\frac{\partial \eta}{\partial x}(t,z)\xi(t) + \frac{\partial \eta}{\partial u}(t,z)\beta(t)\right\}\widetilde{\mathcal{N}}_\alpha(dt,dz) \\ &+ \left\{\frac{\partial \gamma}{\partial x}(t)\xi(t) + \frac{\partial \gamma}{\partial u}(t)\beta(t)\right\}d\widetilde{\Phi}(t), \end{aligned}$$

where, for simplicity of notation, we define

$$\frac{\partial b}{\partial x}(t) := \frac{\partial b}{\partial x}(t,\ X(t),\ \alpha(t),\ u(t)).$$

Note that

$$\xi(0) = 0.$$

(A4) Assume that f satisfies a Lipschitz condition of the form

$$|f(x_1,u_1,e_j) - f(x_2,u_2,e_j)| \leq C(t)\left(|x_1 - x_2| + |u_1 - u_2|\right),$$

for any $t, x_i, u_i,\ i = 1,2, e_j \in \mathcal{S}$.

We have the following theorem.

Theorem 5.1. (Partial Information Necessary Maximum Principle) Suppose $u^* \in \mathcal{A}_\varepsilon$ is a local maximum for $J(u)$ meaning that for all bounded $\beta \in \mathcal{A}_\varepsilon$ there exists a $\delta > 0$ such that $u^* + \epsilon\beta \in \mathcal{A}_\varepsilon$ for all $\epsilon \in (-\delta, \delta)$ and $h(\epsilon) := J(u^* + \epsilon\beta), \epsilon \in (-\delta, \delta)$ is maximal at $\epsilon = 0$. Let $(p^*(t), q^*(t), r^*(t,z), s^*(t))$ be the solution to the adjoint equation

$$dp^*(t) = -\frac{\partial H}{\partial x}(t, X^*(t), u^*(t), \alpha(t), p^*(t), q^*(t), r^*(t,.), s^*(t))\,dt \\ + q^*(t)\,dB(t) + \int_{\mathbb{R}_0} r^*(z,t)\widetilde{\mathcal{N}}_\alpha(dz,dt) + s^*(t)\,d\widetilde{\Phi}(t).$$

Moreover assume that if $\xi^*(t) = \xi^{(u^*,\beta)}(t)$, with corresponding coefficients π_t^*, τ_t^*, $\varsigma_{t,z}^*$, φ_t^*, where

$$\begin{aligned}\pi_t &= \left(\frac{\partial b_{t,X,u,\alpha}}{\partial x}\right)\xi(t) + \left(\frac{\partial b_{t,X,u,\alpha}}{\partial u}\right)\beta(t),\\ \tau_t &= \left(\frac{\partial \sigma_{t,X,u,\alpha}}{\partial x}\right)\xi(t) + \left(\frac{\partial \sigma_{t,X,u,\alpha}}{\partial u}\right)\beta(t),\\ \varsigma_{t,z} &= \left(\frac{\partial \eta_{t,X,u,z,\alpha}}{\partial x}\right)\xi(t) + \left(\frac{\partial \eta_{t,X,u,z,\alpha}}{\partial u}\right)\beta(t),\\ \varphi_t &= \left(\frac{\partial \gamma_{t,X,u,\alpha}}{\partial x}\right)\xi(t) + \left(\frac{\partial \gamma_{t,X,u,\alpha}}{\partial u}\right)\beta(t),\end{aligned}$$

we have

$$\lim_{T\to\infty} E\left[p^*(T)\,\xi^*(T)\right] = 0, \tag{5.2}$$

$$E\left[\int_0^\infty C(t)\left(1 + |\xi^*(t)|\right)dt\right] < \infty, \tag{5.3}$$

$$E\left[\int_0^T (\xi^*(t))^2\left\{(q^*)^2(t) + \int_{\mathbb{R}_0}(r^*(t,z))^2 v_\alpha(dz) + \sum_{j=1}^D \left(\gamma^j\right)^2(t)\,\lambda_j(t)\right\}dt\right] < \infty \tag{5.4}$$

where $\lambda(t) = (\lambda_1(t), .., \lambda_D(t))^T$,and

$$\left[\int_0^T (p^*(t))^2\left[(\tau^*)^2(t, X^*(t), \alpha(t), u^*(t)) + \int_{\mathbb{R}_0}(\varsigma^*)^2(t, X^*(t), \alpha(t), u^*(t), z)\,v_\alpha(dz) \right.\right. \\ \left.\left. + \sum_{j=1}^D \left(\varphi^{j*}\right)^2(t, X^*(t), \alpha(t), u^*(t))\,\lambda_j(t)\right]dt\right] < \infty, \tag{5.5}$$

for all $T < \infty$. Then u^* is a stationary point for $E[H \,/\, \varepsilon_t]$ in the sense that for all $t \geq 0$,

$$E\left[\frac{\partial}{\partial u}H(t, X^*(t), e_i, u^*, p^*(t), q^*(t), r^*(t,.), s^*(t)) \;/\varepsilon_t\right] = 0. \tag{5.6}$$

Proof. First note that by $(A3)$, $(A4)$ and (5.3) we have that

$$\begin{aligned} 0 &= \left.\frac{\partial}{\partial \epsilon}J(u^* + \epsilon\beta)\right|_{\epsilon=0} \qquad (5.7)\\ &= \frac{\partial}{\partial \epsilon}E\left[\int_0^\infty f\left(t, X^{u^*+\epsilon\beta}(t), u^*(t) + \epsilon\beta, \alpha(t)\right)dt\right]\Bigg|_{\epsilon=0}\\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}E\left[\int_0^\infty \left\{f\left(t, X^{u^*+\epsilon\beta}(t), u^*(t) + \epsilon\beta, \alpha(t)\right) - f\left(t, X^{u^*}(t), u^*(t), \alpha(t)\right)\right\}dt\right]\\ &= E\left[\int_0^\infty \left\{\frac{\partial f}{\partial x}\left(t, X^{u^*}(t), u^*(t), \alpha(t)\right)\xi^*(t) + \frac{\partial f}{\partial u}\left(t, X^{u^*}(t), u^*(t), \alpha(t)\right)\beta(t)\right\}dt\right].\end{aligned}$$

We Know by the definition of H that

$$\frac{\partial f}{\partial x}(t)=\frac{\partial H}{\partial x}(t)-\frac{\partial b}{\partial x}(t)\,p(t)-\frac{\partial \sigma}{\partial x}(t)\,q(t)-\int_{\mathbb{R}_0}\frac{\partial \eta}{\partial x}(t,z)\,r(t,z)\,v_\alpha(dz)-\sum_{j=1}^{D}\frac{\partial \gamma^j}{\partial x}(t)\,s_j(t)\,\lambda_j(t) \tag{5.8}$$

and the same for $\frac{\partial f}{\partial u}(t)$.

Applying the Itô formula to

$$p^*(t)\,\xi^*(t),$$

we obtain by (5.2), (A2), (5.4) and (5.5)

$$\begin{aligned}
0&=\lim_{T\to\infty}E\left[p^*(T)\,\xi(T)\right]\\
&=\lim\nolimits_{T\to\infty}E\left[\int_0^T p^*(t)\left\{\frac{\partial b}{\partial x}(t)\,\xi^*(t)+\frac{\partial b}{\partial u}(t)\,\beta(t)\right\}dt+\int_0^T\xi^*(t)\left(-\frac{\partial H^*(t)}{\partial x}\right)dt\right.\\
&+\int_0^T q^*(t)\left\{\frac{\partial \sigma}{\partial x}(t)\,\xi^*(t)+\frac{\partial \sigma}{\partial u}(t)\,\beta(t)\right\}dt\\
&+\int_0^T\int_{\mathbb{R}_0}r^*(t,z)\left\{\frac{\partial \eta}{\partial x}(t,z)\,\xi^*(t)+\frac{\partial \eta}{\partial u}(t,z)\,\beta(t)\right\}v_\alpha(dz)\,dt\\
&\left.+\int_0^T\sum_{j=1}^{D}s_j^*(t)\left\{\frac{\partial \gamma^j}{\partial x}(t)\,\xi^*(t)++\frac{\partial \gamma^j}{\partial u}(t)\,\beta(t)\,\lambda_j(t)\right\}dt\right]\\
&=\lim\nolimits_{T\to\infty}E\left[\int_0^T\xi^*(t)\left\{\frac{\partial b}{\partial x}(t)\,p^*(t)+\frac{\partial \sigma}{\partial x}q^*(t)+\int_{\mathbb{R}_0}\frac{\partial \eta}{\partial x}(t,z)\,r^*(t,z)\,v_\alpha(dz)\right.\right.\\
&\left.+\sum_{j=1}^{D}\tfrac{\partial \gamma^j}{\partial x}(t)\,s_j^*(t)-\tfrac{\partial H^*(t)}{\partial x}\right\}dt\\
&+\int_0^T\beta(t)\left\{\frac{\partial b}{\partial u}(t)\,p^*(t)+\frac{\partial \sigma}{\partial u}q^*(t)+\int_{\mathbb{R}_0}\frac{\partial \eta}{\partial u}(t,z)\,r^*(t,z)\,v_\alpha(dz)+\sum_{j=1}^{D}\frac{\partial \gamma^j}{\partial u}(t)\,s_j^*(t)\right\}dt\\
&=\lim\nolimits_{T\to\infty}E\left[\int_0^T\xi^*(t)\left\{-\frac{\partial f}{\partial x}(t)\right\}dt+\int_0^T\beta(t)\left\{\frac{\partial H^*(t)}{\partial u}-\frac{\partial f}{\partial u}(t)\right\}dt\right]\\
&=-\lim\nolimits_{T\to\infty}E\left[\int_0^T\left\{\frac{\partial f}{\partial x}(t)\,\xi^*(t)+\frac{\partial f}{\partial u}(t)\,\beta(t)\right\}dt\right]+\lim\nolimits_{T\to\infty}E\left[\int_0^T\beta(t)\,\frac{\partial H^*(t)}{\partial u}\right]
\end{aligned}$$

Hence

$$\frac{d}{d\epsilon}J\left(u^*+\epsilon\beta\right)|_{\epsilon=0}=\lim_{T\to\infty}E\left[\int_0^T\frac{\partial H^*}{\partial u}(t)\,\beta(t)\,dt\right]$$

If

$$\beta(s)=\theta 1_{[t,t+h]}(s),$$

then

$$E\left[\int_t^{t+h}\frac{\partial}{\partial u}H^*\left(s,\ X_s^*,\ e_i,\ u_s^*,\ p_s^*,\ q_s^*,\ r^*(s,.),\ s_s^*\right)\theta ds\right]=0.$$

Differentiating with respect to h at $h=0$, we have

$$E\left[\frac{\partial}{\partial u}H^*\left(t,\ X_t^*,\ e_i,\ u_t^*,\ p_t^*,\ q_t^*,\ r^*(t,.),\ s_t^*\right)\theta\right]=0.$$

This holds for all ε_t-measurable θ and hence we obtain that

$$E\left[\frac{\partial}{\partial u}H^*\left(t,\ X_t^*,\ e_i,\ u_t^*,\ p_t^*,\ q_t^*,\ r^*(t,.),\ s_t^*\right)\ /\varepsilon_t\right]=0.$$

Which proves the theorem. ■

6 Applications

Example 01(Optimal portfolio and consumption with regime switching)

We consider a continuous-time, finite-state, hidden Markov chain $\alpha = \{\alpha(t), t \in [0,\infty[\}$ taking values in a finite-state space $S = \{1, 2, ..., n\}$.

The financial market consists of two assets with S_0 the prices of the risk-free asset and S_1 of the stock are given

$$dS_0(t) = \rho S_0(t)\,dt \quad \text{for all } t \in [0,\infty[\,, S_0(0) > 0, \tag{6.1}$$

and

$$dS_k(t) = S_k(t)\{b(t,\alpha(t))\,dt + \sigma(t,\alpha(t))\,dB(t)\}, \tag{6.2}$$

respectively, where the interest rate ρ is a constant, the appreciation rate $b(t,i)$ and the volatility $\sigma(t,i) \neq 0$ are assumed to be deterministic and bounded.

The wealth of an agent $x(t)$ defined as

$$\begin{cases} dx(t) &= x(t)\left[(\pi(t)(b(t,\alpha(t)) - \rho) + \rho - c(t))\,dt + \pi(t)\sigma(t,\alpha(t))\right]dB(t), \\ x(0) &= x_0 > 0, \end{cases} \tag{6.3}$$

where $\pi(.)$ is the fraction of the agent's wealth that is invested in the risky asset and $c(.)$ is the consumption of the agent and the control process $u(t) = (\pi(t), c(t))$, we have that

$$x(t) = x_0 \exp\left[\int_0^t \left\{\rho + \pi(s)(b(s,\alpha(s)) - \rho) - c(s) - \frac{1}{2}\pi^2(s)\sigma^2(s,\alpha(s))\right\} ds + \int_0^t \pi(s)\sigma(s,\alpha(s))\,dB(s)\right], \tag{6.4}$$

and the associated cost functional is

$$J(u) = E\left[\int_0^\infty \mathrm{e}^{-\delta t}\ln(c(t)x(t))\,dt\right], \tag{6.5}$$

where $\delta > 0$.The objective is to find an optimal control $\hat{u}(.) = (\hat{c}(.), \hat{\pi}(.))$ that maximizes (6.5).

Now the Hamiltonian is

$$H(t,x,c,\pi,i,p,q) = \mathrm{e}^{-\delta t}\ln(cx) + (\pi(b(t,i) - \rho) + \rho - c)\,xp + \pi\sigma(t,i)\,xq, \tag{6.6}$$

then

$$\nabla_x H(t,x,c,\pi,i,p,q) \quad = e^{-\delta t}\tfrac{1}{x} + (\pi(b(t,i) - \rho) + \rho - c)\,p + \pi\sigma(t,i)\,q,$$

on the other hand we have

$$\begin{aligned} dp(t) \quad &= -\left(e^{-\delta t}\tfrac{1}{x(t)} + (\pi(b(t,\alpha(t)) - \rho) + \rho - c(t))\,p(t) + \pi\sigma(t,\alpha(t))\,q(t)\right)dt \\ &+ q(t)\,dB(t)\,s(t) + s(t)\,d\widetilde{\Phi}(t), \end{aligned} \tag{6.7}$$

$$\nabla_\pi H(t,x,c,\pi,i,p,q) \quad = (b(t,i) - \rho)\,px + \sigma(t,i)\,qx \quad , \tag{6.8}$$

$$\nabla_c H(t,x,c,\pi,i,p,q) \quad = e^{-\delta t}\tfrac{1}{c} - px \tag{6.9}$$

so that

$$q(t) \quad = -\tfrac{(b(t,i)-\rho)}{\sigma(t,i)}\,p(t), \tag{6.10}$$

and

$$\hat{c}(t) \quad = e^{-\delta t}\tfrac{1}{p(t)x(t)} \tag{6.11}$$

then

$$\begin{aligned} dp(t) \quad &= -\left[\left(e^{-\delta t}\tfrac{1}{x(t)} + \pi(b(t,\alpha(t)) - \rho) + \rho - e^{-\delta t}\tfrac{1}{p(t)x(t)}\right)p(t) - \pi(b(t,\alpha(t)) - \rho)\,p(t)\right]dt \\ &- \tfrac{(b(t,\alpha(t))-\rho)}{\sigma(t,\alpha(t))}\,p(t)\,dB(t) + s(t)\,d\widetilde{\Phi}(t) \\ &= -\rho p(t)\,dt - \tfrac{(b(t,\alpha(t))-\rho)}{\sigma(t,\alpha(t))}\,p(t)\,dB(t) + s(t)\,d\widetilde{\Phi}(t) \\ &= -p(t)\left(\rho dt + \tfrac{(b(t,\alpha(t))-\rho)}{\sigma(t,\alpha(t))}\,dB(t)\right) + s(t)\,d\widetilde{\Phi}(t), \end{aligned}$$

Let us try to choose $s(t)=0$. Then we have that

$$p(t)=p(0)\exp\left[\int_0^t\left\{-\rho-\frac{1}{2}\frac{(b(s,\alpha(s))-\rho)^2}{\sigma^2(s,\alpha(s))}\right\}ds-\int_0^t\frac{(b(s,\alpha(s))-\rho)}{\sigma(s,\alpha(s))}dB(s)\right]. \tag{6.12}$$

So to ensure that the requirement

$$E\left[\overline{\lim_{t\to\infty}}\left[p(t)(x(t)-\hat{x}(t))\right]\right]\geq 0,$$

is satisfied it suffices that

$$E\left[\overline{\lim_{t\to\infty}}\left[\hat{p}(t)(\hat{x}(t))\right]\right]\leq 0. \tag{6.13}$$

Let us try to choose $\hat{c}(t,\omega)=\hat{c}$ and $\hat{\pi}(t,\omega)=\hat{\pi}$.

Then from (6.11) we get

$$\begin{aligned}p(t)&=e^{-\delta t}\tfrac{1}{\hat{c}x(t)}\\&=\tfrac{1}{\hat{c}x_0}\exp\left[\int_0^t-\left\{\rho+\hat{\pi}(b(s,\alpha(s))-\rho)-\hat{c}-\tfrac{1}{2}\hat{\pi}^2\sigma^2(s,\alpha(s))+\delta\right\}ds-\int_0^t\hat{\pi}\sigma(s,\alpha(s))dB(s)\right]\end{aligned} \tag{6.14}$$

comparing (6.12) with (6.14) we get

$$\begin{aligned}&\rho+\hat{\pi}(b(t,i)-\rho)-\hat{c}-\tfrac{1}{2}\hat{\pi}^2\sigma^2(t,i)+\delta=\rho+\tfrac{1}{2}\tfrac{(b(t,i)-\rho)^2}{\sigma^2(t,i)}\\&\hat{\pi}\sigma(t,i)=\tfrac{(b(t,i)-\rho)}{\sigma(t,i)}\end{aligned}$$

then

$$\hat{c}=\hat{\pi}(b(t,i)-\rho)-\frac{1}{2}\left(\hat{\pi}^2\sigma^2(t,i)+\frac{(b(t,i)-\rho)^2}{\sigma^2(t,i)}\right)+\delta \tag{6.15}$$

$$\hat{\pi}=\frac{(b(t,i)-\rho)}{\sigma^2(t,i)} \tag{6.16}$$

Substituting into (6.15) this gives

$$\hat{c}=\frac{(b(t,i)-\rho)^2}{\sigma^2(t,i)}-\frac{1}{2}\left(\frac{(b(t,i)-\rho)^2}{\sigma^2(t,i)}+\frac{(b(t,i)-\rho)^2}{\sigma^2(t,i)}\right)+\delta=\delta \tag{6.17}$$

By (6.4) and (6.14) we have

$$\begin{aligned}p(t)\hat{x}(t)&=p(0)\exp\left[\int_0^t\left\{-\rho(s,\alpha(s))-\tfrac{1}{2}\tfrac{(b(t,i)-\rho)^2}{\sigma^2(t,i)}\right\}ds-\int_0^t\tfrac{(b(t,i)-\rho)}{\sigma(t,i)}dB(s)\right]\\&\cdot x_0\exp\left[\int_0^t\left\{\rho(s,\alpha(s))+\tfrac{(b(t,i)-\rho)^2}{\sigma^2(t,i)}-\hat{c}-\tfrac{1}{2}\tfrac{(b(t,i)-\rho)^2}{\sigma^2(t,i)}\right\}ds+\int_0^t\tfrac{(b(t,i)-\rho)}{\sigma(t,i)}dB(s)\right]\\&=p(0)x_0\exp[-\hat{c}t]\end{aligned}$$

Therefore (6.13) holds.

We have proved the following theorem.

Theorem 6.1 The optimal control of $(6.3)-(6.5)$ are given by (6.16) and (6.17).

Example 02

We consider the following optimization problem which is to maximize the performance functional:

$$J(u)=E\left[2\int_0^\infty e^{-\beta t}\sqrt{u(t)}dt\right], \tag{6.18}$$

where $x(t)$ is subject to

$$\begin{cases}dx(t)&=(A(t,\alpha(t))x(t)-u(t))dt-C(t,\alpha(t))x(t)dB(t),\\x(t)&=x_0,\end{cases} \tag{6.19}$$

where $\beta,x_0>0$, $A(t,i),C(t,i)>0$, for all $i\in\mathcal{S}=\{1,2,...,n\}$.

In this case the Hamiltonian function takes the form

$$H(t,x,u,i,p,q) = 2\sqrt{u}e^{-\beta t} + (A(t,i)x - u)p - C(t,i)xq,$$

then

$$\begin{aligned} H_u(t,x,u,i,p,q) &= e^{-\beta t}\tfrac{1}{\sqrt{u}} - p \\ H_x(t,x,u,i,p,q) &= (A(t,i))p - C(t,i)q. \end{aligned}$$

Therefore, if $H_u = 0$ we get

$$e^{-\beta t}\frac{1}{\sqrt{u}} - p = 0 \tag{6.20}$$

The adjoint equation is given by

$$\begin{aligned} dp(t) &= -[A(t,\alpha(t))p(t) - C(t,\alpha(t))q(t)]dt \\ &+q(t)dB(t) + s(t)d\widetilde{\Phi}(t). \end{aligned}$$

Let us try to choose $q(t) = s(t) = 0$. So

$$dp(t) = -A(t,\alpha(t))p(t)dt,$$

this leads to

$$p(t) = p(0)e^{-\int_0^t A(s,\alpha(s))ds}, \tag{6.21}$$

for some constant $p(0)$ and by (6.20),

$$\hat{u}(t) = \frac{e^{-2\beta t}}{\left(p(0)e^{-\int_0^t A(s,\alpha(s))ds}\right)^2} \tag{6.22}$$

Inserting $\hat{u}(t)$ into (6.19), we get

$$\begin{cases} d\hat{x}(t) &= \hat{x}(t)A(t,\alpha(t)) - p(0)^{-2}e^{2\int_0^t (A(s,\alpha(s))-\beta)ds}dt - \hat{x}(t)C(t,\alpha(t))dB(t), \\ x(t) &= x_0, \end{cases}$$

Let us consider the process $\Gamma(.)$ defined by

$$\Gamma(t) = \exp\left(\int_0^t -C(s,\alpha(s))dB(s) + A(s,\alpha(s))ds - \frac{1}{2}\int_0^t C^2(s,\alpha(s))ds\right),$$

Using integration by part we get

$$\hat{x}(t) = \hat{x}(0)\Gamma(t) - p(0)^{-2}\int_0^t \frac{e^{2\int_0^s (A(r,\alpha(r))-\beta)dr}}{\Gamma(s)}\Gamma(t)ds.$$

Hence

$$E\left[\hat{x}(t)e^{-\int_0^t A(s,\alpha(s))ds}\right] = \hat{x}(0) - p(0)^{-2}\int_0^t E\left(e^{\int_0^s (A(r,\alpha(r))-2\beta)dr}\right)ds,$$

Therefore to ensure the positivity condition , we get the optimal $p(0)$ as

$$\hat{p}(0) = \left[\frac{\hat{x}(0)}{\int_0^\infty E\left(e^{\int_0^s (A(r,\alpha(r))-2\beta)dr}\right)ds}\right]^{-\frac{1}{2}}, \tag{6.23}$$

and we can verify that

$$\lim_{T\to\infty} E[\hat{x}(T)\hat{p}(T)] = 0.$$

Therefore the transversality condition is verified, then with $p(0) = \hat{p}(0)$ given by (6.23), the control $\hat{u}$ given by (6.22) is optimal.

7 Conclusions

In this paper we have been studied an optimal control problem with regime switching and infinite horizon. More precisely under partial information we gave a necessary and sufficient conditions of optimality. As an illustration we have given two examples of applications where in the both case, the state equation is linear and the objective function is of utility form.

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