Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 1553–1563
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.6504

Solving differential equations and integral problems using wavelets

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(Communicated by Javad Vahidi)

Abstract

Due to benefit of wavelets through numerical and other estimation methods and edge through Fourier analysis, the wavelet hypothesis has expanded broad significance at the time of previous years basically because of their application in comparing areas of science and masterminding, for instance, viscoelasticity, scattering of a natural people, signal taking care of, electromagnetism, fluid mechanics, electrochemistry, and some more. Wavelet has been fundamentally a wave design whose graph oscillates just through a short separation and dumps extremely quick. It tends to be utilized as equipment for taking care of such mathematical problems as differential conditions and integral issues. We have been utilizing wavelet techniques for fathoming the request differential condition; likewise, consider their accuracy and efficiency.

Keywords: Differential equation, wavelet, integral equation, Fourier transform analysis
2020 MSC: Primary 90C33, Secondary 26B25

1 Introduction

Wavelet speculation has been everything considered new and creating a field under numerical analysis. Wavelet frameworks have been utilized to create exact and quick calculations for illuminating numerical integral and differential equations, particularly those whose blueprints have been genuinely constrained in position and scale. The possibility of "wavelets" began from the examination of time-recurrence signal analysis, wave spread, and inspecting hypothesis. The fundamental explanation for the exposure of wavelets and wavelet transform has been that the Fourier transform analysis doesn't contain the close through the information of signs. So, the Fourier transform can't be utilized for inspecting signals in joint time and repeat space. The plausibility of wavelets transforms for the examination of non-stationary (signals containing homeless people and fractal structures). Wavelet system has been a stimulating procedure for dealing along with troublesome issues in arithmetic, material science, and building, along with current applications in various fields. For instance, wave engendering, information pressure, picture handling, design acknowledgement, PC illustrations, the recognizable proof of airplanes, and submarines and improvement under Feline inspects and other therapeutic advancements.

Wavelet has been utilized under Electronics (signal compression and denoising; image and speech analysis), Computer (computer graphics, neural network), Mathematics (approximation theory, matrix theory, numerical analysis of

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Received: April 2021 *Accepted:* July 2021

ODEs and PDEs, operator theory, inverse problems), Mathematical Statistics (sampling theory, relapse, thickness and capacity estimation, factor analysis demonstrating and determining under time arrangement analysis, spatial insights, pattern recognition), Meteorology (structure of the clouds), Universe (structure of galaxies and universe), Biomedical (bio-acoustics, electro-cardiograph (ECG)), Biomedical Imaging (biomedical picture processing, for example, clamour decrease, picture upgrade and identification of miniaturized scale classification under mammograms, computer helped magnetic reverberation imaging (X-ray), utilitarian picture analysis), Fluid (turbulence), Mathematical finance and many more.

2 Literature review

In [1], the author proposed two calculations dependent on Haar wavelets. The required computation is proposed for the numerical process of nonlinear Fredholm integral equations of the following type, and the second for the numerical course of action of the accompanying nonlinear Volterra integral equations. These outlines need to misuse the spectacular features of Harar waves underestimates one and two. The formulas for estimating Haar coefficients without illuminating the process of equations have stabilized. These formulas are then used under the proposed numerical structures. Instead of other numerical procedures, the inverse of the proposed methodology is that it avoids any transitional numerical framework for evaluating the integral present under integral equations.

In [12], another technique has been proposed for the numerical technique of nonlinear Fredholm is a successful type of integral equation and the second one with the numerical blueprint of Nonlayer Voltra. These outlines require misusing the horrific properties of Haar wavelets underestimates one and two. The formulas for finding Harar coefficients without illuminating the technique of equations are relentless. These formulas are then used under the proposed numerical systems. Rather than other numerical strategies, the upside of the proposed strategy has been that it excludes any transitional numerical system for assessment of the integral present under integral equations.

In [5] author discussed the wavelet methods for solving the equation. Paper exhibits that the wavelet strategy has been capable and ground-breaking under fathoming a wide class of straight and nonlinear response dissipating conditions. This examination intends to give the shocking utility of wavelets to science and engineering issues which owes its starting stage to 1919. Similarly, future expansion and extension engaged along working up the wavelet count for understanding response dissemination conditions has been tended to.

In [10] author presented a survey of the theory of coherent states (CS) and a portion of their speculations, along with an emphasis on the numerical structure, as an alternative to physical applications. Beginning from the standard hypothesis of CS through Lie groups, they build up a general formalism, under which CS has been related to group portrayals which has been square-integrable through a homogeneous space. A further advance permits us to dispense along with the group context through and through and hence obtain the supposed duplicating considerably increases and nonstop casings presented under some earlier work. They examined under detail various solid models, under particular, semi straightforward Lie groups, the relativity groups and different sorts of wavelets. At long last go to some physical applications, fixating on quantum estimation and the quantization/ dequantization issue that has been, the transition from the traditional to the quantum level and the other way around.

In [6] the maker proposed another methodology subject to the wavelets and, new mother wavelets appear, while the relating wavelet transforms has been settled and helpful to differential equations. Many strategies, for example, the wavelet-Galerkin strategy, the wavelet technique for the minute lead to surmised or numerical arrangements. The strategy can be used for Tributes and PDEs from each solicitation and under like the way the logical arrangements has been gotten.

In [13] author the introduced scheme which has been increasingly steady and effective for numerically settling the fourth-request fractional integrodifferential condition along a feebly particular portion. They in like manner attempted the methodology proposed on two or three ones and two-dimensional issues along with promising results. Additionally, under order to show the intensity of the semi wavelets methodology under relationship along with standard discretization strategies, they also consider the high-continue floundering issues along with the integrodifferential term.

In [9] author introduced, a collocation system subject to Haar wavelet and Kronecker tensor thing for settling three-dimensional fragmentary differential conditions. The framework depends after taking after a 6th solicitation a mixed backup through a movement of Haar wavelet premise limits. A collocation procedure was reliant on Haar wavelet and Kronecker tensor thing for settling three-dimensional fragmentary differential conditions. Numerical outcomes got has been superior when showed up distinctively corresponding to the numerical outcomes got beforehand.

In [2] author proposed a strategy for the numerical methodology of elliptic inadequate differential conditions having oscillatory and non-oscillatory direct. The present approach has been made in fewer than two stages. Under the central stage, they have been made for Haar wavelets. Under order to accomplish higher accuracy, Haar wavelets have been removed through Legendre wavelets at the consequent stage. A for every intent and purposes indistinguishable assessment of the introduction of the Haar wavelets collocation system and Legendre wavelets collocation method has been finished. Likewise, relative appraisals of the exhibit of the Legendre wavelet collocation technique and quadratic spline collocation framework, and work fewer systems and Sinc–Galerkin reasoning has been under the like way done.

In [4] author exhibited a Legendre–Galerkin procedure for illuminating second-demand elliptic differential conditions subject to the broadest non-homogeneous Robin limit conditions has been appeared. The homogeneous Robin limit conditions have been fulfilled precisely through growing the dim variable utilizing a polynomial explanation behind limits that rely upon the Legendre polynomials. The speedy course of action estimation here made for the homogeneous Robin issue under two-estimations depends on a tensor thing process. Non-homogeneous Robin data has been considered through procedures for lifting. Such lifting has been acted under two unique advances, the first to speak to the data chose at the corners and the subsequent one to speak as far as possible respects upheld under within the sides. Numerical outcomes indicating the high exactness and adequacy of these estimations have been shown.

In [7] author displayed the Chebyshev wavelets procedure for the course of action of fractional differential equations along with limit conditions of the transmission type. Under the proposed procedure they have used generally the operational networks of integration and detachment to get numerical game plans of such equations. The intensity of this reasonable technique has been affirmed. Under addition, the utilization of the Chebyshev wavelet has been seen as exact, straightforward and quick.

In [11] the creator displayed a computational methodology for understanding 2D and 3D Poisson conditions and biharmonic conditions has been subject to the use of Haar wavelets. The most brought backup showing up under the differential condition meanders into the Haar plan, this speculation has been made while the point wild conditions has been met through using integration constants. Under 2D the first transform the uncommon coefficients into the nodal variable characteristics and a brief time allotment later use Kronecker things to gather the approximations for subordinates through a tensor thing cross-section of the level and vertical squares. Finally, reactions for four test issues have been inspected.

3 Methodology

1. **Defining Wavelets**

   A wavelet has been a wave pattern of small size that has been, its graph oscillates only through the short distance or damps very fast; it means the value through the whole domain equates to zero. A wavelet has been localizable both under time (position) and frequency (scale).

   Wavelet: An oscillatory function $\psi(x) \in L^2(R)$ through zero means has been a wavelet if it has the enviable properties:

   1.1. **Smoothness:** $\psi(x)$ has been n times differentiable and that their derivatives has been continuous.

   1.2. **Localization:** $\psi(x)$ has been well-localized both under time and frequency domains, i.e. $\psi(x)$ and its derivatives must decay very quickly. For frequency localization $\widehat{\psi}(\omega)$ must decay adequately fast as $|\omega| \to \infty$ and that $\widehat{\psi}(\omega)$ becomes flat under the locality of $0 = \omega$. The evenness has been related to the number of declining moments of $\psi(x)$, for example,

   $$\int_{\infty}^{\infty} x^k \psi(x) dk = 0$$

   Or equivalently

   $$\frac{d^k \hat{\psi}(\omega)}{d\omega^{\mathrm{k}}} = 0 \quad \text{for } k = 0, 1, \ldots .n$$

   As under bigger the quantity of declining minutes more has been the consistency when ω has been little.

   1.3. **The admissibility condition:**

   For

   $$\int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty \tag{3.1}$$

Suggests that $|\widehat{\psi}(\omega)|^2$ decays at slightest as $|\omega|^{-1}$ or $|x|^{\epsilon-1}$ for $\epsilon > 0$.
Let $\psi_{a,b}(x), a\epsilon R$ and $b \in R$ be a family of functions generated as of mother wavelet $\psi(x)$ through scaling (a) and translation (b) and defined through

$$\psi_{a,b}(x) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{x-b}{a}\right), \quad \|\psi_{a,b}(x)\| = \| \psi\|_2 \tag{3.2}$$

Notice that a has been a measure of the degree of compression and b signifies that $\psi_{a,b}$ has been centred (localized) around b. $\{\psi_{a,b}(x)\}$ has been an orthonormal basis of $L^2(R)$

2. **Wavelets Models**

i. Gaussian wavelet:

$$\psi(x) = cxe^{-\pi x^2}$$

ii. Mexican Hat or Maar's Wavelet:

$$\psi(x) = \frac{d}{dx}\left[\frac{cxe^{-\pi x^2}}{2\pi}\right] = \mathrm{c}\left(\frac{1}{2\pi} - x^2\right)e^{-\pi x^2}$$

iii. Haar Wavelet:

$$\psi(x) = \begin{cases} 1 & 0 \leq x < \frac{1}{2} \\ -1 & \frac{1}{2} \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

iv. Poisson Wavelet:

$$\psi(x) = -\left(1 + \frac{d}{dx}\right)\frac{1}{\pi} \cdot \frac{1}{1+x^2}$$

v. Morlet wavelet:

$$\psi(x) = \exp\left(i\omega x - \frac{x^2}{2}\right)$$

3. **Mathematical Representation of Wavelet**

3.1 **Continuous Wavelet Transform [CWT]:** The wavelet transform (continuous or discrete) or wavelet analysis has been likely the latest answer for conquering the deficiencies of the FT. The CWT can adequately treat signals or capacity along with spikes whose Fourier arrangement would require some high-frequency segments. Wavelet establishes a group of capacities got from one single capacity along listed through two names, one for the position and the other for frequency. That has been the wavelet transform of a one-dimensional capacity has been two dimensional, the wavelet transform of a two-dimensional capacity has been four-dimensional. One forces some extra conditions on the wavelet work to cause the wavelet to transform decline rapidly along a diminishing scale. These have been the normality conditions and express that the wavelet capacity ought to have some smoothness and fixation under both the time and frequency domain.

The CWT of a function $f(x) \in L^2(R)$ at a scale a and position b along with respect to $\psi(x) \in L^2(R)$

$$W_\psi[f](a,b) = \langle f, \psi_{a,b}\rangle = \int_{-\infty}^{\infty} f(x)\bar{\psi}_{a,b}(\mathrm{x})\mathrm{dx} = \frac{1}{\sqrt{|a|}}\int_{-\infty}^{\infty} f(x)\bar{\psi}\left(\frac{x-b}{a}\right)dx \tag{3.3}$$

Under $\psi_{a,b}$ the parameter b gives the position of the wavelet, while the dilation parameter governs frequency. For little values of a (> 0), the wavelet has been contracted under the time area and the wavelet transform gives data about the better subtleties of the sign; while for tremendous estimation of the wavelet extends and the wavelet transform gives a worldwide perspective on the sign. Fig.1. shows two dilations of the Morlet wavelet. If $a > 1$ there has been an extending of $\psi(x)$ along the time axis whereas if $0 < a < 1$ there has been a withdrawal of $\psi(x)$ If a and b assume only the discrete values then the corresponding wavelet transform will be Discrete wavelet transform [DWT.]

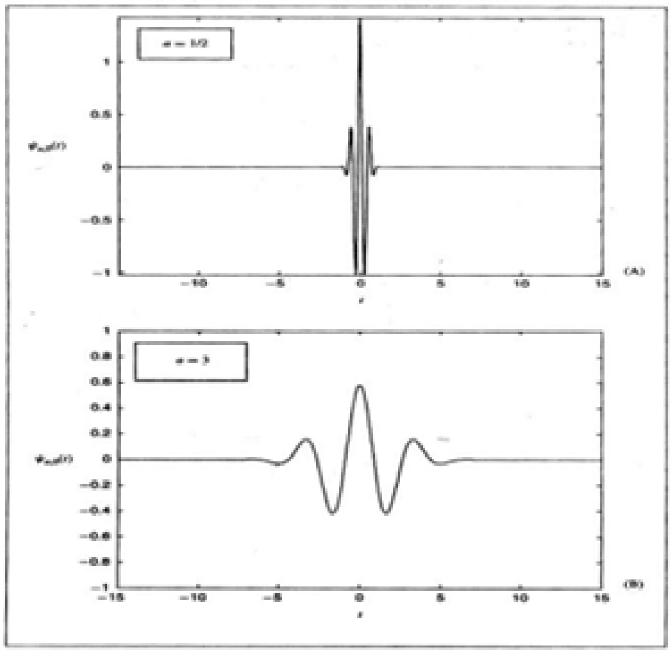

Figure 1: A Morlet wavelet dilated through the factor of 1/ 2=a and 3.=a

3.2 **Inverse CWT:**
$f(x)$ Can be recovered from its CWT as,

$$f(x) = \frac{1}{c}\int_{a=0}^{\infty}\int_{b=-\infty}^{\infty}\frac{1}{|a|^2}W(a,b)\psi_{a,b}dadb \tag{3.4}$$

$$\mathrm{C} = \int_{0}^{\infty}\frac{|\psi(\omega)|^2}{|\omega|}\mathrm{d}\omega \tag{3.5}$$

3.3 **The CWT as an Operator:** The CWT takes a member of the set of the squareintegrable function of one actual variable under $L^2(R)$ and transforms it to a member of the set of functions of two actual variables. Thus, it can be seen as a mapping operator from $L^2(R)$ to the latter set. Define $W_\psi[f(x)] = W(a,b)$ Then $W_c[f]$ has been to be read CWT with respect to $\psi(x)$ of f. The notation for the operator uses ψ as a subscript to remind us of the fact that the transform depends not only on the function $f(x)$ but also on the mother wavelet. We now enumerate various properties of CWT using the operator notation.
Linearity:

$$W_\psi[\alpha f(x) + \beta g(x)] = \alpha W_\psi[f(x)] + \beta W_\psi g(x) \tag{3.6}$$

For scalar (β, α) and functionf $f(x), g(x) \in L^2(R)$.
Translation:

$$W_\psi[f(x-\tau)] = W[a, b-\tau]. \tag{3.7}$$

Scaling:

$$W_\psi\left[\frac{1}{\sqrt{a}}f\left(\frac{1}{\alpha}\right)\right] = \mathrm{W}\left[\frac{a}{\alpha}, \frac{b}{\alpha}\right] \quad \text{for } \alpha > 0. \tag{3.8}$$

Wavelet shifting:
Let, $\bar{\psi}(\mathrm{x}) = \psi(x-\tau)$ then,

$$W_{\psi}[f(x)] = w(a, b + a\tau). \tag{3.9}$$

Observe that the CWT obtained through shifting the wavelet has been different from the one obtained through shifting the signal.

Energy Conservation:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{c} \int_{-\infty}^{\infty} \int_{0}^{\infty} |< f, \psi_{\mathrm{a,b}} >|^2 \frac{dadb}{a^2}. \tag{3.10}$$

Localization:

Let, $f(x) = \mathrm{f}\left(\mathrm{x} - \mathrm{x}_0\right)$ Be the Dirac pulse at the point x_0. Then,

$$W_{\psi}[f](a, b) = \frac{1}{\sqrt{a}} \psi\left(\frac{\mathrm{x}_0 - \mathrm{b}}{\mathrm{a}}\right). \tag{3.11}$$

3.4 Wavelet Series:

A function $\psi \in L^2(R)$ has been said to be an orthonormal wavelet if the family $\{\psi_{j,k}\}_{j,l \in Z}$, where,

$$\psi_{j,k} = 2^{\frac{j}{2}} \psi\left(2^{\mathrm{j}}\mathrm{x} - \mathrm{k}\right), \quad \|\psi_{j,k}\| = \|\psi\|_2 \tag{3.12}$$

Satisfies the condition

$$< \psi_{j,k}, \psi_{i,m} >= \delta_{i,j}, \delta_{\mathrm{k,m}}; \quad \mathrm{i, j, k, m} \in \mathrm{Z} \tag{3.13}$$

Wavelet series expansion of $f \in L^2(R)$. Has been defined through

$$f(x) = \sum_{j,k=-\infty}^{\infty} \beta_{j,k} \psi_{j,k}(x) \tag{3.14}$$

Where wavelet coefficients

$$\beta_{j,k} =< f, \psi_{j,k} >= \int_{-\infty}^{\infty} f(x) \psi_{j,k}(x) dx \tag{3.15}$$

4. Solving Mathematical Problems using Wavelets

Wavelets have the capacity of speaking to the solutions at various degrees of resolution, which makes them especially valuable for creating progressive solutions to building issues. Among the approximations, the wavelet-Galerkin procedure has been most of the time utilized plan nowadays. Under spite of the fact that the wavelet approach has been given a productive elective procedure to illuminating differential equations.

4.1 Wavelet Methods for ODEs:

4.1.1 Wavelet-Galerkin Method:

Consider the equation

$$\frac{\partial^2 u}{\partial x^2} + \alpha u = f \tag{3.16}$$

Where u, f has been periodic under x of period $d \in Z$.The wavelet-Galerkin arrangement of the periodic issue has been somewhat more confused than the finite difference approach as the previous includes explaining a ton of synchronous conditions under wavelet space and not under physical space. The course of action under wavelet space has been then transformed back to physical space through FFT.

Let the wavelet expansion $u(x)$ at scale j be

$$u(x) = \sum_k c_k 2^{\frac{j}{2}} \varphi\left(2^j x - k\right), \mathrm{k} \in \mathrm{Z} \tag{3.17}$$

Where c_k has been periodic wavelet coefficients of u, periodic under x. Put $y = 2^j x$ so as to

$$U(y) = u(x) = \sum_k c_k \varphi(\mathrm{y} - \mathrm{k}), c_k = c_k 2^{\frac{j}{2}} \tag{3.18}$$

If d has been the period of u, then $U(y)$ and so also c_k has been periodic under y along with period 2 d. j Let us discretize $U(y)$ at every dyadic point $x = 2^{-j} y, \quad y \in Z$

$$U_i \sum_k c_k \varphi_{i-k} = \sum_k c_{i-k} \varphi_{k'} i = 0, 1, 2, \ldots, n-1 \tag{3.19}$$

The matrix demonstration has been $U = k_\varphi * C$, where k_φ has been the convolution kernel, i.e., the first column of the scaling function matrix. Likewise, the wavelet expansion for $f(x)$

$$f(x) = \sum_k d_k 2^{\frac{j}{2}} \varphi\left(2^j x - k\right), k \in Z \tag{3.20}$$

$$f(y) = f(x) = \sum_k D_k \varphi(y-k), D_k = 2^{\frac{j}{2}} d_k \tag{3.21}$$

and the matrix demonstration has been

$$\mathrm{F} = k_\varphi{}^* \mathrm{D} \tag{3.22}$$

Alternative expansions of $u(x)$ and $f(x)$ under eq. (3.22) and then take the inner product on both sides along $\varphi(y-j), j \in Z$ Use

$$\Omega_{j-k} = \int \varphi^{(n)}(y-k)\varphi(y-j)dy \tag{3.23}$$

and

$$\delta_{jk} = \int \varphi(y-k)\varphi(y-j)dx \tag{3.24}$$

we obtain

$$k_\Omega \cdot C = g \tag{3.25}$$

Now take FT's,

$$\hat{U} = \hat{k}_\varphi \hat{C} \tag{3.26}$$

$$\hat{F} = \hat{k}_\varphi \cdot \widehat{D} \tag{3.27}$$

$$\hat{k}_\Omega \cdot \hat{C} = \hat{g} \tag{3.28}$$

Subsequently, $\hat{U} = \hat{F}/\hat{k}_\Omega$. Inverse FT will give U. Where under. and / indicate component through component multiplication and division correspondingly.

4.1.2 **Fictitious Boundary Approach**
Consider the equation

$$\frac{\partial^2 u}{\partial x^2} + \alpha u = 0, x \in [0,1] \tag{3.29}$$

Along Dirichlet's boundaries $u(0), u(1)$.
U under (3.19) under has been periodic under x of period $Z \in k$ (k varies from $-N+1$ to 2^j).

The boundaries of the support of (3.28) have been $\frac{-N+1}{2^j}$ and $\frac{N-1+2^j}{2^j}$. Later, the original boundaries 0 and 1, now changes to the Fictitious Boundaries, i.e. boundary on both sides of 0 and 1 has been extended through an amount $\frac{N-1}{2^j}$.

$$\varphi(\mathrm{u}) = \mathrm{u}\left(\frac{-N+1}{2^j}\right) \tag{3.30}$$

$$\varphi(\mathrm{N}-1) = \mathrm{u}\left(\frac{N-1+2^j}{2^j}\right) \tag{3.31}$$

Without disturbing the solution within $[0,1]$, the affected solution has been within $\left[\frac{-N+1}{2^j}, 0\right]$ and $\left[1, \frac{N-1+2^j}{2^j}\right]$.
The eq. (3.29) now reduces to

$$2^{2j} \sum_{k=-N+1}^{2^j} C_k \varphi''(\mathrm{X}-\mathrm{k}) + \alpha \sum_{k=-N+1}^{2^j} C_k \varphi(\mathrm{X}-\mathrm{k}) = 0 \quad 2^j x = X \tag{3.32}$$

Inner product has been taken on both sides of (3.31) along $\varphi(x-n)$ taking the integration limits to $\frac{-N+1}{2^j}, \frac{N-1+2^j}{2^j}$.
We obtain

$$2^{2j} \sum_{k=-N+1}^{2^j} C_k \Omega_{n-k} + \alpha \sum_{k=-N+1}^{2^j} C_k \delta_{n,k} = 0 \tag{3.33}$$

Where under,

$$\Omega_{n-k} = \int \varphi_{\mathrm{xx}}(\mathrm{x}-\mathrm{n})\varphi(\mathrm{x}-\mathrm{n})\mathrm{dx} \ \text{ and } \ \delta_{\mathrm{n,k}} = \int \varphi(\mathrm{x}-\mathrm{n})\varphi(\mathrm{x}-\mathrm{k})\mathrm{dx} \tag{3.34}$$

$$C_k = 2^{j/2} C_k \tag{3.35}$$

The Dirichlet boundary conditions give equations

$$\sum_{k=-N+1}^{2^j} C_k \varphi(-\mathrm{k}) = \mathrm{u}(0) \tag{3.36}$$

$$\sum_{k=-N+\mathbf{1}}^{2^j} C_k \varphi(1-\mathrm{k}) = \mathrm{u}(1) \tag{3.37}$$

The first and last equations have been replaced through the boundary conditions under Equ. (3.34). The place of equivalent connection coefficients under first and last rows has been resolute through the inner product of equations (3.35) and (3.36). Suitable connection coefficients have been used to solve the ill-conditioned system for C_k.

4.1.3 Wavelet-Galerkin Finite Difference Method

Here we build up a wavelet-Galerkin limited distinction technique [WGFDM], under the view of a limited contrast approach, to discover wavelet arrangement of specific Tributes

Lemma. For large j ∈ Z,

$$f^{(n)}(x) = 2^{nj} \sum_{i=0}^{n} (-1)^{n+1} \begin{pmatrix} n \\ i \end{pmatrix} f\left(x + \frac{1}{2^j}\right) \tag{3.38}$$

Proof: For $\frac{1}{2^j}$ small

$$f(x) = \frac{f\left(x + \frac{1}{2^j}\right) - f(x)}{\frac{1}{2^j}} = 2^j \left[f\left(x + \frac{1}{2^j}\right) - f(x) \right]$$

Using forward difference Taylor expansion

$$\begin{aligned} f''(x) &= 2^j \left[f'\left(x + \frac{1}{2^j}\right) - f'(x) \right] \\ &= 2^{2j} \left[\left\{ f\left(x + \frac{1}{2^j}\right) - 2f\left(x + \frac{1}{2^j}\right) + f(x) \right\} \right] \end{aligned}$$

$$\begin{aligned} f''(x) &= 2^{2j} \left[f'\left(x + \frac{2}{2^j}\right) - 2f'\left(x + \frac{1}{2^j}\right) + f(x) \right] \\ &= 2^{3j} \left[f'\left(x + \frac{3}{2^j}\right) - 3f\left(x + \frac{2}{2^j}\right) + 3f\left(x + \frac{2}{2^j}\right) - f(x) \right] \end{aligned}$$

Proceeding under this way,

$$\begin{aligned} f^{(n)}(x) = 2^{nj} \Bigg[& f\left(x + \frac{n}{2^j}\right) - \begin{pmatrix} n \\ 1 \end{pmatrix} f\left(x + \frac{n-1}{2^j}\right) + \begin{pmatrix} n \\ 2 \end{pmatrix} f\left(x + \frac{n-1}{2^j}\right) + \cdots + \\ & \left(n-1 \right) f\left(x + \frac{1}{2^j}\right) - f(x) \Bigg] \\ = 2^{nj} & \sum_{i=0}^{n} (-1)^{n+1} f\left(x + \frac{n}{2^j}\right), j \in Z \end{aligned}$$

This can easily be proved by letting

$$\text{f(x)} = 2^j \left[f(x) - f\left(x - \frac{i}{2j}\right) \right]$$

<u>EXPERIMENTAL STUDY</u>

Solving PDE (partial differential equations) using Wavelet-Galerkin Method: Wavelet-Galerkin Method

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad c > 0, 0 \leq x \leq 1 \tag{3.39}$$

Along with initial condition $u(x, 0) = u_0(x)$ and the boundary condition $u(0, t) = u(1, t)$.
The comparable solution along with the exact one

$$\sin \pi x e^{-tc^2\pi^2}$$

Along $t_{\min} = 0, t_{\max} = 1$
Number of time steps = 10,
$u(0, t) = u(1, t), c = 2 + 10^{-6}$ at $N = 6, j = 7$ has been shown under figure 2.

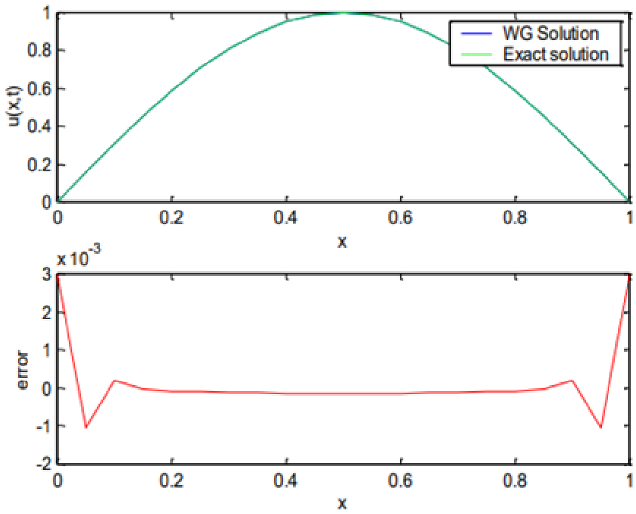

Figure 2: Wavelet solution of 1D wave equation along error estimation for N=6, j=7.

4 Conclusion

The essential target of this paper was to show that the wavelet Galerkin procedure has been an incredible resource for understanding different sorts of integral equations and partial differential equations. The technique along far fewer degrees of chance and along more diminutive CPU time gives ideal courses of action through old-style ones. The key supported circumstance of this framework has been its straightforwardness and small figuring costs, happening Due to the sparsity of the transform grids and the unpretentious number of essential wavelet coefficients. The procedure has been under like manner invaluable for handling the utmost regard issues since the point of confinement conditions has been managed normally.

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