Int. J. Nonlinear Anal. Appl. 14 (2023) 1, 541–555
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.23840.2621

Approximation of common fixed points of a finite family of total asymptotically demicontraction semigroup

Austine Efut Ofem[a,*], Donatus Ikechi Igbokwe[b]

[a]*Department of Mathematics, University of Uyo, Uyo, Nigeria*

[b]*Department of Mathematics, Michael Okpara University of Agriculture, Umudike, Nigeria*

(Communicated by Madjid Eshaghi Gordji)

Abstract

The purpose of this article is to introduce an implicit iterative algorithm which contains several well known existing iterative methods. We study the strong convergence of the proposed iteration process to the common fixed points of a finite family of uniformly L-Lipschitzian total asymptotically demicontraction semigroup in Banach spaces. The results presented in this article extend, generalize, unify and improve the corresponding results of several authors in the existing literature.

Keywords: Fixed points, normalized duality mapping, implict iterative scheme, Banach space, total asymptotically demicontractive semigroup, strong convergence
2020 MSC: Primary 90C33; Secondary 26B25.

1 Introduction

Let X be a real Banach space with dual X^* and C a nonempty closed convex subset of X. We denote by J the *normalized duality* mapping from X into 2^{X^*} defined by

$$J(\psi) = \{f^* \in X^* : \langle \psi, f^* \rangle = \|\psi\|^2 = \|f^*\|^2\}, \ \forall \psi \in X, \tag{1.1}$$

where $\langle ., . \rangle$ denotes the generalized duality pairing. Let j denote the single-valued-normalized duality mapping, $\Re^+$ the set of real numbers, $\mathbb{N}$ the set of natural numbers and $F(S)$ denotes the set of fixed points of mapping $S : C \to C$, i.e., $F(S) = \{\psi \in X : S\psi = \psi\}$.

In the sequel, we give the following definitions which will be useful in this study.

Definition 1.1. A one parameter family $\Im = \{S(s) : s \geq 0\}$ of self mappings of C is said to be *nonexpansive semigroup*; if the following conditions are satisfied:

(i) $S(s_1 + s_2)\psi = S(s_1)S(s_2)\psi$, for any $s_1, s_2 \in \Re^+$ and $\psi \in C$;

(ii) $S(0)\psi = \psi$, for each $\psi \in C$;

*Corresponding author
Email addresses: ofemaustine@gmail.com (Austine Efut Ofem), igbokwedi@yahoo.com (Donatus Ikechi Igbokwe)

Received: May 2021 *Accepted:* July 2022

(iii) for each $\psi \in C$, $s \mapsto S(s)\psi$ is continuous;

(iv) for any $s \geq 0$ and $\psi, \zeta \in C$,

$$\|S(s)\psi - S(s)\zeta\| \leq \|\psi - \zeta\|. \tag{1.2}$$

If the family $\Im = \{S(s) : s \geq 0\}$ satisfies conditions (i)-(iii), then it is called:

(1) *uniformly $L(s)$-Lipschitzian semigroup*, if there exists a bounded measurable function $L : [0, \infty) \to [0, \infty)$ such that, for any $\psi, \zeta \in C$ and $s \geq 0$,

$$\|S^n(s)\psi - S^n(s)\zeta\| \leq L(s)\|\psi - \zeta\|, \quad n \geq 1; \tag{1.3}$$

(2) *strictly pseudocontractive semigroup* (see Yang et al. [18]), if there exists a bounded function $\lambda : [0, \infty) \to [0, \infty)$ and for any given $\psi, \zeta \in C$, there exists $j(\psi - \zeta) \in J(\psi - \zeta)$ such that

$$\langle S(s)\psi - S(s)\zeta, j(\psi - \zeta)\rangle \quad \leq \quad \|\psi - \zeta\|^2 - \lambda(s)\|(I - S(s))\psi - (I - S(s))\zeta\|^2, \tag{1.4}$$

for any $s \geq 0$;

(3) *asymptotically strictly pseudocontractive semigroup* (see Yang el al. [19]), if there exists a bounded function $\lambda : [0, \infty) \to [0, \infty)$ and a sequence $\{k_n\} \in [1, \infty)$ with $k_n \to 1$ as $n \to \infty$; for any given $\psi, \zeta \in C$, there exists $j(\psi - \zeta) \in J(\psi - \zeta)$ such that

$$\langle S^n(s)\psi - S^n(s)\zeta, j(\psi - \zeta)\rangle \quad \leq \quad k_n\|\psi - \zeta\|^2 - \lambda(s)\|(I - S^n(s))\psi - (I - S^n(s))\zeta\|^2, \tag{1.5}$$

for all $n \geq 1$ and for any $s \geq 0$;

(4) *total asymptotically strictly pseudocontractive semigroup* (see Yang et al. [19]), if there exists a bounded function $\lambda : [0, \infty) \to [0, \infty)$ and sequences $\{\mu_n\} \in [0, \infty)$ and $\{\xi_n\} \in [0, \infty)$ with $\mu_n \to 0$ and $\xi_n \to 0$ as $n \to \infty$. For any given $\psi, \zeta \in C$, there exists $j(\psi - \zeta) \in J(\psi - \zeta)$ such that

$$\langle S^n(s)\psi - S^n(s)\zeta, j(\psi - \zeta)\rangle \leq \|\psi - \zeta\|^2 - \lambda(s)\|(I - S^n(s))\psi - (I - S^n(s))\zeta\|^2 + \mu_n\phi(\|\psi - \zeta\|) + \xi_n, \tag{1.6}$$

for all $n \geq 1$ and for any $s \geq 0$;

(5) *demicontractive semigroup* (see Chang et al. [3]), if $\bigcap_{s \geq 0} F(S(s)) \neq \emptyset$ and there exists a bounded function $\lambda : [0, \infty) \to [0, \infty)$ and for any given $\psi \in C$, $s \geq 0$ and $q \in \bigcap_{s \geq 0} F(S(s))$, there exists $j(\psi - q) \in J(\psi - q)$ such that

$$\langle S(s)\psi - q, j(\psi - q)\rangle \leq \|\psi - q\|^2 - \lambda(s)\|(I - S(s))\psi\|^2; \tag{1.7}$$

(6) *asymptotically demicontractive semigroup* (see Yang et al. [18]), if $\bigcap_{s \geq 0} F(S(s)) \neq \emptyset$ and there exists a bounded function $\lambda : [0, \infty) \to [0, \infty)$ and a sequence $\{k_n\} \in [1, \infty)$ with $k_n \to 1$ as $n \to \infty$; for any $s \geq 0$, $\psi \in C$ and $q \in \bigcap_{s \geq 0} F(S(s))$, there exists $j(\psi - q) \in J(\psi - q)$ such that

$$\langle S^n(s)\psi - q, j(\psi - q)\rangle \leq k_n\|\psi - q\|^2 - \lambda(s)\|(I - S^n(s))\psi\|^2, \ \forall n \geq 1, \tag{1.8}$$

for any $s \geq 0$.

(7) *total asymptotically demicontractive semigroup* (see Yang et al. [18]), if $\bigcap_{s \geq 0} F(S(s)) \neq \emptyset$ and there exists a bounded function $\lambda : [0, \infty) \to [0, \infty)$ and sequences $\{\mu_n\} \in [0, \infty)$ and $\{\xi_n\} \in [0, \infty)$ with $\mu_n \to 0$ and $\xi_n \to 0$ as $n \to \infty$. For any given $s \geq 0$, $\psi \in C$ and $q \in \bigcap_{s \geq 0} F(S(s))$, there exists $j(\psi - q) \in J(\psi - q)$ such that

$$\begin{aligned}\langle S^n(s)\psi - q, j(\psi - q)\rangle \quad \leq \quad & \|\psi - q\|^2 - \lambda(s)\|(I - S^n(s))\psi\|^2 \\ & + \mu_n\phi(\|\psi - \zeta\|) + \xi_n, \ \forall n \geq 1,\end{aligned} \tag{1.9}$$

and for any $s \geq 0$.

Remark 1.2. From the above definitions we have the following implications:

(i) If $\phi(t) = t^2$, $\xi_n = 0$, then the class of total asymptotically strictly pseudocontractive semigroup reduces to the class of asymptotically strictly pseudocontractive semigroup and the class of total asymptotically demicontractive semigroup reduces to the class of asymptotically demicontarctive semigroup;

(ii) If $k_n = 1$, $\forall n \geq 1$, the class asymptotically strictly pseudocontractive semigroup reduces to the class of strictly pseudocontractive semigroup and the class asymptotically demicontractive semigroup reduces to then the class of demicontractive semigroup;

(iii) clearly, every strictly pseudocontractive semigroup with $\bigcap_{s\geq 0} F(S(s)) \neq \emptyset$ is a demicontractive semigroup, every asymptotically strictly pseudocontractive semigroup with $\bigcap_{s\geq 0} F(S(s)) \neq \emptyset$ is an asymptotically demicontractive semigroup and every total strictly pseudocontractive semigroup with $\bigcap_{s\geq 0} F(S(s)) \neq \emptyset$ is a total demicontractive semigroup.

From the above implications, it easy to notice that the class of total demicontractive semigroup is the most general of all the classes of semigroup mentioned above.

On the other hand, the convergence problems of semigroup have been considered by many authors in the past three decades. Several implicit and explicit iterative schemes have been introduced and studied by many researchers in nonlinear analysis for approximating the common fixed points of nonexpansive seemigroup, strictly pseudocontractive semigroup, demicontractive semigroup and total asymptotically strictly pseudocontractive semigroup (see for example, [1, 2, 3, 5, 10, 11, 12, 16, 18, 19, 20, 22, 24, 25] and the references there in).

In [24]-[25], Zhang considered the following implicit iteration scheme and proved that it converges strongly to the common fixed point of strictly pseudocontractive semigroups in reflexive Banach spaces:

$$\begin{cases} \psi_0 \in C, \\ \psi_n = (1-\gamma_n)\psi_{n-1} + \gamma_n S(s_n)\psi_n, \end{cases} \quad \forall n \geq 1, \tag{1.10}$$

where $\{\gamma_n\}$ is a sequence in [0,1] and $\{s_n\}$ is an increasing sequence in $[0,\infty)$. Many authors have studied (1.10) and proved its convergence to the common fixed points of nonexpansive semigroup, strictly pseudocontractive semigroup and pseudocontractive semigroup, respectively (see for example Kim [6], Quan et al. [9], Thong [13]–[14] and the references there in).

In 2011, Chang et al. [3] introduced the following Man-type iteration process:

$$\begin{cases} \psi_0 \in C, \\ \psi_{n+1} = (1-\gamma_n)\psi_n + \gamma_n S(s_n)\psi_n, \end{cases} \quad \forall n \geq 1, \tag{1.11}$$

where $\{\gamma_n\}$ is a sequence in [0,1] and $\{s_n\}$ is an increasing sequence in $[0,\infty)$. The authors proved that (1.11) converges strongly the common fixed point of Lipschitzian and demicontraction semigroup $\{S(s) : s \geq 0\}$.

In 2012, Yang and Zhau [19] introduced the following modified Mann-type iteration process for approximating common fixed points of total asymptotically strictly pseudocontractive mappings in Banach spaces:

$$\begin{cases} \psi_0 \in C, \\ \psi_{n+1} = (1-\gamma_n)\psi_n + \gamma_n S^n(s_n)\psi_n \end{cases} \quad \forall n \geq 1, \tag{1.12}$$

where $\{\gamma_n\}$ is a sequence in [0,1] and $\{s_n\}$ is an increasing sequence in $[0,\infty)$.

In 2013, Yang et al. [18] introduced and studied the following averaging modified Mann-type iteration process for finite family of total asymptotically strictly pseudocontractive mappings in the frame work of Banach spaces:

$$\begin{cases} \psi_0 \in C, \\ \psi_{n+1} = (1-\gamma_n)\psi_n + \gamma_n S_{i(n)}^{k(n)}(s_n)\psi_n, \end{cases} \quad \forall n \geq 1, \tag{1.13}$$

where $\{\gamma_n\}$ is a real sequence in [0,1], $\{s_n\}$ a increasing function in $[0,\infty)$ and $n = (k-1)N + i$, $i = n(i) \in I = \{1, 2, ..., N\}$, $k = k(n) \geq 1$ is some positive integers and $k(n) \to \infty$ as $n \to \infty$.

Recently, Wang and Wang [15] studied the convergence of the following modified Ishikawa-type iteration process for common fixed Points of total asymptotically strictly pseudocontractive semigroups in the settings of Banach spaces:

$$\begin{cases} \psi_0 \in C, \\ \psi_{n+1} = (1-\gamma_n)\psi_n + \gamma_n S^n(s_n)\zeta_n \\ \zeta_n = (1-\delta_n)\psi_n + \delta_n S^n(s_n)\psi_n \end{cases} \quad \forall n \geq 1, \tag{1.14}$$

where $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences in [0,1] and $\{s_n\}$ is an increasing sequence in $[0,\infty)$.

Motivated and inspired by the above results, we introduce the following one step implicit iteration process for finite family of total asymptotically demicontractive semigroup in Banach spaces:

$$\begin{cases} \psi_0 \in C, \\ \psi_n = (1-\gamma_n-\delta_n)\psi_{n-1} + \gamma_n S_{i(n)}^{k(n)}(s_n)\psi_{n-1} + \delta_n S_{i(n)}^{k(n)}(s_n)\psi_n, \end{cases} \quad \forall n \geq 1, \tag{1.15}$$

where $\{\gamma_n\}$ and $\{\delta_n\}$ are real sequences in [0,1] satisfying $\gamma_n + \delta_n \leq 1$, $\{s_n\}$ is an increasing sequence in $[0,\infty)$ and $n = (k-1)N + i$, $i = n(i) \in I = \{1, 2, ..., N\}$, $k = k(n) \geq 1$ is some positive integers and $k(n) \to \infty$ as $n \to \infty$.

Notice that the iteration processes (1.10)-(1.13) are special cases of our new iterative scheme (1.15), i.e.,

- (1.15) reduces to (1.10) when $\gamma_n = 0$, $S^n = S$, $N = 1$.
- (1.15) reduces to (1.11) when $\delta_n = 0$, $S^n = S$, $N = 1$.
- (1.15) reduces to (1.12) when $\delta_n = 0$, $N = 1$.
- (1.15) reduces to (1.13) when $\delta_n = 0$.

The aim of this paper is to prove strong convergence theorem of our new iteration process (1.15) for common fixed points of total asymptotically uniformly $L(s)$-Lipsschitzian demicontractive semigroup in Banach spaces. Since our new implicit iteration process (1.15) properly includes the iterative processes (1.10)-(1.13) which has been considered by well known authors, hence, our results extend, generalize and unify the corresponding results of Chang et al. [3], Thong [13]–[14], Yang and Zhao [19], Yang et al. [18], Zhang [24]-[25], and several others in the existing literature.

2 Preliminaries

In the sequel, we will need the following Lemmas.

Lemma 2.1. [4] Let C be a nonempty closed convex subset of a Banach space X and $S: C \to C$ be a continuous and strongly pseudocontractive mapping. Then S has a unique fixed point.

Lemma 2.2. [20] Let $J: X \to 2^{X^*}$ be the normalized duality mapping. Then for any $\psi, \zeta \in X$, one has

$$\|\psi + \zeta\|^2 \leq \|\psi\|^2 + 2\langle \zeta, j(\psi+\zeta)\rangle, \ \forall j(\psi+\zeta) \in J(\psi+\zeta). \tag{2.1}$$

Lemma 2.3. [15] Let $\{\vartheta_n\}$ and $\{\Lambda_n\}$, $\{\sigma_n\}$ be sequences of nonnegative real numbers satisfying the following inequality:

$$\vartheta_n \leq (1+\Lambda_n)\vartheta_n + \sigma_n, \ n \geq 1. \tag{2.2}$$

If $\sum_{n=1}^{\infty} \Lambda_n < \infty$ and $\sum_{n=1}^{\infty} \sigma_n < \infty$ then $\lim\limits_{n\to\infty} \vartheta_n$ exists. Additionally, if $\{\vartheta_n\}$ has a subsequence $\{\vartheta_{n_i}\}$ such that $\vartheta_{n_i} \to 0$, then $\lim\limits_{n\to\infty} \vartheta_n = 0$.

3 Main results

Now we show that our new iteration process (1.15) can employed to approximate the common fixed points of finite family of uniformly $L_i(s)$-Lipschitzian total asymptotically demicontractive semigroup. To see this, we need to prove that it is well defined.

Let C be a nonempty closed convex subset of a real Banach spaces X. For some fixed $i \in \mathbb{N}$, let $\Im_i = \{S_i(s): s \geq 0\}$ be a finite family of uniformly $L_i(s)$-Lipschitzian total asymptotically demicontractive semigroup with a bounded measurable function $L_i: [0,\infty) \to [0,\infty)$ and a bounded function $\lambda_i: [0,\infty) \to [0,\infty)$ and sequences $\{\mu_{in}\}, \{\xi_{in}\} \in [0,\infty)$ with $\mu_{in} \to 0$ and $\xi_{in} \to 0$ as $n \to \infty$, such that

$$L^i = \sup_{s\geq 0} L_i(s) < \infty, \ \lambda^i = \inf_{s\geq 0} \lambda_i(s) > 0, \ F^i = \bigcap_{s\geq 0} F(S_i(s)) \neq \emptyset.$$

Then, for $\psi, \zeta \in C$, $q \in F^i$ and $s \geq 0$,

$$\langle S_i^n(s)\psi - q, j(\psi - q)\rangle \quad \leq \quad \|\psi - q\|^2 - \lambda^i\|(I - S_i^n(s))\psi\|^2 + \mu_{in}\phi(\|\psi - \zeta\|) + \xi_{in}, \forall n \geq 1, \tag{3.1}$$

where $\phi_i : [0, \infty) \to [0, \infty)$ is a continuous and strictly increasing function with $\phi_i(0) = 0$, and

$$\|S_i^n(s)\psi - S_i^n(s)\zeta\| \leq L^i\|\psi - \zeta\|, \quad n \geq 1. \tag{3.2}$$

Consider a family $\{\Im_i\}_{i=1}^N$ of uniformly $L_i(s)$-Lipschitzian total asymptotically demicontractive semigroup of C and let

$$L = \max_{1\leq i\leq N} L^i < \infty, \;\; \lambda = \max_{1\leq i\leq N} \lambda^i > 0, \;\; F = \bigcap_{1\leq i\leq N} \bigcap_{s\geq 0} F(S_i(s)) \neq \emptyset,$$

$$\mu_n = \max_{1\leq i\leq N} \mu_{in}, \;\; \xi_n = \max_{1\leq i\leq N} \xi_{in}, \;\; \phi = \max_{1\leq i\leq N} \phi_i$$

For $\psi, \zeta \in C$, $q \in F$ and $s \geq 0$ and any $i \in \{1, 2, ..., N\}$,

$$\langle S_i^n(s)\psi - q, j(\psi - q)\rangle \quad \leq \quad \|\psi - q\|^2 - \lambda\|(I - S_i^n(s))\psi\|^2 + \mu_n\phi(\|\psi - \zeta\|) + \xi_n, \forall n \geq 1, \tag{3.3}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and strictly increasing function with $\phi(0) = 0$, and

$$\|S_i^n(s)\psi - S_i^n(s)\zeta\| \leq L\|\psi - \zeta\|, \;\; n \geq 1. \tag{3.4}$$

Let $\{\psi_n\}$ be defined by (1.15). For $s \geq 0$, define a mapping $T_n : C \to C$ by

$$T_n\psi = (1 - \gamma_n - \delta_n)\psi_{n-1} + \gamma_n S_{i(n)}^{k(n)}(s_n)\psi_{n-1} + \delta_n S_{i(n)}^{k(n)}(s_n)\psi \tag{3.5}$$

$\forall\psi \in C$ and $n \geq 1$.

Now for any $\psi, \zeta \in C$ and $\forall n \geq 1$, we have,

$$\begin{aligned}
\langle T_n\psi - T_n\zeta, j(\psi - \zeta)\rangle &= \langle \delta_n S_{i(n)}^{k(n)}(s_n)\psi - \delta S_{i(n)}^{k(n)}(s_n)\zeta, j(\psi - \zeta)\rangle \\
&= \langle \delta_n (S_{i(n)}^{k(n)}(s_n)\psi - S_{i(n)}^{k(n)}(s_n)\zeta), j(\psi - \zeta)\rangle \\
&= \delta_n\langle S_{i(n)}^{k(n)}(s_n)\psi - S_{i(n)}^{k(n)}(s_n)\zeta, j(\psi - \zeta)\rangle \\
&\leq \delta_n\|S_{i(n)}^{k(n)}(s_n)\psi - S_{i(n)}^{k(n)}(s_n)\zeta\|\|\psi - \zeta\| \\
&\leq \delta_n L\|\psi - \zeta\|^2,
\end{aligned}$$

since $\delta_n L \in (0, 1)$, we see that T_n is a strongly pseudocontractive mapping, which is also continuous, so from Lemma 2.1, T_n has a unique fixed point $\psi_n \in C$, that is

$$\psi_n = (1 - \gamma_n - \delta_n)\psi_{n-1} + \gamma_n S_{i(n)}^{k(n)}(s_n)\psi_{n-1} + \delta_n S_{i(n)}^{k(n)}(s_n)\psi_n \tag{3.6}$$

has a unique solution for each $n \geq 1$. Therefore, $\{\psi_n\}$ is well defined. Now, our main result is given as follows:

Theorem 3.1. Let C be a nonempty closed convex subset of a real Banach space X. For some fixed $i \in \mathbb{N}$, let $\Im_i = \{S_i(s) : s \geq 0\}$ be a finite family of uniformly $L_i(s)$-Lipschitzian total asymptotically demicontractive semigroup with a bounded measurable function $L_i : [0, \infty) \to [0, \infty)$ and a bounded function $\lambda_i : [0, \infty) \to [0, \infty)$ and sequences $\{\mu_{in}\}, \{\xi_{in}\} \in [0, \infty)$ with $\mu_{in} \to 0$ and $\xi_{in} \to 0$ as $n \to \infty$, such that

$$L^i = \sup_{s\geq 0} L_i(s) < \infty, \;\; \lambda^i = \inf_{s\geq 0} \lambda_i(s) > 0, \;\; F^i = \bigcap_{s\geq 0} F(S_i(s)) \neq \emptyset.$$

Let $L, \lambda, F, \phi, \{\mu_n\}$ and $\{\xi_n\}$ be same as above. Assume that there exist M and M^* which are constants such that $\phi(\psi) \leq M^*\psi^2$, for all $\psi \geq M$. For arbitrary $\psi_0 \in C$, let $\{\psi_n\}$ be the sequence generated by (1.15). Let $\{\gamma_n\}, \{\delta_n\}$ be two real sequences in $[0, 1]$ such that $\gamma_n + \delta_n \leq 1$ and $\{s_n\}$ be an increasing sequence in $[0, \infty)$. If the following conditions are satisfied:

(i) $\sum_{n=1}^{\infty}(\gamma_n+\delta_n)^2<\infty$;
(ii) $\sum_{n=1}^{\infty}(\gamma_n+\delta_n)=\infty$;
(iii) $\sum_{n=1}^{\infty}(\gamma_n+\delta_n)\mu_n<\infty,\ \sum_{n=1}^{\infty}(\gamma_n+\delta_n)\xi_n<\infty$;
(iv) $\delta_n L<1$;
(v) assume for any $i\in\{1,2,...,N\}$ and for any subset $D\in C$,

$$\lim_{n\to\infty}\sup_{\psi\in D,t\in\Re^+}\|S_i^n(t+s_n)\psi-S_i^n(s_n)\psi\|=0; \tag{3.7}$$

(vi) there exists a compact subset Ψ of X such that $\bigcap_{s\geq 0}S_g(s)(C)\subset\Psi$ for some $g\in\{1,2,...,N\}$.

Then the sequence $\{\psi_n\}$ converges to a point in F.

Proof . We dive the proof of Theorem 3.1 into five steps.

Step 1. Since the common fixed point set F is nonempty, let $q\in F$. For each $q\in F$, we prove that $\lim_{n\to\infty}\|\psi_n-q\|$ exists.

Using (1.15) and Lemma 2.2, we obtain

$$\begin{aligned}
\|\psi_n-q\|^2 &= \|(1-\gamma_n-\delta_n)\psi_{n-1}+\gamma_n S_{i(n)}^{k(n)}(s_n)\psi_{n-1}+\delta_n S_{i(n)}^{k(n)}(s_n)\psi_n-q\|^2\\
&= \|(1-\gamma_n-\delta_n)(\psi_{n-1}-q)+\gamma_n(S_{i(n)}^{k(n)}(s_n)\psi_{n-1}-q)\\
&\quad +\delta_n(S_{i(n)}^{k(n)}(s_n)\psi_n-q)\|^2\\
&\leq (1-\gamma_n-\delta_n)^2\|\psi_{n-1}-q\|^2+2\langle\gamma_n(S_{i(n)}^{k(n)}(s_n)\psi_{n-1}-q)\\
&\quad +\delta_n(S_{i(n)}^{k(n)}(s_n)\psi_n-q),j(\psi_n-p)\rangle\\
&= (1-\gamma_n-\delta_n)^2\|\psi_{n-1}-q\|^2+2\gamma_n\langle S_{i(n)}^{k(n)}(s_n)\psi_{n-1}-q,j(\psi_n-q)\rangle\\
&\quad +2\delta_n\langle S_{i(n)}^{k(n)}(s_n)\psi_n-q,j(\psi_n-q)\rangle\\
&= (1-\gamma_n-\delta_n)^2\|\psi_{n-1}-q\|^2+2\gamma_n\langle S_{i(n)}^{k(n)}(s_n)\psi_{n-1}-S_{i(n)}^{k(n)}(s_n)\psi_n\\
&\quad +S_{i(n)}^{k(n)}(s_n)\psi_n-q,j(\psi_n-q)\rangle+2\delta_n\langle S_{i(n)}^{k(n)}(s_n)\psi_n-q,j(\psi_n-q)\rangle\\
&= (1-\gamma_n-\delta_n)^2\|\psi_{n-1}-q\|^2+2\gamma_n\langle S_{i(n)}^{k(n)}(s_n)\psi_{n-1}-S_{i(n)}^{k(n)}(s_n)\psi_n,j(\psi_n-q)\rangle\\
&\quad +2\gamma_n\langle S_{i(n)}^{k(n)}(s_n)\psi_n-q,j(\psi_n-q)\rangle+2\delta_n\langle S_{i(n)}^{k(n)}(s_n)\psi_n-q,j(\psi_n-q)\rangle\\
&\leq (1-\gamma_n-\delta_n)^2\|\psi_{n-1}-q\|^2+2\gamma_n\|S_{i(n)}^{k(n)}(s_n)\psi_{n-1}-S_{i(n)}^{k(n)}(s_n)\psi_n\|\|\psi_n-q\|\\
&\quad +2\gamma_n\langle S_{i(n)}^{k(n)}(s_n)\psi_n-q,j(\psi_n-q)\rangle+2\delta_n\langle S_{i(n)}^{k(n)}(s_n)\psi_n-q,j(\psi_n-q)\rangle\\
&\leq (1-\gamma_n-\delta_n)^2\|\psi_{n-1}-q\|^2+2\gamma_n L\|\psi_{n-1}-\psi_n\|\|\psi_n-q\|\\
&\quad +2(\gamma_n+\delta_n)\langle S_{i(n)}^{k(n)}(s_n)\psi_n-q,j(\psi_n-q)\rangle.
\end{aligned} \tag{3.8}$$

Notice from (1.15) that

$$\begin{aligned}
\|\psi_n-\psi_{n-1}\| &= \|\gamma_n(S_{i(n)}^{k(n)}(s_n)\psi_{n-1}-\psi_{n-1})+\delta_n(S_{i(n)}^{k(n)}(s_n)\psi_n-\psi_{n-1})\|\\
&= \|\gamma_n(S_{i(n)}^{k(n)}(s_n)\psi_{n-1}-\psi_{n-1})+\delta_n(S_{i(n)}^{k(n)}(s_n)\psi_n-\psi_{n-1})\|\\
&\leq \gamma_n\|S_{i(n)}^{k(n)}(s_n)\psi_{n-1}-q\|+\gamma_n\|\psi_{n-1}-q\|\\
&\quad +\delta_n\|S_{i(n)}^{k(n)}(s_n)\psi_n-q\|+\delta_n\|\psi_{n-1}-q\|\\
&\leq \gamma_n L\|\psi_{n-1}-q\|+\gamma_n\|\psi_{n-1}-q\|\\
&\quad +\delta_n L\|\psi_n-q\|+\delta_n\|\psi_{n-1}-q\|\\
&= (\gamma_n(1+L)+\delta_n)\|\psi_{n-1}-q\|+\delta_n L\|\psi_n-q\|.
\end{aligned} \tag{3.9}$$

By putting (3.9) into (3.8), we obtain

$$\begin{aligned}\|\psi_n - q\|^2 &\leq (1-\gamma_n-\delta_n)^2\|\psi_{n-1}-q\|^2 + 2\gamma_n L\{(\gamma_n(1+L)+\delta_n)\|\psi_{n-1}-q\| \\ &\quad +\delta_n L\|\psi_n - q\|\}\|\psi_n - q\| + 2(\gamma_n+\delta_n)\langle S_{i(n)}^{k(n)}(s_n)\psi_n - q), j(\psi_n - q)\rangle \\ &= (1-\gamma_n-\delta_n)^2\|\psi_{n-1}-q\|^2 + 2\gamma_n L(\gamma_n(1+L)+\delta_n)\|\psi_{n-1}-q\|\|\psi_n - q\| \\ &\quad +2\gamma_n\delta_n L^2\|\psi_n - q\|^2 + 2(\gamma_n+\delta_n)\langle S_{i(n)}^{k(n)}(s_n)\psi_n - q, j(\psi_n - q)\rangle. \end{aligned} \tag{3.10}$$

Since

$$\|\psi_{n-1}-q\|\|\psi_n - q\| \leq \frac{1}{2}(\|\psi_{n-1}-q\|^2 + \|\psi_n - q\|^2), \tag{3.11}$$

then from (3.10) and (3.11) we obtain

$$\begin{aligned}\|\psi_n - q\|^2 &\leq (1-\gamma_n-\delta_n)^2\|\psi_{n-1}-q\|^2 + 2\gamma_n L(\gamma_n(1+L)+\delta_n) \\ &\quad \times\frac{1}{2}(\|\psi_{n-1}-q\|^2 + \|\psi_n - q\|^2) + 2\gamma_n\delta_n L^2\|\psi_n - q\|^2 \\ &\quad +2(\gamma_n+\delta_n)\langle S_{i(n)}^{k(n)}(s_n)\psi_n - q, j(\psi_n - q)\rangle \\ &= [(1-\gamma_n-\delta_n)^2 + \gamma_n L(\gamma_n(1+L)+\delta_n)]\|\psi_{n-1}-q\|^2 \\ &\quad +[\gamma_n L(\gamma_n(1+L)+\delta_n) + 2\gamma_n\delta_n L^2]\|\psi_n - q\|^2 \\ &\quad +2(\gamma_n+\delta_n)\langle S_{i(n)}^{k(n)}(s_n)\psi_n - q, j(\psi_n - q)\rangle. \end{aligned} \tag{3.12}$$

Since $\Im_i = \{S_i(s) : s \geq 0\}$, $i \in \{1, 2, ..., \}$ is a finite family of uniformly $L(s)$-Lipschitzian total asymptotically demicontractive semigroup, then from (3.3) and (3.12) we obtain

$$\begin{aligned}\|\psi_n - q\|^2 &\leq [(1-\gamma_n-\delta_n)^2 + \gamma_n L(\gamma_n(1+L)+\delta_n)]\|\psi_{n-1}-q\|^2 \\ &\quad +[\gamma_n L(\gamma_n(1+L)+\delta_n) + 2\gamma_n\delta_n L^2]\|\psi_n - q\|^2 + 2(\gamma_n+\delta_n)\times \\ &\quad \{\|\psi_n - q\|^2 - \lambda\|\psi_n - S_{i(n)}^{k(n)}(s_n)\psi_n\| + \mu_n\phi(\|\psi_n - q\|) + \xi_n\} \end{aligned} \tag{3.13}$$

Since ϕ is a strictly increasing function, it follows that $\phi(\psi) \leq \phi(M)$, if $\psi \leq M$; $\phi(\psi) \leq M^*\psi^2$, if $\psi \geq M$. In either case, we can obtain

$$\phi(\psi) \leq \phi(M) + M^*\psi^2. \tag{3.14}$$

Hence from (3.13) and (3.14), we have

$$\begin{aligned}\|\psi_n - q\|^2 &\leq [(1-\gamma_n-\delta_n)^2 + \gamma_n L(\gamma_n(1+L)+\delta_n)]\|\psi_{n-1}-q\|^2 \\ &\quad +[\gamma_n L(\gamma_n(1+L)+\delta_n) + 2\gamma_n\delta_n L^2]\|\psi_n - q\|^2 + 2(\gamma_n+\delta_n)\times \\ &\quad \{\|\psi_n - q\|^2 - \lambda\|\psi_n - S_{i(n)}^{k(n)}(s_n)\psi_n\|^2 + \mu_n\phi(M) + \mu_n M^*\|\psi_n - q\|^2 + \xi_n\} \\ &= [(1-\gamma_n-\delta_n)^2 + \gamma_n L(\gamma_n(1+L)+\delta_n)]\|\psi_{n-1}-q\|^2 \\ &\quad +[\gamma_n L(\gamma_n(1+L)+\delta_n) + 2\gamma_n\delta_n L^2 + 2(\gamma_n+\delta_n) + 2(\gamma_n+\delta_n)\mu_n M^*]\|\psi_n - q\|^2 \\ &\quad -2(\gamma_n+\delta_n)\lambda\|\psi_n - S_{i(n)}^{k(n)}(s_n)\psi_n\|^2 + 2(\gamma_n+\delta_n)\mu_n\phi(M) + 2(\gamma_n+\delta_n)\xi_n \\ &= \nu_n\|\psi_{n-1}-q\|^2 + \zeta_n\|\psi_n - q\|^2 + 2(\gamma_n+\delta_n)\mu_n\phi(M) + 2(\gamma_n+\delta_n)\xi_n \\ &\quad -2(\gamma_n+\delta_n)\lambda\|\psi_n - S_{i(n)}^{k(n)}(s_n)\psi_n\|^2, \end{aligned} \tag{3.15}$$

where

$$\nu_n = (1-\gamma_n-\delta_n)^2 + \gamma_n L(\gamma_n(1+L)+\delta_n), \tag{3.16}$$

$$\zeta_n = \gamma_n L(\gamma_n(1+L)+\delta_n) + 2\gamma_n\delta_n L^2 + 2(\gamma_n+\delta_n) + 2(\gamma_n+\delta_n)\mu_n M^*. \tag{3.17}$$

From simplification of (3.15) we get

$$\begin{aligned}\|\psi_n - q\|^2 &\le \frac{\nu_n}{1-\zeta_n}\|\psi_{n-1} - q\|^2 + \frac{2(\gamma_n+\delta_n)\mu_n\phi(M)}{1-\zeta_n} + \frac{2(\gamma_n+\delta_n)\xi_n}{1-\zeta_n}\\ &\quad -\frac{2(\gamma_n+\delta_n)\lambda}{1-\zeta_n}\|\psi_n - S_{i(n)}^{k(n)}(s_n)\psi_n\|^2\\ &= \left(1+\frac{\nu_n+\zeta_n-1}{1-\zeta_n}\right)\|\psi_{n-1} - q\|^2 + \frac{2(\gamma_n+\delta_n)\mu_n\phi(M)}{1-\zeta_n} + \frac{2(\gamma_n+\delta_n)\xi_n}{1-\zeta_n}\\ &\quad -\frac{2(\gamma_n+\delta_n)\lambda}{1-\zeta_n}\|\psi_n - S_{i(n)}^{k(n)}(s_n)\psi_n\|^2. \end{aligned} \tag{3.18}$$

Observe that

$$\nu_n+\zeta_n-1 = (\gamma_n+\delta_n)^2 + 2\gamma_n L(\gamma_n(1+L)+\delta_n) + 2\gamma_n\delta_n L^2 + 2(\gamma_n+\delta_n)\mu_n M^*. \tag{3.19}$$

Set

$$\vartheta_n = \nu_n+\zeta_n-1. \tag{3.20}$$

Since by condition (i) we have that $\sum_{n=1}^{\infty}(\gamma_n+\delta_n)^2<\infty$, then this implies that $\gamma_n+\delta_n, \gamma_n^2, \gamma_n\delta_n \to 0$ as $n\to\infty$ and recalling condition (iii), we obtain

$$\zeta_n = \gamma_n L(\gamma_n(1+L)+\delta_n) + 2\gamma_n\delta_n L^2 + 2(\gamma_n+\delta_n) + 2(\gamma_n+\delta_n)\mu_n M^* \to 0 \text{ as } n\to\infty,$$

therefore, there exists a positive integer n_0 such that

$$\frac{1}{2} < 1-\zeta_n \le 1,\ \forall\, n\ge n_0. \tag{3.21}$$

Thus, from (3.18) we have that

$$\begin{aligned}\|\psi_n - q\|^2 &\le (1+2\vartheta_n)\|\psi_{n-1} - q\|^2 + 4(\gamma_n+\delta_n)\mu_n\phi(M)\\ &\quad +4(\gamma_n+\delta_n)\xi_n - 2(\gamma_n+\delta_n)\lambda\|\psi_n - S_{i(n)}^{k(n)}(s_n)\psi_n\|^2\\ &= (1+\alpha_n)\|\psi_{n-1} - q\|^2 + \beta_n\\ &\quad -2(\gamma_n+\delta_n)\lambda\|\psi_n - S_{i(n)}^{k(n)}(s_n)\psi_n\|^2 \qquad (3.22)\\ &\le (1+\alpha_n)\|\psi_{n-1} - q\|^2 + \beta_n, \qquad (3.23)\end{aligned}$$

where

$$\begin{aligned}\alpha_n &= 2\vartheta_n,\\ \beta_n &= 4(\gamma_n+\delta_n)\mu_n\phi(M) + 4(\gamma_n+\delta_n)\xi_n.\end{aligned}$$

Again, since by condition (i) we have that $\sum_{n=1}^{\infty}(\gamma_n+\delta_n)^2<\infty$, which implies that $\sum_{n=1}^{\infty}\gamma_n^2<\infty$, $\sum_{n=1}^{\infty}\gamma_n\delta_n<\infty$, recalling from condition (iii) that $\sum_{n=1}^{\infty}(\gamma_n+\delta_n)\mu_n<\infty$ and $\sum_{n=1}^{\infty}(\gamma_n+\delta_n)\xi_n<\infty$, then it follows that $\sum_{n=1}^{\infty}\alpha_n<\infty$ and $\sum_{n=1}^{\infty}\beta_n<\infty$.

Notice from (3.23) that all the conditions of Lemma 2.3 are satisfied. Thus, $\lim_{n\to\infty}\|\psi_n - q\|$ exists, therefore, the sequence $\{\|\psi_n - q\|\}$ is bounded.

Step 2. We now show that $\liminf_{n\to\infty}\|\psi_n - S_{i(n)}^{k(n)}(s_n)\psi_n\| = 0$.

From (3.22), we have

$$2(\gamma_n+\delta_n)\lambda\|\psi_n - S_{i(n)}^{k(n)}(s_n)\psi_n\|^2 \leq \|\psi_{n-1}-q\|^2 - \|\psi_n - q\|^2 + \alpha_n\|\psi_{n-1}-q\|^2 + \beta_n. \quad (3.24)$$

Let $\varphi = \sup_{n\geq 1}\|\psi_{n-1}-q\|$. Then, for some $m \geq 1$ we get

$$\begin{aligned} 2\lambda\sum_{n=1}^{m}(\gamma_n+\delta_n)\|\psi_n - S_{i(n)}^{k(n)}(s_n)\psi_n\|^2 &\leq \sum_{n=1}^{m}(\|\psi_{n-1}-q\|^2 - \|\psi_n-q\|^2) \\ &\quad +\varphi^2\sum_{n=1}^{m}\alpha_n + \sum_{n=1}^{m}\beta_n \quad (3.25)\\ &\leq \|\psi_0 - q\|^2 + \varphi^2\sum_{n=1}^{m}\alpha_n + \sum_{n=1}^{m}\beta_n. \quad (3.26)\end{aligned}$$

Letting $m\to\infty$, we obtain

$$\begin{aligned} 2\lambda\sum_{n=1}^{\infty}(\gamma_n+\delta_n)\|\psi_n - S_{i(n)}^{k(n)}(s_n)\psi_n\|^2 &\leq \sum_{n=1}^{\infty}(\|\psi_{n-1}-q\|^2 - \|\psi_n-q\|^2) \\ &\quad +\varphi^2\sum_{n=1}^{\infty}\alpha_n + \sum_{n=1}^{\infty}\beta_n \\ &\leq \|\psi_0 - q\|^2 + \varphi^2\sum_{n=1}^{\infty}\alpha_n + \sum_{n=1}^{\infty}\beta_n. \quad (3.27)\end{aligned}$$

Since $\sum_{n=1}^{\infty}\alpha_n < \infty$ and $\sum_{n=1}^{\infty}\beta_n < \infty$, it follows from (3.27) that

$$\sum_{n=1}^{\infty}(\gamma_n+\delta_n)\|\psi_n - S_{i(n)}^{k(n)}(s_n)\psi_n\|^2 < \infty. \quad (3.28)$$

Since from condition (ii) we have $\sum_{n=1}^{\infty}(\gamma_n+\delta_n) = \infty$, then we must have that

$$\liminf_{n\to\infty}\|\psi_n - S_{i(n)}^{k(n)}(s_n)\psi_n\| = 0. \quad (3.29)$$

Step 3. We now prove that $\liminf_{n\to\infty}\|\psi_n - S_{i(n)}^{k(n)}(s)\psi_n\| = 0$.

Since from condition (i) we have $\sum_{n=1}^{\infty}(\gamma_n+\delta_n)^2 < \infty$, which implies that $\sum_{n=1}^{\infty}\gamma_n^2 < \infty$, $\sum_{n=1}^{\infty}\delta_n^2 < \infty$, then we know that $\gamma_n, \delta_n \to 0$ as $n\to\infty$. Also, since $\lim_{n\to\infty}\|\psi_n - q\|$ exists for all $q\in F$, from (3.9) we have

$$\lim_{n\to\infty}\|\psi_n - \psi_{n-1}\| = 0. \quad (3.30)$$

This implies that

$$\lim_{n\to\infty}\|\psi_{n+i} - \psi_n\| = 0, \ \ \forall i \in \{1,2,...,N\}. \quad (3.31)$$

For any $s \geq 0$, we have

$$\begin{aligned}\|\psi_n - S_{i(n)}^{k(n)}(s)\psi_n\| &\leq \|\psi_n - S_{i(n)}^{k(n)}(s_n)\psi_n\| + \|S_{i(n)}^{k(n)}(s_n)\psi_n - S_{i(n)}^{k(n)}(s_n+t)\psi_n\| \\ &\quad + \|S_{i(n)}^{k(n)}(s_n+s)\psi_n - S_{i(n)}^{k(n)}(s)\psi_n\| \\ &\leq (1+L)\|\psi_n - S_{i(n)}^{k(n)}(s_n)\psi_n\| \\ &\quad + \sup_{t\in\{\psi_n\}, s\in\Re^+} \|S_{i(n)}^{k(n)}(s+s_n)t - S_{i(n)}^{k(n)}(s_n)t\|. \end{aligned} \tag{3.32}$$

Now from (3.29) and with the help of condition (v), we have

$$\lim_{n\to\infty} \|\psi_n - S_{i(n)}^{k(n)}(s)\psi_n\| = 0. \tag{3.33}$$

Step 4. We will now show that

$$\lim_{n\to\infty} \|\psi_n - S_\ell(s)\psi_n\| = 0, \ \forall\, \ell \in \{1,2,...,N\}. \tag{3.34}$$

Since for each $n > N$, $n = (k(n)-1)N + i(n)$, where $i(n) \in \{1,2,...,N\}$, then $n - N = (k(n)-1)N + i(n) - N = [(k(n)-1)-1]N + i(n) = (k(n-N)-1)N + i(n-N)$, thus $k(n-N) = k(n)-1$ and $i(n-N) = i(n)$, hence, we see that

$$\begin{aligned}\|\psi_n - S_{i(n)}(s)\psi_n\| &\leq \|\psi_n - S_{i(n)}^{k(n)}(s)\psi_n\| + \|S_{i(n)}^{k(n)}(s)\psi_n - S_{i(n)}(s)\psi_n\| \\ &\leq \|\psi_n - S_{i(n)}^{k(n)}(s)\psi_n\| + L\|S_{i(n)}^{k(n)-1}(s)\psi_n - \psi_n\| \\ &\leq \|\psi_n - S_{i(n)}^{k(n)}(s)\psi_n\| + L(\|S_{i(n)}^{k(n)-1}(s)\psi_n - S_{i(n-N)}^{k(n)-1}(s)\psi_{n-N}\| \\ &\quad + \|S_{i(n-N)}^{k(n)-1}(s)\psi_{n-N} - \psi_{n-N}\| + \|\psi_{n-N} - \psi_n\|). \end{aligned} \tag{3.35}$$

Notice that $k(n-N) = k(n)-1$ and $i(n-N) = i(n)$. Thus, it implies that

$$\begin{aligned}\|S_{i(n)}^{k(n)-1}(s)\psi_n - S_{i(n-N)}^{k(n)-1}(s)\psi_{n-N}\| &= \|S_{i(n)}^{k(n)-1}(s)\psi_n - S_{i(n)}^{k(n)-1}(s)\psi_{n-N}\| \\ &\leq L\|\psi_n - \psi_{n-N}\| \end{aligned} \tag{3.36}$$

and

$$\|S_{i(n-N)}^{k(n)-1}(s)\psi_{n-N} - \psi_{n-N}\| = \|S_{i(n-N)}^{k(n-N)}(s)\psi_{n-N} - \psi_{n-N}\|. \tag{3.37}$$

Substituting (3.36) and (3.37) into (3.35)

$$\begin{aligned}\|\psi_n - S_{i(n)}(s)\psi_n\| &\leq \|\psi_n - S_{i(n)}^{k(n)}(s)\psi_n\| + L(L\|\psi_n - \psi_{n-N}\| \\ &\quad + \|S_{i(n-N)}^{k(n-N)}(s)\psi_{n-N} - \psi_{n-N}\| + \|\psi_{n-N} - \psi_n\|). \end{aligned}$$

It follows from (3.31) and (3.33) that

$$\lim_{n\to\infty} \|\psi_n - S_{i(n)}(s)\psi_n\| = 0. \tag{3.38}$$

In particular, we see that

$$\begin{cases} \lim_{k\to\infty} \|\psi_{kN+1} - S_1(s)\psi_{kN+1}\| = 0, \\ \lim_{k\to\infty} \|\psi_{kN+2} - S_2(s)\psi_{kN+2}\| = 0, \\ \quad\vdots \\ \lim_{k\to\infty} \|\psi_{kN+N} - S_N(s)\psi_{kN+N}\| = 0. \end{cases} \tag{3.39}$$

For any $\ell, \tau = 1, 2, ...N$, we obtain that

$$\begin{aligned}\|\psi_{kN+\tau} - S_\ell(s)\psi_{kN+\tau}\| &\leq \|\psi_{kN+\tau} - \psi_{kN+\ell}\| + \|\psi_{kN+\ell} - S_\ell(s)\psi_{kN+\ell}\| \\ &\quad + \|S_\ell(s)\psi_{kN+\ell} - S_\ell(s)\psi_{kN+\tau}\| \\ &\leq (1+L)\|\psi_{kN+\tau} - \psi_{kN+\ell}\| + \|\psi_{kN+\ell} - S_\ell(s)\psi_{kN+\ell}\|.\end{aligned}$$

Letting $k \to \infty$, we obtain

$$\lim_{k\to\infty} \|\psi_{kN+\tau} - S_\ell(s)\psi_{kN+\tau}\| = 0, \tag{3.40}$$

which is equivalent to

$$\lim_{n\to\infty} \|\psi_n - S_\ell(s)\psi_n\| = 0, \forall \ell \in \{1, 2, ..., N\}. \tag{3.41}$$

This completes the proof.

Step 5. We end the proof of Theorem 3.1 by showing that $\{\psi_n\}$ converges strongly to a point in F.

Since from (3.41) we have $\lim\limits_{n\to\infty} \|\psi_n - S_\ell(s)\psi_n\| = 0, \forall \ell \in \{1, 2, ..., N\}$. Suppose that $\bigcap\limits_{s\geq 0} S_g(s)(C) \subset \Psi$ for some compact subset Ψ of X and some $g \in \{1, 2, ..., N\}$, then we know that a subsequence $\{\psi_{n_k}\}$ of $\{\psi_n\}$ exists and $z \in C$ such that

$$\lim_{k\to\infty} S_g(s)\psi_{n_k} = z, \ \ \lim_{k\to\infty} \|\psi_{n_k} - S_g(s)\psi_{n_k}\| = 0. \tag{3.42}$$

Hence, from (3.42) it follows that

$$\lim_{k\to\infty} \psi_{n_k} = z. \tag{3.43}$$

Now, for any $\ell \in \{1, 2, ..., N\}$, since $\lim\limits_{k\to\infty} \|\psi_{n_k} - S_\ell(s)\psi_{n_k}\| = 0$, then a subsequence $\{\psi_{n_{k_j}}\}$ of $\{\psi_{n_k}\}$ exists such that $\lim\limits_{j\to\infty} \|\psi_{n_{k_j}} - S_\ell(s)\psi_{n_{k_j}}\| = 0$. From (3.43) and recalling that S_ℓ is Lipschitzian, we have $z \in \bigcap\limits_{s\geq 0} F(S_\ell(s))$. Since $\ell \in \{1, 2, ..., N\}$ is arbitrary taken, we have $z \in F$.

Again, since $\psi_{n_k} \to z$ as $k \to \infty$ and the limit $\lim\limits_{n\to\infty} \|\psi_n - z\|$ exists, this implies that $\psi_n \to z \in F$ as $n \to \infty$. This completes the proof of Theorem 3.1. □ The following results can be obtain from Theorem 3.1 immediately.

Corollary 3.2. Let C be a nonempty closed convex subset of a real Banach space X. For some fixed $i \in \mathbb{N}$, let $\Im_i = \{S_i(s) : s \geq 0\}$ be a finite family of uniformly $L_i(s)$-Lipschitzian total asymptotically demicontractive semigroup with a bounded measurable function $L_i : [0, \infty) \to [0, \infty)$ and a bounded function $\lambda_i : [0, \infty) \to [0, \infty)$ and sequences $\{\mu_{in}\}, \{\xi_{in}\} \in [0, \infty)$ with $\mu_{in} \to 0$ and $\xi_{in} \to 0$ as $n \to \infty$, such that

$$L^i = \sup_{s\geq 0} L_i(s) < \infty, \ \lambda^i = \inf_{s\geq 0} \lambda_i(s) > 0, \ F^i = \bigcap_{s\geq 0} F(S_i(s)) \neq \emptyset.$$

Let $L, \lambda, F, \phi, \{\mu_n\}$ and $\{\xi_n\}$ be same as in Theorem 3.1. Assume that there exist M and M^* which are constants such that $\phi(\psi) \leq M^*\psi^2$, for all $\psi \geq M$. Let $\{\psi_n\}$ be the sequence generated by

$$\begin{cases} \psi_0 \in C, \\ \psi_n = (1-\gamma_n)\psi_{n-1} + \gamma_n S_{i(n)}^{k(n)}(s_n)\psi_{n-1}, \end{cases} \qquad \forall n \geq 1, \tag{3.44}$$

where $\{\gamma_n\}$ is a real sequence in $[0, 1]$, $\{s_n\}$ is an increasing sequence in $[0, \infty)$ and $n = (k-1)N + i$, $i = n(i) \in I = \{1, 2, ..., N\}$, $k = k(n) \geq 1$ is some positive integers and $k(n) \to \infty$ as $n \to \infty$. If the following conditions are satisfied:

(i) $\sum\limits_{n=1}^{\infty} \gamma_n^2 < \infty$;

(ii) $\sum\limits_{n=1}^{\infty} \gamma_n = \infty$;

(iii) $\sum\limits_{n=1}^{\infty} \gamma_n\mu_n < \infty, \ \sum\limits_{n=1}^{\infty} \gamma_n\xi_n < \infty$;

(iv) assume for any $i \in \{1, 2, ..., N\}$ and for any subset $D \in C$,

$$\lim_{n\to\infty} \sup_{\psi\in D, t\in\Re^+} \|S_i^n(t + s_n)\psi - S_i^n(s_n)\psi\| = 0; \tag{3.45}$$

(v) there exists a compact subset Ψ of X such that $\bigcap\limits_{s\geq 0} S_g(s)(C) \subset \Psi$ for some $g \in \{1, 2, ..., N\}$.

Then the sequence $\{\psi_n\}$ converges to a point in F.

Proof . Set $\delta_n = 0$ in Theorem 3.1. □

Corollary 3.3. Let C be a nonempty closed convex subset of a real Banach space X. For some fixed $i \in \mathbb{N}$, let $\Im_i = \{S_i(s) : s \geq 0\}$ be a finite family of uniformly $L_i(s)$-Lipschitzian total asymptotically demicontractive semigroup with a bounded measurable function $L_i : [0, \infty) \to [0, \infty)$ and a bounded function $\lambda_i : [0, \infty) \to [0, \infty)$ and sequences $\{\mu_{in}\}, \{\xi_{in}\} \in [0, \infty)$ with $\mu_{in} \to 0$ and $\xi_{in} \to 0$ as $n \to \infty$, such that

$$L^i = \sup_{s\geq 0} L_i(s) < \infty, \ \lambda^i = \inf_{s\geq 0} \lambda_i(s) > 0, \ F^i = \bigcap_{s\geq 0} F(S_i(s)) \neq \emptyset.$$

Let $L, \lambda, F, \phi, \{\mu_n\}$ and $\{\xi_n\}$ be same as in Theorem 3.1. Assume that there exist M and M^* which are constants such that $\phi(\psi) \leq M^*\psi^2$, for all $\psi \geq M$. Let $\{\psi_n\}$ be the sequence generated by

$$\begin{cases} \psi_0 \in C, \\ \psi_n = (1 - \delta_n)\psi_{n-1} + \delta_n S_{i(n)}^{k(n)}(s_n)\psi_n, \end{cases} \quad \forall n \geq 1, \tag{3.46}$$

where $\{\delta_n\}$ is a real sequence in $[0, 1]$, $\{s_n\}$ is an increasing sequence in $[0, \infty)$ and $n = (k - 1)N + i$, $i = n(i) \in I = \{1, 2, ..., N\}$, $k = k(n) \geq 1$ is some positive integers and $k(n) \to \infty$ as $n \to \infty$. If the following conditions are satisfied:

(i) $\sum\limits_{n=1}^{\infty} \delta_n^2 < \infty$;

(ii) $\sum\limits_{n=1}^{\infty} \delta_n = \infty$;

(iii) $\sum\limits_{n=1}^{\infty} \delta_n\mu_n < \infty, \ \sum\limits_{n=1}^{\infty} \delta_n\xi_n < \infty$;

(iv) $\delta_n L < 1$;

(v) assume for any $i \in \{1, 2, ..., N\}$ and for any subset $D \in C$,

$$\lim_{n\to\infty} \sup_{\psi\in D, t\in\Re^+} \|S_i^n(t + s_n)\psi - S_i^n(s_n)\psi\| = 0; \tag{3.47}$$

(vi) there exists a compact subset Ψ of X such that $\bigcap\limits_{s\geq 0} S_g(s)(C) \subset \Psi$ for some $g \in \{1, 2, ..., N\}$.

Then the sequence $\{\psi_n\}$ converges to a point in F.

Proof . Set $\gamma_n = 0$ in Theorem 3.1. □

Corollary 3.4. Let C be a nonempty closed convex subset of a real Banach space X. Let $\Im = \{S(s) : s \geq 0\}$ be a uniformly $L(s)$-Lipschitzian total asymptotically demicontractive semigroup with a bounded measurable function $L : [0, \infty) \to [0, \infty)$ and a bounded function $\lambda : [0, \infty) \to [0, \infty)$ and sequences $\{\mu_n\}, \{\xi_n\} \in [0, \infty)$ with $\mu_n \to 0$ and $\xi_n \to 0$ as $n \to \infty$, such that

$$L = \sup_{s\geq 0} L(s) < \infty, \ \lambda = \inf_{s\geq 0} \lambda(s) > 0, \ F = \bigcap_{s\geq 0} F(S(s)) \neq \emptyset.$$

Assume that there exist M and M^* which are constants such that $\phi(\psi) \leq M^*\psi^2$, for all $\psi \geq M$. Let $\{\psi_n\}$ be the sequence generated by

$$\begin{cases} \psi_0 \in C, \\ \psi_n = (1-\gamma_n-\delta_n)\psi_{n-1} + \gamma_n S^n(s_n)\psi_{n-1} + \delta_n S^n(s_n)\psi_n, \end{cases} \quad \forall n \geq 1. \tag{3.48}$$

Let $\{\gamma_n\}, \{\delta_n\}$ be two real sequences in $[0,1]$ such that $\gamma_n + \delta_n \leq 1$ and $\{s_n\}$ be an increasing sequence in $[0,\infty)$. If the following conditions are satisfied:

(i) $\sum\limits_{n=1}^{\infty}(\gamma_n+\delta_n)^2 < \infty$;

(ii) $\sum\limits_{n=1}^{\infty}(\gamma_n+\delta_n) = \infty$;

(iii) $\sum\limits_{n=1}^{\infty}(\gamma_n+\delta_n)\mu_n < \infty,\ \sum\limits_{n=1}^{\infty}(\gamma_n+\delta_n)\xi_n < \infty$;

(iv) $\delta_n L < 1$;

(v) assume for any subset $D \in C$,

$$\lim_{n\to\infty} \sup_{\psi\in D, t\in\Re^+} \|S^n(t+s_n)\psi - S^n(s_n)\psi\| = 0; \tag{3.49}$$

(vi) there exists a compact subset Ψ of X such that $\bigcap\limits_{s\geq 0} S(s)(C) \subset \Psi$.

Then the sequence $\{\psi_n\}$ converges to a point in F.

Proof . Set $N = 1$ in Theorem 3.1. □

Corollary 3.5. Let C be a nonempty closed convex subset of a real Banach space X. Let $\Im = \{S(s) : s \geq 0\}$ be a uniformly $L(s)$-Lipschitzian total asymptotically demicontractive semigroup with a bounded measurable function $L : [0,\infty) \to [0,\infty)$ and a bounded function $\lambda : [0,\infty) \to [0,\infty)$ and sequences $\{\mu_n\}, \{\xi_n\} \in [0,\infty)$ with $\mu_n \to 0$ and $\xi_n \to 0$ as $n \to \infty$, such that

$$L = \sup_{s\geq 0} L(s) < \infty,\ \lambda = \inf_{s\geq 0} \lambda(s) > 0,\ F = \bigcap_{s\geq 0} F(S(s)) \neq \emptyset.$$

Assume that there exist M and M^* which are constants such that $\phi(\psi) \leq M^*\psi^2$, for all $\psi \geq M$. Let $\{\psi_n\}$ be the sequence generated by

$$\begin{cases} \psi_0 \in C, \\ \psi_n = (1-\gamma_n)\psi_{n-1} + \gamma_n S^n(s_n)\psi_{n-1}, \end{cases} \quad \forall n \geq 1, \tag{3.50}$$

where $\{\gamma_n\}$ is a real sequence in $[0,1]$ and $\{s_n\}$ is an increasing sequence in $[0,\infty)$. If the following conditions are satisfied:

(i) $\sum\limits_{n=1}^{\infty}\gamma_n^2 < \infty$;

(ii) $\sum\limits_{n=1}^{\infty}\gamma_n = \infty$;

(iii) $\sum\limits_{n=1}^{\infty}\gamma_n\mu_n < \infty,\ \sum\limits_{n=1}^{\infty}\gamma_n\xi_n < \infty$;

(iv) assume for any subset $D \in C$,

$$\lim_{n\to\infty} \sup_{\psi\in D, t\in\Re^+} \|S^n(t+s_n)\psi - S^n(s_n)\psi\| = 0; \tag{3.51}$$

(v) there exists a compact subset Ψ of X such that $\bigcap\limits_{s\geq 0} S(s)(C) \subset \Psi$.

Then the sequence $\{\psi_n\}$ converges to a point in F.

Proof . Set $\delta_n = 0$ in Corollary 3.4. □

Corollary 3.6. Let C be a nonempty closed convex subset of a real Banach space X. Let $\Im = \{S(s) : s \geq 0\}$ be a uniformly $L(s)$-Lipschitzian total asymptotically demicontractive semigroup with a bounded measurable function $L : [0,\infty) \to [0,\infty)$ and a bounded function $\lambda : [0,\infty) \to [0,\infty)$ and sequences $\{\mu_n\}, \{\xi_n\} \in [0,\infty)$ with $\mu_n \to 0$ and $\xi_n \to 0$ as $n \to \infty$, such that

$$L = \sup_{s\geq 0} L(s) < \infty,\ \lambda = \inf_{s\geq 0} \lambda(s) > 0,\ F = \bigcap_{s\geq 0} F(S(s)) \neq \emptyset.$$

Assume that there exist M and M^* which are constants such that $\phi(\psi) \leq M^*\psi^2$, for all $\psi \geq M$. Let $\{\psi_n\}$ be the sequence generated by

$$\begin{cases} \psi_0 \in C, \\ \psi_n = (1-\delta_n)\psi_{n-1} + \delta_n S^n(s_n)\psi_n, \end{cases} \quad \forall n \geq 1, \tag{3.52}$$

where $\{\delta_n\}$ is a real sequence in $[0,1]$ and $\{s_n\}$ is an increasing sequence in $[0,\infty)$. If the following conditions are satisfied:

(i) $\sum_{n=1}^{\infty} \delta_n^2 < \infty$;

(ii) $\sum_{n=1}^{\infty} \delta_n = \infty$;

(iii) $\sum_{n=1}^{\infty} \delta_n\mu_n < \infty,\ \sum_{n=1}^{\infty} \delta_n\xi_n < \infty$;

(iv) $\delta_n L < 1$;

(v) assume for any subset $D \in C$,

$$\lim_{n\to\infty} \sup_{\psi\in D, t\in\Re^+} \|S^n(t+s_n)\psi - S^n(s_n)\psi\| = 0; \tag{3.53}$$

(vi) there exists a compact subset Ψ of X such that $\bigcap_{s\geq 0} S(s)(C) \subset \Psi$.

Then the sequence $\{\psi_n\}$ converges to a point in F.

Proof . Set $\gamma_n = 0$ in Corollary 3.4. □

4 conclusion

In this article, we have seen that the class of total asymptotically demicontractive semigroup is more general than all of the classes of nonexpansive semigroup, strictly pseudocontractive semigroup, demicontractive semigroup, asymptotically strictly pseudocontractive semigroup, asymptotically demicontractive semigroup and total asymptotically strictly pseudocontractive semigroup. Also, owing the fact that our new iteration process (1.15) properly includes the iteration processes (1.10)-(1.13) which has been considered by Chang et al. [3], Thong [13]–[14], Yang and Zhao [19], Yang et al. [18], Zhang [24]-[25], it follows that their results are special cases of our results. Hence, our results generalize, extend, complement and improve the corresponding results of Chang et al. [3], Thong [13]–[14], Yang and Zhao [19], Yang et al. [18], Zhang [24]-[25] and several other results in these directions.

Acknowledgement

The authors are grateful to the reviewers for their useful suggestions and comments that helped in improving this paper.

References

[1] A. Aleyner and S. Reich, *An explicit construction of sunny nonexpansive retractions in Banach spaces*, Fixed Point Theory Appl. **3** (2005), 295–305.

[2] S. S. Chang, C. K. Chan, H. W. Joseph Lee and L. Yang, *A system of mixed equilibrium problems, fixed point problems of strictly pseudocontractive mappings and nonexpansive semigroups*, Appl. Math. Comput. **216** (2010), no. 1, 51–60.

[3] S.S. Chang, Y.J. Cho, H.W. Joseph Lee and C.K. Chan, *Strong convergence theorems for Lipschitzian demicontraction semigroups in Banach spaces*, Fixed Point Theory Appl. **2011** (2011), doi:10.1155/2011/583423.

[4] K. Deimling, *Zeros of accretive operators*, Manuscripta Math. **13** (1974), 365–374.

[5] S. Li, L. H. Li and F. Su, *General iteration methods for a one-parameter nonexpansive semigroups in Hilbert spaces*, Nonlinear Anal. **70** (2009), no. 9, 3065–3071.

[6] G.E. Kim, *Approximating common fixed points of Lipschitzian pseudocontraction semigroups*, J. Fixed Point Theory Appl. **2016** (2016), doi: 10.1007/s11784-016-0299-7.

[7] L. Maruster and Ş. Maruster, *Strong convergence of the Mann iteration for α-demicontractive mappings*, Math. Comput. Modell. **54** (2011), 2486–2492.

[8] M.O. Osilike and A.C. Onah, *Strong convergence of the Ishikawa iteration for Lipschitz α-hemicontractive mappings*, Ann. West Univ. Timisoara-Math. Comput. Sci. **53** (2015), no. 1, 151–161.

[9] J. Quan, S. Chang and M. Liu, *Strong and weak convergence of an implicit iterative process for pseudocontractive semigroups in Banach space*, Fixed Point Theory and Appl. **2012** (2012), 16.

[10] N. Shioji and W. Takahashi, *Strong convergence theorems for asymptotically nonexpansive mappings in Hilbert spaces*, Nonlinear Anal. **34** (1998), no. 1, 87–99.

[11] T. Suzuki, *On strong convergence to a common fixed point of nonexpansive semigroups in Hilbert spaces*, Proc. Amer. Math Soc. **131** (2003), no. 7, 2133–2136.

[12] T. Suzuki, *Fixed point property for nonexpansive mappings versus that for nonexpansive semigroups*, Nonlinear Anal. **70** (2009), 3358–3361.

[13] D. V. Thong, *An implicit iteration process for nonexpansive semigroups*, Nonlinear Anal. **74** (2011), 6116–6120.

[14] D. V. Thong, *On Mann type implicit iteration process for strictly pseudocontraction semigroups*, Ann. Univ. Craiova Math. Comput. Sci. Ser. **38** (2011), no. 3, 101–108.

[15] Y. Wang and C. Wang, *Convergence of a new modified Ishikawa type iteration for common fixed Points of total asymptotically strict pseudocontractive semigroups*, Abstr. Appl. Anal. **2013** (2013), Article ID 319241, 7 pages.

[16] H.K. Xu, *Strong convergence theorem for contraction semigroups in Banach spaces*, Bull. Aust. Math. Soc. **72**(2005), no. 3, 371–379.

[17] H.K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal. **16** (1991), no. 12, 1127–1138.

[18] L. Yang, S. Chang and F.H. Zhao, *Strong convergence theorems for a finite family of total asymptotically strict pseudocontractive semigroups in Banach spaces*, Fixed Point Theory Appl. **2013** (2013), 178.

[19] L. Yang and F.H. Zhao, *Large strong convergence theorems for total asymptotically strict pseudocontractive semigroup in banach spaces*, Fixed Point Theory Appl. **2012** (2012), 24.

[20] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal. **16** (1991), 1127–1138.

[21] S. S. Zhang, L. Yang, H. W. Joseph Lee and C. K. Chan, *Strong convergence theorem for nonexpansive semigroups in Hilbert spaces*, Acta Math. Sinica **52** (2009), no. 2, 337–342.

[22] S.S. Zhang, L. Yang and J.A. Liu, *Strong convergence theorem for nonexpansive semigroups in Banach spaces*, Appl. Math. Mech. **28** (2007), no. 10, 1287–1297.

[23] S. S. Zhang, *Convergence theorem of common fixed points for Lipshitzian pseudocontraction semigroups in Banach spaces*, Appl. Math. Mech. **30** (2009), 145–152.

[24] S.S. Zhang, *Convergence theorem of common fixed points for Lipschitzian pseudo-contraction semi-groups in Banach spaces*, Appl. Math. Mech. Engl. Ed. **30** (2009), 145–152.

[25] S.S. Zhang, *Weak convergence theorem for Lipschitzian pseudocontraction semigroups in Banach spaces*, Acta Math. Sinica Eingl. Ser. **26** (2010), 337–344.