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Inequalities for a class of rational functions

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Abstract

In this paper, we consider a more general class of rational functions $r(s(z))$, of degree mn, with $s(z)$ being a polynomial of degree m. Our results not only generalize the results due to Wali and Shah [JOA, **25** (2017), no. 1, 43-53] but also improve the results obtained by Qasim and Liman [Indian. J. Pure Appl. Math. **46** (2015), no. 3, 337-348] and Mir [Indian J. Pure Appl. Math. **50** (2019), no. 2, 315-331].

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1 Introduction

For a positive integer n, let $\mathcal{P}_n$ be the linear space of all polynomials $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree atmost n over the field $\mathbb{C}$ of complex numbers and P' be its derivative. For any positive real number k, we denote

$$D_k^- = \{z \in \mathbb{C} : |z| < k\}$$
$$D_k^+ = \{z \in \mathbb{C} : |z| > k\}$$
$$T_k = \{z \in \mathbb{C} : |z| = k\}.$$

Let

$$\mathcal{R}_n = \mathcal{R}_n(z_1, z_2, ..., z_n) := \left\{ \frac{P(z)}{w(z)} : P \in \mathcal{P}_n \right\},$$

where

$$w(z) := \prod_{j=1}^{n} (z - z_j), \quad z_j \in D_k^+, j = 1, 2, ..., n.$$

Thus $\mathcal{R}_n$ is the set of all rational functions with poles $z_1, z_2, ..., z_n$ and with finite limit at ∞. Throughout this paper, we shall assume that all poles $z_1, z_2, ..., z_n$ lie in D_k^+. We observe that the Blaschke product $B(z) \in \mathcal{R}_n$, where

$$B(z) := \prod_{j=1}^{n} \left(\frac{1 - \overline{z_j} z}{z - z_j} \right) = \frac{w^*(z)}{w(z)},$$

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with $w^*(z) = z^n\overline{w\left(\frac{1}{\bar{z}}\right)} = \prod_{j=1}^{n}(1-\overline{z_j}z)$ and satisfying $|B(z)| = 1$ for $z \in T_1$.

Li, Mohapatra and Rodriguez [10] obtained the following Bernstein-type inequalities for rational functions $r \in \mathcal{R}_n$ with prescribed poles $z_1, z_2, ..., z_n$.

Theorem 1.1. Suppose that $r \in \mathcal{R}_n$ has exactly n zeros and all lie in $T_1 \cup D_1^+$. Then for $z \in T_1$

$$|r'(z)| \leq \frac{|B'(z)|}{2}|r(z)|. \tag{1.1}$$

In the same paper they also proved the following.

Theorem 1.2. Suppose $r \in \mathcal{R}_n$ has exactly n zeros and all lie in $T_1 \cup D_1^-$. Then for $z \in T_1$

$$|r'(z)| \geq \frac{|B'(z)|}{2}|r(z)|. \tag{1.2}$$

Wali and Shah [9] improved Theorem 1.2 by proving the following result.

Theorem 1.3. If $r \in \mathcal{R}_n$ and all zeros of r lie in $T_1 \cup D_1^-$, then for $z \in T_1$

$$|r'(z)| \geq \frac{1}{2}\left\{|B'(z)| + \frac{|a_n| - |a_0|}{|a_n| + |a_0|}\right\}|r(z)|.$$

2 Composition of Rational Functions

Let $r \in \mathcal{R}_n$ and $s \in \mathcal{P}_m$, then their composition $r \circ s \in \mathcal{R}_{mn}$ is defined as $(r \circ s)z = r(s(z))$. Here

$$r(s(z)) = \frac{P(s(z))}{w(s(z))},$$

where $P(s(z))$ denote the composition of polynomials P and s, and

$$w(s(z)) = \prod_{j=1}^{mn}(z - z_j).$$

Also the Blaschke product in this case is defined as

$$B(s(z)) = \frac{w^*(s(z))}{w(s(z))} = z^{mn}\frac{\overline{w(s(\frac{1}{\bar{z}}))}}{w(s(z))} = \prod_{j=1}^{mn}\frac{(1-\overline{z_j}z)}{z - z_j}.$$

Recently, Qasim and Liman [5] considered this class of rational functions and proved several Bernstein type inequalities for rational functions. Among other things they proved.

Theorem 2.1. Let $r \circ s \in \mathcal{R}_{mn}$ and if $t_1, t_2, ..., t_n$ are zeros of $B(s(z)) + \lambda$ and $s_1, s_2, ..., s_n$ are zeros of $B(s(z)) - \lambda, \lambda \in T_1$, then for $z \in T_1$

$$\left|r'(s(z))\right| \leq \frac{1}{2mm'}\left|\left(B(s(z))\right)'\right|\left[\left(\max_{1\leq k\leq mn}|r(s(t_k))|\right) + \left(\max_{1\leq k\leq mn}|r(s(\mu_k))|\right)\right],$$

where $m' = \min_{z\in T_1}|s(z)|$.

If $P(z)=\sum_{j=0}^{n} a_j z^j$ and $s(z)=\sum_{j=0}^{m} b_j z^j$, then $P\circ s\in\mathcal{P}_{mn}$ and

$$\begin{aligned}(P\circ s)(z) &= P(s(z))\\ &= a_n\big(b_m z^m + b_{m-1}z^{m-1}+...+b_0\big)^n +\\ & a_{n-1}\big(b_m z^m + b_{m-1}z^{m-1}+...+b_0\big)^{n-1} + ... + a_0\\ &= a_n\left[\binom{n}{0}b_m^n z^{mn} + \left(\binom{n}{1}b_m^{n-1}b_{m-1}\right)z^{mn-1} + ... + b_0^n\right] + ... + a_1 b_0 + a_0\\ &= \sum_{j=0}^{mn} c_j z^j, \quad \text{where} \quad c_0=\sum_{j=0}^{n} a_j b_0^j, ..., c_{mn} = a_n b_m^n.\end{aligned}$$

3 Lemmas and Proofs of Lemmas

Lemma 3.1. Let $f : D \to D$ be holomorphic. Suppose $f(0)=0$ and also assume there exists $b\in\delta D$, such that f extends continuously to $b, |f(b)|=1$ and $f'(b)$ exists, then

$$|f'(b)| \geq \frac{2}{1+|f'(0)|}.$$

The above lemma is due to Osserman [6].

Lemma 3.2. Let $r\circ s\in\mathcal{R}_{mn}$ be such that all the zeros of $r(s(z))$ lie in $T_1\cup D_1^-$, then for some $z\in T_1$

$$Re\left\{\frac{z\big(r(s(z))\big)'}{r(s(z))}\right\} \geq \frac{1}{2}\left\{\left|\big(B(s(z))\big)'\right| + \frac{|a_n b_m^n| - \left|\sum_{j=0}^{n} a_j b_0^j\right|}{|a_n b_m^n| + \left|\sum_{j=0}^{n} a_j b_0^j\right|}\right\}.$$

Proof . Since all zeros of $r(s(z))$ lie in $T_1\cup D_1^-$, and

$$r(s(z)) = \frac{P(s(z))}{w(s(z))}.$$

Therefore we have

$$\frac{z\big(r(s(z))\big)'}{r(s(z))} = \frac{z\big(P(s(z))\big)'}{P(s(z))} - \frac{z\big(w(s(z))\big)'}{w(s(z))}.$$

Equivalently

$$Re\left(\frac{z\big(r(s(z))\big)'}{r(s(z))}\right) = Re\left(\frac{z\big(P(s(z))\big)'}{P(s(z))}\right) - Re\left(\frac{z\big(w(s(z))\big)'}{w(s(z))}\right). \tag{3.1}$$

If $z_1, z_2, ...z_{mn}$ are the zeros of $P(s(z))$, then

$$P(s(z)) = c_{mn}\prod_{j=1}^{mn}(z-z_j), z_j\in D_1^-,$$

where $c_{mn}=a_n b_m^n$.
Therefore, $Q(s(z)) = z^{mn}\overline{P\big(s\big(\frac{1}{\overline{z}}\big)\big)}$, has all zeros in D_1^+.
Hence,

$$G(s(z)) = \frac{zP(s(z))}{Q(s(z))} = z\frac{c_{mn}}{\overline{c_{mn}}}\prod_{j=1}^{mn}\left(\frac{z-z_j}{1-\overline{z_j}z}\right)$$

is analytic in $T_1\cup D_1^-$, with $G(s(0))=0, |G(s(z))|=1$ for $z\in T_1$. Therefore applying Lemma 3.1 to $G(s(z))$, we have

$$\left|\big(G(s(z))\big)'\right| \geq \frac{2}{1+\left|\big(G(s(0))\big)'\right|}. \tag{3.2}$$

Now

$$G(s(z)) = \frac{zP(s(z))}{Q(s(z))} = \frac{zP(s(z))}{z^{mn}\overline{P\big(s(\frac{1}{\overline{z}})\big)}} = \frac{z^{-(mn-1)}P(s(z))}{\overline{P\big(s(\frac{1}{\overline{z}})\big)}}.$$

Therefore, for $z \in T_1$

$$\frac{z\big(G(s(z))\big)'}{G(s(z))} = -(mn-1) + 2Re\left(\frac{z\big(P(s(z))\big)'}{P(s(z))}\right). \tag{3.3}$$

Also

$$G(s(z)) = \frac{zP(s(z))}{Q(s(z))} = z\frac{c_{mn}}{\overline{c_{mn}}}\prod_{j=1}^{mn}\left(\frac{z-z_j}{1-\overline{z_j}z}\right).$$

This gives

$$\frac{z\big(G(s(z))\big)'}{G(s(z))} = 1 + \sum_{j=1}^{mn}\frac{1-|z_j|^2}{|z-z_j|^2} \geq 0.$$

Therefore for $z \in T_1$

$$\left|\frac{z\big(G(s(z))\big)'}{G(s(z))}\right| = \big|\big(G(s(z))\big)'\big|. \tag{3.4}$$

Also by vieta's formula

$$\big|\big(G(s(0))\big)'\big| = |\prod_{j=1}^{mn} z_j| = \frac{|\sum_{j=0}^{n} a_j b_0^j|}{|a_n b_m^n|}. \tag{3.5}$$

On combining 3.2, 3.3, 3.4 and 3.5, and noting $P(s(z)) \neq 0$ for $z \in T_1$, we get

$$-(mn-1) + 2Re\left(\frac{z\big(P(s(z))\big)'}{P(s(z))}\right) \geq \frac{2|a_n b_m^n|}{|a_n b_m^n| + |\sum_{j=0}^{n} a_j b_0^j|}.$$

Hence for $z \in T_1$,

$$Re\left(\frac{z\big(P(s(z))\big)'}{P(s(z))}\right) \geq \frac{mn-1}{2} + \frac{|a_n b_m^n|}{|a_n b_m^n| + |\sum_{j=0}^{n} a_j b_0^j|}. \tag{3.6}$$

Again since

$$B(s(z)) = \frac{w^*(s(z))}{w(s(z))},$$

therefore

$$\frac{z\big(B(s(z))\big)'}{B(s(z))} = \frac{z\big(w^*(s(z))\big)'}{w(s(z))} - \frac{z\big(w(s(z))\big)'}{w(s(z))}.$$

Now using the fact that for $z \in T_1$

$$\frac{z\big(B(s(z))\big)'}{B(s(z))} = \big|\big(B(s(z))\big)'\big|,$$

we get

$$Re\left(\frac{z\big(w^*(s(z))\big)'}{w^*(s(z))}\right) - Re\left(\frac{z\big(w(s(z))\big)'}{w(s(z))}\right) = \big|\big(B(s(z))\big)'\big|. \tag{3.7}$$

Also

$$\big(w^*(s(z))\big)' = mnz^{mn-1}\overline{w\big(s(\tfrac{1}{\overline{z}})\big)} - z^{mn-2}\overline{\big(w\big(s(\tfrac{1}{\overline{z}})\big)\big)'}.$$

Therefore it can be easily be verified that for $z \in T_1$

$$Re\left(\frac{z\big(w^*(s(z))\big)'}{w^*(s(z))}\right) = nm - Re\left(\frac{z\big(w(s(z))\big)'}{w(s(z))}\right). \tag{3.8}$$

From 3.7 and 3.8 ,we get

$$Re\left(\frac{z\big(w(s(z))\big)'}{w(s(z))}\right)=\frac{nm-|\big(B(s(z))\big)'|}{2}. \tag{3.9}$$

Using 3.8 and 3.9 in 3.1, we get

$$Re\left(\frac{z\big(r(s(z))\big)'}{r(s(z))}\right)\geq\frac{mn-1}{2}+\frac{|a_nb_m^n|}{|a_nb_m^n|+|\sum\limits_{j=0}^{n}a_jb_0^j|}-\frac{nm-|\big(B(s(z))\big)'|}{2}.$$

Equivalently

$$Re\left\{\frac{z\big(r(s(z))\big)'}{r(s(z))}\right\}\geq\frac{1}{2}\left\{\big|\big(B(s(z))\big)'\big|+\frac{|a_nb_m^n|-|\sum\limits_{j=0}^{n}a_jb_0^j|}{|a_nb_m^n|+|\sum\limits_{j=0}^{n}a_jb_0^j|}\right\}.$$

This completely proves the Lemma. □

Lemma 3.3. Let $r\circ s\in\mathcal{R}_{mn}$. if $t_1,t_2,...,t_n$ are zeros of $B(s(z))+\lambda$ and $s_1,s_2,...,s_n$ are zeros of $B(s(z))-\lambda,\lambda\in T_1$, then for $z\in T_1$

$$\big|\big(r(s(z))\big)'\big|^2+\big|\big(r^*(s(z))\big)'\big|^2\leq\frac{1}{2}\big|\big(B(s(z))\big)'\big|^2\left[\left(\max_{1\leq k\leq mn}|r(s(t_k))|\right)^2+\left(\max_{1\leq k\leq mn}|r(s(\mu_k))|\right)^2\right].$$

The above lemma is due to Mir [1].

Lemma 3.4. If $P(z)$ is a polynomial of degree n, having all zeros in $T_1\cup D_1^-$, then

$$\min_{z\in T_1}|P'(z)|\geq n\min_{z\in T_1}|P(z)|.$$

The inequality is sharp and equality holds for the polynomials having all zeros at origin.

The above Lemma is due to Aziz and Dawood [3].

4 Main Results

In this paper, we first prove the following.

Theorem 4.1. Let $r\circ s\in\mathcal{R}_{mn}$. If all the zeros of $r(s(z))$ lie in $T_1\cup D_1^-$, then for any $z\in T_1$

$$\big|r'(s(z))\big|\geq\frac{1}{2mM'}\left\{\big|\big(B(s(z))\big)'\big|+\frac{|a_nb_m^n|-|\sum\limits_{j=0}^{n}a_jb_0^j|}{|a_nb_m^n|+|\sum\limits_{j=0}^{n}a_jb_0^j|}\right\}|r(s(z))|,$$

where m denote the degree of $s(z)$ and $M'=\max\limits_{z\in T_1}|s(z)|$.
The result is sharp and equality holds for $r(s(z))=\alpha B(s(z))+\beta$, where $s(z)=z$ with $\alpha,\beta\in T_1$.

Proof . Since for $z\in T_1$, we have

$$\left|\frac{z\big(r(s(z))\big)'}{r(s(z))}\right|\geq Re\left\{\frac{z\big(r(s(z))\big)'}{r(s(z))}\right\}.$$

Therefore, we get by using Lemma 3.2, for $z \in T_1$

$$\big|\big(r(s(z))\big)'\big| \geq \frac{1}{2}\Bigg\{\big|\big(B(s(z))\big)'\big| + \frac{|a_n b_m^n| - \big|\sum\limits_{j=0}^{n} a_j b_0^j\big|}{|a_n b_m^n| + \big|\sum\limits_{j=0}^{n} a_j b_0^j\big|}\Bigg\}|r(s(z))|. \tag{4.1}$$

Now

$$\big|\big(r(s(z))\big)'\big| = |r'(s(z))s'(z)| \leq |r'(s(z))| \max_{z\in T_1} |s'(z)|.$$

This gives by using Bernstein's inequality [4],

$$\big|\big(r(s(z))\big)'\big| \leq |r'(s(z))| m \max_{z\in T_1} |s(z)|.$$

Equivalently

$$\big|\big(r(s(z))\big)'\big| \leq mM' |r'(s(z))|.$$

Therefore from 4.1, we get

$$\big|r'(s(z))\big| \geq \frac{1}{2mM'}\Bigg\{\big|\big(B(s(z))\big)'\big| + \frac{|a_n b_m^n| - \big|\sum\limits_{j=0}^{n} a_j b_0^j\big|}{|a_n b_m^n| + \big|\sum\limits_{j=0}^{n} a_j b_0^j\big|}\Bigg\}|r(s(z))|.$$

□

Remark 4.2. Since all the zeros of $r(s(z))$ lie in $T_1 \cup D_1^-$, we have $|a_n b_m^n| \geq |\sum_{j=0}^{n} a_j b_0^j|$, showing that Theorem 4.1 improves the result due to Qasim and Liman [5].

Remark 4.3. For $s(z) = z$, Theorem 4.1 reduces to a result due to Wali and Shah [9].

We next prove the following.

Theorem 4.4. Let $r \circ s \in \mathcal{R}_{mn}$, be such that all the zeros of $r(s(z))$ lie in $T_1 \cup D_1^+$, then for any $z \in T_1$

$$Re\Bigg\{\frac{z\big(r(s(z))\big)'}{r(s(z))}\Bigg\} \leq \frac{1}{2}\Bigg\{\big|\big(B(s(z))\big)'\big| - \frac{\big|\sum\limits_{j=0}^{n} a_j b_0^j\big| - |a_n b_m^n|}{|a_n b_m^n| + \big|\sum\limits_{j=0}^{n} a_j b_0^j\big|}\Bigg\}.$$

Proof . Since all zeros of $r(s(z))$ lie in $T_1 \cup D_1^+$, and

$$r(s(z)) = \frac{P(s(z))}{w(s(z))},$$

therefore,

$$Re\left(\frac{z\big(r(s(z))\big)'}{r(s(z))}\right) = Re\left(\frac{z\big(P(s(z))\big)'}{P(s(z))}\right) - Re\left(\frac{z\big(w(s(z))\big)'}{w(s(z))}\right). \tag{4.2}$$

If $z_1, z_2, ...z_{mm}$ are the zeros of $P(s(z))$, then

$$P(s(z)) = c_{mn} \prod_{j=1}^{mn} (z - z_j), z_j \in D_1^+,$$

where $c_{mn} = a_n b_m^n$.
Therefore, $Q(s(z)) = z^{mn}\overline{P\big(s(\frac{1}{\overline{z}})\big)}$ has all zeros in D_1^-.
Hence,

$$H(s(z)) = \frac{zQ(s(z))}{P(s(z))} = z\frac{\overline{c_{mn}}}{c_{mn}} \prod_{j=1}^{mn} \left(\frac{1 - \overline{z_j}z}{z - z_j}\right)$$

is analytic in $T_1 \cup D_1^-$, with $H(s(0)) = 0, |H(s(z))| = 1$ for $z \in T_1$. Thus applying Lemma 3.1 for $H(s(z))$, we have for $z \in T_1$

$$\left|\left(H(s(z))\right)'\right| \geq \frac{2}{1+\left|\left(H(s(0))\right)'\right|}. \tag{4.3}$$

Now

$$H(s(z)) = \frac{zQ(s(z))}{P(s(z))} = \frac{z^{mn+1}\overline{P\left(s\left(\frac{1}{\overline{z}}\right)\right)}}{P(s(z))}.$$

Therefore, for $z \in T_1$

$$\frac{z\left(H(s(z))\right)'}{H(s(z))} = (mn+1) - 2Re\left(\frac{z\left(P(s(z))\right)'}{P(s(z))}\right). \tag{4.4}$$

Also for $z \in D_1^+$,

$$H(s(z)) = \frac{zQ(s(z))}{P(s(z))} = z\frac{\overline{c_{mn}}}{c_{mn}}\prod_{j=1}^{mn}\left(\frac{1-\overline{z_j}z}{z-z_j}\right).$$

Therefore, it can easily be verified that for $z \in T_1$

$$\left|\frac{z\left(H(s(z))\right)'}{H(s(z))}\right| = \left|\left(H(s(z))\right)'\right|. \tag{4.5}$$

Also by vieta's formula

$$\left|\left(H(s(0))\right)'\right| = \frac{1}{|\prod_{j=1}^{mn} z_j|} = \frac{|a_n b_m^n|}{|\sum_{j=0}^{n} a_j b_0^j|}. \tag{4.6}$$

On combining 4.4, 4.5 and 4.6 and noting $P(s(z)) \neq 0$, for $z \in T_1$, we get

$$(mn+1) - 2Re\left(\frac{z\left(P(s(z))\right)'}{P(s(z))}\right) \geq \frac{2|\sum_{j=0}^{n} a_j b_0^j|}{|a_n b_m^n| + |\sum_{j=0}^{n} a_j b_0^j|}.$$

Hence for $z \in T_1$,

$$Re\left(\frac{z\left(P(s(z))\right)'}{P(s(z))}\right) \leq \frac{mn+1}{2} - \frac{|\sum_{j=0}^{n} a_j b_0^j|}{|a_n b_m^n| + |\sum_{j=0}^{n} a_j b_0^j|}. \tag{4.7}$$

Also by 3.9, we have

$$Re\left(\frac{z\left(w(s(z))\right)'}{w(s(z))}\right) = \frac{nm - |\left(B(s(z))\right)'|}{2}. \tag{4.8}$$

Using 4.7 and 4.8 in 4.2, we get

$$Re\left\{\frac{z\left(r(s(z))\right)'}{r(s(z))}\right\} \leq \frac{1}{2}\left\{\left|\left(B(s(z))\right)'\right| + \frac{|a_n b_m^n| - |\sum_{j=0}^{n} a_j b_0^j|}{|a_n b_m^n| + |\sum_{j=0}^{n} a_j b_0^j|}\right\}.$$

□

Remark 4.5. For $s(z) = z$, Theorem 4.4 reduces a result due to Wali and Shah [9].

We also prove the following improvement of a result due to Qasim and Liman [5] and hence a generalization of the result due to Wali and Shah [9].

Theorem 4.6. Let $r\circ s\in\mathcal{R}_{mn}$ be such that all the zeros of $r(s(z))$ lie in $T_1\cup D_1^+$. If $t_1,t_2,...,t_n$ are zeros of $B(s(z))+\lambda$ and $s_1,s_2,...,s_n$ are zeros of $B(s(z))-\lambda$, $\lambda\in T_1$, then for $z\in T_1$

$$\big|r'(s(z))\big|\le\frac{1}{2mm'}\big|\big(B(s(z))\big)'\big|\Bigg[\Bigg(\max_{1\le k\le mn}|r(s(t_k))|\Bigg)^2+\Bigg(\max_{1\le k\le mn}|r(s(\mu_k))|\Bigg)^2 - 2\frac{|\sum_{j=0}^{n}a_jb_0^j|-|a_nb_m^n|}{|\sum_{j=0}^{n}a_jb_0^j|+|a_nb_m^n|}\frac{|r(s(z))|^2}{\big|\big(B(s(z))\big)'\big|}\Bigg]^{\frac{1}{2}},$$

where $m'=\min_{z\in T_1}|s(z)|$.

The result is sharp and equality holds for $r(s(z))=B(s(z))+\lambda,\lambda\in T_1$, where $s(z)=z$.

Proof . We assume $r(s(z))\neq 0$ for $z\in T_1$. Now for $z\in T_1$, it can be verified that if

$$\big(r^*(s(z))\big)=B(s(z))\overline{r(s(\frac{1}{\bar{z}}))},$$

then

$$\big|\big(r^*(s(z))\big)'\big|=\Big|\big|\big(B(s(z))\big)'\big|r(s(z))-z\big(r(s(z))\big)'\Big|.$$

Hence for $z\in T_1$, with $r(s(z))\neq 0$, we get by using Theorem 4.4.

$$\begin{aligned}\left|\frac{z\big(r^*(s(z))\big)'}{r(s(z))}\right|^2&=\left|\big|\big(B(s(z))\big)'\big|-\frac{z\big(r(s(z))\big)'}{r(s(z))}\right|^2\\&=\big|\big(B(s(z))\big)'\big|^2+\left|\frac{z\big(r(s(z))\big)'}{r(s(z))}\right|^2-2\big|\big(B(s(z))\big)'\big|Re\left\{\frac{z\big(r(s(z))\big)'}{r(s(z))}\right\}\\&\ge\big|\big(B(s(z))\big)'\big|^2+\left|\frac{z\big(r(s(z))\big)'}{r(s(z))}\right|^2-\big|\big(B(s(z))\big)'\big|\left\{\big|\big(B(s(z))\big)'\big|-\frac{|\sum_{j=0}^{n}a_jb_0^j|-|a_nb_m^n|}{|\sum_{j=0}^{n}a_jb_0^j|+|a_nb_m^n|}\right\}\\&=\left|\frac{z\big(r(s(z))\big)'}{r(s(z))}\right|^2+\frac{|\sum_{j=0}^{n}a_jb_0^j|-|a_nb_m^n|}{|\sum_{j=0}^{n}a_jb_0^j|+|a_nb_m^n|}\big|\big(B(s(z))\big)'\big|.\end{aligned}$$

Equivalently for $z\in T_1$, we get

$$\big|\big(r(s(z))\big)'\big|^2+\frac{|\sum_{j=0}^{n}a_jb_0^j|-|a_nb_m^n|}{|\sum_{j=0}^{n}a_jb_0^j|+|a_nb_m^n|}\big|\big(B(s(z))\big)'\big|\big|\big(r(s(z))\big)\big|^2\le\big|\big(r^*(s(z))\big)'\big|^2.$$

Using Lemma 3.3 in the above inequality, we get

$$\begin{aligned}2\big|\big(r(s(z))\big)'\big|^2&\le\big|\big(r(s(z))\big)'\big|^2+\big|\big(r^*(s(z))\big)'\big|^2-\frac{|\sum_{j=0}^{n}a_jb_0^j|-|a_nb_m^n|}{|\sum_{j=0}^{n}a_jb_0^j|+|a_nb_m^n|}\big|\big(B(s(z))\big)'\big|\big|\big(r(s(z))\big)\big|^2\\&\le\frac{1}{2}\big|\big(B(s(z))\big)'\big|^2\Bigg[\Bigg(\max_{1\le k\le mn}|r(s(t_k))|\Bigg)^2+\Bigg(\max_{1\le k\le mn}|r(s(\mu_k))|\Bigg)^2\Bigg]\\&\quad-\frac{|\sum_{j=0}^{n}a_jb_0^j|-|a_nb_m^n|}{|\sum_{j=0}^{n}a_jb_0^j|+|a_nb_m^n|}\big|\big(B(s(z))\big)'\big|\big|\big(r(s(z))\big)\big|^2.\end{aligned}$$

Equivalently, we get by using Lemma 3.4

$$\begin{aligned}|r'(s(z))| \leq \frac{1}{2mm'}\left|\left(B(s(z))\right)'\right|\Bigg[\left(\max_{1\leq k\leq mn}|r(s(t_k))|\right)^2+\left(\max_{1\leq k\leq mn}|r(s(\mu_k))|\right)^2\\ -2\frac{|\sum_{j=0}^{n}a_jb_0^j|-|a_nb_m^n|}{|\sum_{j=0}^{n}a_jb_0^j|+|a_nb_m^n|}\frac{|r(s(z))|^2}{\left|\left(B(s(z))\right)'\right|}\Bigg]^{\frac{1}{2}}.\end{aligned}$$

□

Remark 4.7. A result due to Wali and Shah [[9], Theorem 1] is a special case of Theorem 4.6, where $s(z) = z$.

References

[1] A. Mir, *Inequalities concerning rational functions with prescribed poles,* J. Pure Appl. Math. **50** (2019), no. 2, 315–331.

[2] A. Mir, *Some inequalities concerning rational functions with fixed poles,* J. Contemp. Math. Anal. **55** (2020), no. 2, 105–114.

[3] A. Aziz and Q.M. Dawood, *Inequalities for polynomials and its derivative,* Indian J.Approx. Theory **54** (1988), 119–122.

[4] S.N. Bernstein, *Sur la limitation des dérivées des polynomes,* C. R. Acad. Sci. Paris. **190** (1930), 338–340.

[5] I. Qasim and A. Liman, *Bernstein type inequalities for rational functions,* Indian. J. Pure Appl. Math. **46** (2015), no. 3, 337–348.

[6] R. Osserman, *A sharp Schwarz inequality on the boundary for functions regular in disk,* Proc. Amer. Math. Soc. **128** (2000), no. 12, 3513–3517.

[7] P. Turán, *Über die ableitung von polynomen,* Compos. Math. **7** (1939), 89–95.

[8] S.L. Wali and W.M. Shah, *Some applications of Dubinin's lemma to rational functions with prescribed poles,* J. Math. Anal. Appl. **450** (2017), 769–779.

[9] S.L. Wali and W.M. Shah, *Applications of the schwarz lemma to inequalities for rational functions with prescribed poles,* J. Anal. **25** (2017), no. 1, 43–53.

[10] X. Li, R.N. Mohapatra and R.S. Rodriguez, *Bernstein-type inequalities for rational functions with prescribed poles,* J. London Math. Soc. **51** (1995), 523–531.