Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 1643–1648
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2020.20812.2202

Interior inverse problems for discontinuous differential pencils with spectral boundary conditions

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(Communicated by Vicenc Torra)

Abstract

In this work, we investigate the inverse problem for differential pencils with spectral boundary conditions having jump conditions on $(0, 1)$. Taking the Weyl function technique, we prove a uniqueness theorem from the interior spectral data.

Keywords: Interior spectral data, Differential pencil, Boundary condition dependent on the spectrum, Discontinuity, Weyl function
2020 MSC: 34A36, 34B08, 34B20

1 Introduction

Consider the boundary value problem $L = L(p, q, h_0, h_1, H_0, H_1, \alpha, \beta)$ defined by the differential pencil

$$y''(x) + (\rho^2 - 2\rho p(x) - q(x))y(x) = 0, \quad x \in (0, 1), \tag{1.1}$$

subject to boundary conditions

$$U(y) := y'(0) - (h_1\rho + h_0)y(0) = 0, \quad V(y) := y'(1) + (H_1\rho + H_0)y(1) = 0, \tag{1.2}$$

and jump conditions

$$y\left(\frac{1}{2} + 0, \rho\right) = \alpha y\left(\frac{1}{2} - 0, \rho\right), \quad y'\left(\frac{1}{2} + 0, \rho\right) = \alpha^{-1} y'\left(\frac{1}{2} - 0, \rho\right) + \beta y\left(\frac{1}{2} - 0, \rho\right), \tag{1.3}$$

and the boundary value problem $\widetilde{L} := L(\widetilde{p}, \widetilde{q}, \widetilde{h}_0, \widetilde{h}_1, \widetilde{H}_0, \widetilde{H}_1, \alpha, \beta)$ defined by the differential pencil

$$y''(x) + (\rho^2 - 2\rho\widetilde{p}(x) - \widetilde{q}(x))y(x) = 0, \quad x \in (0, 1), \tag{1.4}$$

with boundary conditions

$$\widetilde{U}(y) := y'(0) - (\widetilde{h}_1\rho + \widetilde{h}_0)y(0) = 0, \quad \widetilde{V}(y) := y'(1) + (\widetilde{H}_1\rho + \widetilde{H}_0)y(1) = 0, \tag{1.5}$$

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Received: July 2020 *Accepted:* November 2020

and jump conditions (1.3). The potentials $p(x), q(x), \widetilde{p}(x)$ and $\widetilde{q}(x)$ have complex values, and $p, \widetilde{p} \in W_2^1(0,1)$ and $q, \widetilde{q} \in W_2^0(0,1)$. The parameters $h_0,\ h_1,\ H_0,\ H_1,\ \widetilde{h}_0,\ \widetilde{h}_1,\ \widetilde{H}_0,\ \widetilde{H}_1$, α and β have complex values and the parameter ρ is spectral. Furthermore the coefficients $h_1, \widetilde{h}_1, H_1, \widetilde{H}_1 \neq \pm i$ and $\alpha \neq 0$.

Inverse spectral problems have many applications in various branches of sciences such as mathematics, physics, engineering, etc [10, 11, 18, 21]. Inverse problems for differential operators consist in reconstructing operators and boundary conditions from their spectral data. Direct and inverse problems for differential equations without discontinuity have been investigated in [3, 5, 7, 9, 22]. Discontinuous inverse problems for differential operators have also been studied in some papers [8, 13, 15, 16]. Interior inverse problems have first been established by Mochizuki and Trooshin [12] and the function $q(x)$ for the Sturm-Liouville operator was determined by a set of values on eigenfunctions at some internal points and some spectra. Later, Wang gave some uniqueness theorems for Sturm-Liouville equations by the Mochizuki-Trooshin type theorem using properties of the eigenfunctions and eigenvalues and the Weyl function technique [16, 17, 18, 19]. By using the method of spectral mappings, many researchers have also studied the inverse problem for Sturm-Liouville equations from the Weyl function [7, 8, 21, 22]. As far as we know, interior inverse problems for L taking the Weyl function have not been studied yet. Therefore, we consider the inverse problem for discontinuous differential pencils with boundary conditions dependent on the spectrum on $(0,1)$. By extending the Mochizuki-Trooshin type theorem and the result of Ref. [2], we show that the potentials $p(x)$, $q(x)$ and the boundary conditions are uniquely established using one spectrum and a set of values of eigenfunctions in an interior point $x = \frac{1}{2}$.

This paper is organized as follows. In Sec. 2, we present some preliminaries for L. In Sec. 3, we establish the main result of this article. The technique used has been based on the Mochizuki-Trooshin type theorem and the Weyl function technique which is a combined method in the inverse problem theory.

2 Preliminaries

Consider $\varphi(x,\rho)$ and $\psi(x,\rho)$ as the solution of (1.1) under the initial conditions

$$\varphi(0,\rho) = \psi(1,\rho) = 1,\ \varphi'(0,\rho) = h_1\rho + h_0,\ \psi'(1,\rho) = -H_1\rho - H_0. \tag{2.1}$$

For any fixed x, the functions $\varphi^{(v)}(x,\rho)$ and $\psi^{(v)}(x,\rho),\ v = 0,1$ are entire in ρ. From [1, 14, 20], we have the following formulae for sufficiently large ρ,

$$\varphi(x,\rho) = \sqrt{1+h_1^2}\cos(\kappa_0 - (\rho x - \mathcal{P}(x))) + O\left(\frac{1}{\rho}\exp(|\Im\rho|x)\right), \qquad x < \frac{1}{2}, \tag{2.2}$$

$$\begin{aligned}\varphi(x,\rho) \quad = \quad & \sqrt{1+h_1^2}\big(\alpha^+\cos(\kappa_0 - (\rho x - \mathcal{P}(x))) + \alpha^-\cos(\kappa_0 - (\rho(1-x) + \mathcal{P}(x) - \mathcal{P}(1)))\big) \\ & + O\left(\frac{1}{\rho}\exp(|\Im\rho|x)\right), \qquad x > \frac{1}{2},\end{aligned} \tag{2.3}$$

where $\mathcal{P}(x) = \int_0^x p(t)dt,\ \ \kappa_0 = \frac{1}{2i}ln\frac{i-h_1}{i+h_1}$ and $\alpha^{\pm} = \frac{1}{2}(\alpha \pm \alpha^{-1})$.

Denote $\Delta(\rho) = -V(\varphi) = U(\psi)$, and is called the characteristic function of L. The roots of this entire function coincide with the eigenvalues of L [6]. Now taking (2.3), we can get the asymptotic formula for the characteristic function of the following form as large enough ρ,

$$\Delta(\rho) = \Delta_0(\rho) + O\left(\exp\left(|\Im\rho|\right)\right), \tag{2.4}$$

where

$$\Delta_0(\rho) = \rho\sqrt{(1+h_1^2)(1+H_1^2)}\big(\alpha^+\sin(\rho - \mathcal{P}(1) - (\kappa_0+\kappa_1)) + \alpha^-\sin(\kappa_0 - \kappa_1)\big),$$

in which $\kappa_1 = \frac{1}{2i}ln\frac{i-H_1}{i+H_1}$.

Here we recall the following lemma which helps us to give the eigenvalues.

Lemma 2.1. [23] Let $\{\alpha_i\}_{i=1}^p$ be the set of real numbers satisfying the inequalities $\alpha_0 > \alpha_1 > ... > \alpha_{p-1} > 0$ and $\{\beta_i\}_{i=1}^p$ be the set of complex numbers. If $\beta_p \neq 0$ then the roots of the equation $e^{\alpha_0\lambda} + \beta_1 e^{\alpha_1\lambda} + ... + \beta_{p-1}e^{\alpha_{p-1}\lambda} + \beta_p = 0$ have the form $\lambda_n = \frac{2n\pi i}{\alpha_0} + h(n)$ for any n, where $h(n)$ is a bounded sequence.

Consider ρ_n as zeros of the characteristic function $\Delta(\rho)$. By the well known method in [6], it is trivial that $|\Delta_0(\rho)| \geq |\Delta(\rho) - \Delta_0(\rho)|$. So according to the Rouche theorem [4] and Lemma 2.1, we can see the following roots as sufficiently large n,

$$\rho_n = n\pi + \kappa_1 + \kappa_0 + \mathcal{P}(1) + O(n^{-1}). \tag{2.5}$$

Set $G_\delta := \{\rho; \mid \rho - \rho_n \mid \geq \delta, \ \forall n\}$, for a fixed small $\delta > 0$. By taking the known method [6], we hold the following estimate for large enough ρ,

$$|\Delta(\rho)| \geq C_\delta |\rho| \exp(|\Im \rho|), \tag{2.6}$$

where C_δ is a positive constant. Put the meromorphic function $M(\rho)$ of the form

$$M(\rho) = \frac{\psi(0,\rho)}{\Delta(\rho)}, \tag{2.7}$$

which is called the Weyl function of L. This function is a main tool to solve the inverse problem.

By virtue of Ref. [2], we have the following lemma which is important to prove the uniqueness theorem.

Lemma 2.2. Let $M(\rho)$ be the Weyl function of the boundary value problem (1.1)-(1.3) and $\widetilde{M}(\rho)$ be the Weyl function of the same boundary value problem with tilde. If $M(\rho) = \widetilde{M}(\rho)$, then $p(x) = \widetilde{p}(x)$ and $q(x) = \widetilde{q}(x)$ a.e. on $(0,1)$, and $h_0 = \widetilde{h}_0, \ h_1 = \widetilde{h}_1, \ H_0 = \widetilde{H}_0$ and $H_1 = \widetilde{H}_1$. In other words, the Weyl function $M(\rho)$ uniquely determines the boundary conditions as well as the potentials $p(x)$ and $q(x)$ a.e. on $(0,1)$.

3 Main result

In this section, we state the uniqueness theorem and prove it by taking the Mochizuki-Trooshin type theorem and the Weyl function technique. We note that ρ_n and $y_n(x,\rho)$ are the eigenvalues and the corresponding eigenfunctions of the boundary value problem L, respectively.

Theorem 3.1. If coefficients h_0 and h_1 of the first boundary condition are prescribed a priori and for each n,

$$\rho_n = \widetilde{\rho}_n, \quad < y_n, \widetilde{y}_n >_{x=\frac{1}{2}} = 0.$$

Then $p(x) = \widetilde{p}(x)$ and $q(x) = \widetilde{q}(x)$ a.e. on $(0,1)$ and

$$H_0 = \widetilde{H}_0, \ H_1 = \widetilde{H}_1.$$

Proof . Consider $y(x,\rho)$ as the solution of the equation (1.1) satisfying the initial conditions $y(1,\rho) = 1$ and $y'(1,\rho) = -H_1\rho - H_0$ and also $\widetilde{y}(x,\rho)$ as the solution to the equation (1.4) under the initial conditions $\widetilde{y}(1,\rho) = 1$ and $\widetilde{y}'(1,\rho) = -\widetilde{H}_1\rho - \widetilde{H}_0$. Multiplying (1.1) by $\widetilde{y}(x,\rho)$ and (1.4) by $y(x,\rho)$, and subtracting, we infer that

$$\big(2\rho P(x) + Q(x)\big) y(x,\rho)\widetilde{y}(x,\rho) = y''(x,\rho)\widetilde{y}(x,\rho) - y(x,\rho)\widetilde{y}''(x,\rho),$$

where $P(x) = p(x) - \widetilde{p}(x)$ and $Q(x) = q(x) - \widetilde{q}(x)$. Integrating the above relation on $\left(\frac{1}{2}, 1\right)$, one gets

$$\int_{\frac{1}{2}}^{1} \big(2\rho P(x) + Q(x)\big) y(x,\rho)\widetilde{y}(x,\rho)dx = (y'(x,\rho)\widetilde{y}(x,\rho) - y(x,\rho)\widetilde{y}'(x,\rho))|_{\frac{1}{2}}^{1}.$$

Taking the initial conditions at $x = 1$, we have

$$\int_{\frac{1}{2}}^{1} \big(2\rho P(x) + Q(x)\big) y(x,\rho)\widetilde{y}(x,\rho)dx = (\widetilde{H}_1 - H_1)\rho + (\widetilde{H}_0 - H_0) - G\left(\frac{1}{2}, \rho\right),$$

where

$$G(x,\rho) = y'(x,\rho)\widetilde{y}(x,\rho) - y(x,\rho)\widetilde{y}'(x,\rho).$$

Therefore

$$G\left(\frac{1}{2},\rho\right)=(\widetilde{H}_1-H_1)\rho+(\widetilde{H}_0-H_0)-\int_{\frac{1}{2}}^{1}\big(2\rho P(x)+Q(x)\big)y(x,\rho)\widetilde{y}(x,\rho)dx. \tag{3.1}$$

Using the hypothesis of the theorem, it is clear that $G\left(\frac{1}{2},\rho_n\right)=0$. It is sufficient to show that $G\left(\frac{1}{2},\rho\right)=0$ for $\rho\neq\rho_n$.
From [17], we hold for enough large ρ and $x>\frac{1}{2}$,

$$y(x,\rho)=\sqrt{1+H_1^2}\cos(\rho(1-x)+\mathcal{P}(x)-\mathcal{P}(1)-\kappa_1)+O\left(\frac{1}{\rho}\exp(|\Im\rho|(1-x))\right). \tag{3.2}$$

Because

$$|cos(\rho(1-x))|\le\exp(|\Im\rho|(1-x)),\quad |sin(\rho(1-x))|\le\exp(|\Im\rho|(1-x)),$$

the formula (3.2) results that

$$|y(x,\rho)\widetilde{y}(x,\rho)|\le C_0\exp(2|\Im\rho|(1-x)),\qquad x>\frac{1}{2}, \tag{3.3}$$

for a constant $C_0>0$. Therefore for enough large ρ,

$$\left|G\left(\frac{1}{2},\rho\right)\right|\le(C_1|\rho|+C_2)\exp(|\Im\rho|), \tag{3.4}$$

for constants $C_1,C_2>0$. Put the meromorphic function

$$\phi(\rho):=\frac{G\left(\frac{1}{2},\rho\right)}{\Delta(\rho)}. \tag{3.5}$$

Together with (2.6) and (3.4), this yields that $\phi(\rho)=O(1)$. From this and Liouville's theorem [4], we give that $\phi(\rho)=C$ for all ρ. To get that $C=0$, we rewrite (3.5) as $G\left(\frac{1}{2},\rho\right)=C\Delta(\rho)$. So

$$\begin{aligned}&(\widetilde{H}_1-H_1)\rho+(\widetilde{H}_0-H_0)-\int_{\frac{1}{2}}^{1}\big(2\rho P(x)+Q(x)\big)y(x,\rho)\widetilde{y}(x,\rho)dx\\&=C\rho\sqrt{(1+h_1^2)(1+H_1^2)}\big(\alpha^+\sin(\rho-\mathcal{P}(1)-(\kappa_0+\kappa_1))+\alpha^-\sin(\kappa_0-\kappa_1)\big)\\&\qquad+O\left(\exp\left(|\Im\rho|\right)\right).\end{aligned}$$

That is

$$\begin{aligned}&(\widetilde{H}_1-H_1)+\frac{1}{\rho}(\widetilde{H}_0-H_0)-\int_{\frac{1}{2}}^{1}\big(2P(x)+\frac{1}{\rho}Q(x)\big)y(x,\rho)\widetilde{y}(x,\rho)dx\\&=C\sqrt{(1+h_1^2)(1+H_1^2)}\big(\alpha^+\sin(\rho-\mathcal{P}(1)-(\kappa_0+\kappa_1))+\alpha^-\sin(\kappa_0-\kappa_1)\big)\\&\qquad+O\left(\frac{1}{\rho}\exp\left(|\Im\rho|\right)\right).\end{aligned}$$

According to the Riemann-Lebesgue Lemma, since the limit of the left side of the above equality exists for large enough ρ, we can result that $C=0$. So, it proves that $G\left(\frac{1}{2},\rho\right)=0$ for all ρ.

To complete the proof, we should consider the supplementary problem $\widehat{L}:=L(p_1,q_1,H_0,H_1,h_0,h_1,\alpha,\beta)$ for the differential pencil

$$\begin{aligned}&y''(x)+(\rho^2-2\rho p_1(x)-q_1(x))y(x)=0,\quad x\in(0,1),\\&\quad p_1(x)=p(1-x),\quad q_1(x)=q(1-x),\end{aligned} \tag{3.6}$$

with boundary conditions

$$\widehat{U}(y) := y'(0) - (H_1\rho + H_0)y(0) = 0, \quad \widehat{V}(y) := y'(1) + (h_1\rho + h_0)y(1) = 0, \tag{3.7}$$

and jump conditions

$$y\left(\frac{1}{2}+0,\rho\right) = \alpha^{-1}y\left(\frac{1}{2}-0,\rho\right), \quad y'\left(\frac{1}{2}+0,\rho\right) = \alpha y'\left(\frac{1}{2}-0,\rho\right) - \beta y\left(\frac{1}{2}-0,\rho\right). \tag{3.8}$$

By straightforward computations, we can show $\widehat{y}(x) := y(1-x)$ as the solution of the supplementary problem $\widehat{L}$. We also take the problem $\widetilde{\widehat{L}} := L(\widetilde{p}_1, \widetilde{q}_1, \widetilde{H}_0, \widetilde{H}_1, \widetilde{h}_0, \widetilde{h}_1, \alpha, \beta)$ for the differential pencil

$$y''(x) + (\rho^2 - 2\rho\widetilde{p}_1(x) - \widetilde{q}_1(x))y(x) = 0, \quad x \in (0,1), \tag{3.9}$$

with boundary conditions

$$\widetilde{\widehat{U}}(y) := y'(0) - (\widetilde{H}_1\rho + \widetilde{H}_0)y(0) = 0, \quad \widetilde{\widehat{V}}(y) := y'(1) + (\widetilde{h}_1\rho + \widetilde{h}_0)y(1) = 0, \tag{3.10}$$

and jump conditions (3.8). By repeating the earlier argument to the supplementary problem $\widehat{L}$, we have

$$\int_{\frac{1}{2}}^{1} \big(2\rho P(x) + Q(x)\big)\widehat{y}(x,\rho)\widetilde{\widehat{y}}(x,\rho)dx = (\widehat{y}'(x,\rho)\widetilde{\widehat{y}}(x,\rho) - \widehat{y}(x,\rho)\widetilde{\widehat{y}}'(x,\rho))|_{\frac{1}{2}}^{1}.$$

So

$$\int_{\frac{1}{2}}^{1} \big(2\rho P(x) + Q(x)\big)\widehat{y}(x,\rho)\widetilde{\widehat{y}}(x,\rho)dx = \widehat{G}(1,\rho) - \widehat{G}\left(\frac{1}{2},\rho\right),$$

where

$$\widehat{G}(x,\rho) := \widehat{y}'(x,\rho)\widetilde{\widehat{y}}(x,\rho) - \widehat{y}(x,\rho)\widetilde{\widehat{y}}'(x,\rho).$$

Because of $\widehat{G}(x,\rho) = -G(1-x,\rho)$, we can obtain that $\widehat{G}\left(\frac{1}{2},\rho\right) = -G\left(\frac{1}{2},\rho\right) = 0$. Therefore

$$\widehat{G}(1,\rho) = \int_{\frac{1}{2}}^{1} \big(2\rho P(x) + Q(x)\big)\widehat{y}(x,\rho)\widetilde{\widehat{y}}(x,\rho)dx. \tag{3.11}$$

Moreover we will have $\widehat{G}(1,\rho_n) = -G(0,\rho_n) = 0$, from the hypothesis of the theorem. With the same argument in the boundary value problem L, it proves that $G(0,\rho) = -\widehat{G}(1,\rho) = 0$ for all ρ.

Thus from this result, we will have $M(\rho) = \widetilde{M}(\rho)$. Together with Lemma 2.2, this equality gives that $p(x) = \widetilde{p}(x)$, $q(x) = \widetilde{q}(x)$ a.e. on $(0,1)$, and $H_0 = \widetilde{H}_0$, $H_1 = \widetilde{H}_1$. The proof is completed. □

Acknowledgment

This research work has been supported by a research grant from the University of Mazandaran.

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