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Legendre cardinal functions and their application in solving nonlinear stochastic differential equations

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Abstract

This paper presents a new numerical technique for solving stochastic Itô integral equations. A new operational matrix for integration of cardinal Legendre polynomials are introduced. By using this new operational matrix of integration and the so-called collocation method, stochastic nonlinear Itô integral equations are reduced to systems of algebraic equations with unknown coefficients. Only small dimension of Legnedre operational matrix is needed to obtain a satisfactory results. Some error estimations are provided and illustrative examples are also included to demonstrate the efficiency and applicability of the proposed numerical technique.

Keywords: Cardinal Legendre functions, stochastic operational matrix, Brownian motion, Itô integral, collocation method, numerical solution
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1 Introduction

In the last decades, there has been an increasing interest in applying cardinal basis functions [4] for various types of problems. Spectral methods have been finding an important rôle in numerical analysis. They have a wide range of application in science and engineering. Numerical methods are important tools for calculating approximation solutions of stochastic differential equations. In recent years many numerical methods for deterministic and stochastic integral equations have been designed, for example, Adomian method [37], implicit Taylor methods [15, 23] and recently the operational matrices of integration for orthogonal polynomials, Legendre wavelets, Chebychev polynomials,..etc [2, 6, 16, 19, 20, 25, 26, 27, 28, 29, 30, 31, 32, 36, 38]. Several analytical and numerical methods have been proposed for solving various types of stochastic problems with the classical Brownian motion [21, 22, 24, 26, 29]. Noting that finding the exact solutions for most of these equations is hard, therefore, we have to apply approximate numerical methods to obtain numerical solutions. There is a growing interest in using interpolate approximate base function to deal with various problems. The main characteristic of the approach using this technique is that it reduces these problems to a systems of algebraic equations which simplifying the problem. In recent years, Cardinal functions have been finding an important role in numerical analysis, in particulary for solving integral equations [9, 10, 17, 18]. Integral equation technique is a well known approach for modeling of scattering models. Traditionally, most of the numerical methods for the solutions of these models use basis functions [2, 20, 35, 39]. Some authors have proposed modified or hybrid methods to increase the computational efficiency of the traditional approach [19, 25, 38]. In [9] M.H. Heydari

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& al. used Chebyshev cardinal wavelets and their application in solving nonlinear stochastic differential equations with fractional Brownian motion. M.H.Heydari obtained a new method based on the Chebyshev cardinal functions for variable-order fractional optimal control problems [34]. An effective direct method to determine the numerical solution of Volterra-Fredholm integro-differential equations based on Chebychev cardinal functions and deterministic operational matrices was also found in [10] their method shows good results in solution of nonlinear integro-differential equations. In [14], Kader et al used cardinal Legendre functions for solving $m-$ order linear and nonlinear deterministic integro-differential equations under mixed boundary conditions. There are several advantages to using approximations based on cardinal functions. First, due to their rapid convergence, cardinal numerical methods do not suffer from the common instability problems associated with other numerical methods and secondly, it is now well-established that they are characterized by exponentially decaying error. Finally, cardinal functions is a good method for solving problems with singular equations. In this paper, we use cardinal Legendre functions to find numerical solution of the following stochastic Itô integral equations.

$$X(t) = X_0 + \int_0^t a(s, X(s))ds + \int_0^t b(s, X(s))dB(s), \tag{1.1}$$

under the initial condition $X(0) = X_0$, where $X(t)$ is an unknown process, which shoud be computed. for $0 \le t, s \le 1$, X_0 is a random variable, $B(s)$ is a Brownian motion and where $a(s, X(s,\omega))$, $b(s, X(s,\omega))$ for $s, t \in [0, 1]$ are known stochastic processes defined on the same filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with natural filtration $\mathcal{F}_t$, X_0 is the known random variable with $E|X_0|^2 < +\infty$ and $X(t)$ is unknown stochastic process. The second integral in ((1.1)) is the Itô integral. Furthermore, all Lebesgue's and Itô integrals in ((1.1)) are well defined. Note that the existence and the uniqueness of a solution for the problem (1.1) are investigated in [15]. The organization of this paper is as follows. Section 2 reviews some definitions of stochastic calculus. We introduce Legendre and Legendre cardinal functions and operational matrix of integration in section 3. In sections 4 and 5, we present the numerical procedure of the numerical solution of the proposed problem. Convergence analysis of the method will be investigated in section 6. To show the computational efficiency of the proposed technique, we give some test problems which will be presented in section 7. Conclusion of the article is supplied in section 8.

2 Preliminaries

In this section, we express some basic definitions and mathematical preliminary of stochastic calculus.

Definition 2.1. Let $\mathcal{V} = \mathcal{V}(s, T)$, $0 \le s \le T$ be the class of functions $g(t,\omega) : [0, \infty) \longrightarrow \mathbb{R}$ such that:

1. The function $g(t,\omega)$ be $\mathcal{B} \times \mathcal{F}$ measurable, where $\mathcal{B}$ is the Borel σ-algebra of $\mathbb{R}^+$.
2. The function $g(t,\omega)$ is $\mathcal{F}_t-$ adapted (measurable).
3. $E\left[\int_s^T g^2(t,\omega)dt\right] < \infty$.

Lemma 2.2. (Itô isometry) For each $X(t,\omega) \in \mathcal{V}(s,T)$, we have

$$E\big(\int_s^T X(s,\omega)dB(s)\big)^2 = E\big(\int_s^T X^2(s,\omega)ds\big).$$

Lemma 2.3. (The Gronwall inequality) Let α, $\beta : [t_0, T] \longrightarrow \mathbb{R}$ be integrable with

$$0 \le \alpha(t) \le \beta(t) + L\int_{t_0}^t \alpha(s)ds, \tag{2.1}$$

for $t \in [t_0, T]$ where $L > 0$. Then

$$0 \le \alpha(t) \le \beta(t) + L\int_{t_0}^t e^{L(t-s)}\beta(s)ds, \quad t \in [t_0, T]. \tag{2.2}$$

3 Cardinal functions and Legendre polynomials

Definition 3.1. A cardinal function $C_j(x)$ for a specific interpolation function (e.g polynomail, etc) and for a set of interpolation points x_j is defined as [1]

$$C_j(x_i) = \delta_{ij},\ i, j = 1, 2, \ldots, N, \tag{3.1}$$

where N is the number of the interpolation points and δ_{ij} is the Kronecker delta.

3.1 Legendre polynomials and some properties

Legendre polynomials $L_0(x)$, $L_1(x), \ldots, L_n(x)$ are a special case of Jacobi polynomilas. These polynomials are very attractive to use because of they are orthogonal on the interval $[-1,1]$ with respect to the weight function $w(t) = 1$ and easy to compute. The Legendre polynomials $L_n(x)$, for $-1 \leq t \leq 1$ and $n \geq 0$, are given by the forms [7, 11, 13]

$$L_n(x) = \frac{1}{2^n} \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{n-2k},\ n = 0, 1, \ldots, \tag{3.2}$$

where $[n/2] = n/2$ if n is even, otherwise $(n-1)/2$. To use the Legendre polynomials for our purposes, it is preferable to map this to $[0,1]$. The Legendre basis of degree n in $[0,1]$ or shifted Legendre polynomials are defined by

$$L_{i+1}(x) = \frac{(2i+1)(2x-1)}{i+1} L_i(t) - \frac{i}{i+1} L_{i-1}(x), \quad i = 1, 2, \ldots, \tag{3.3}$$

where $L_0(x) = 1$, $L_1(x) = 2x - 1$. The shifted Legendre polynomials of degree i can be also written as

$$L_i(x) = \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)!}{(i-k)!(k!)^2} x^k. \tag{3.4}$$

3.2 Legendre cardinal functions

To construct the so called Legendre cardinal functions for the set of orthogonal Legendre polynomials $L_N(x)$, we will use the Taylor expansion of $L_{N+1}(x)$ in neighborhood the j-th root of $L_{N+1}(x)$, which gives

$$L_{n+1}(x) \simeq L_{N+1}(x_j) + L_{N+1,x}(x - x_j) + o(x - x_j)^2.$$

Since the first term in the right hand side vanishes, then we can define the cardinal function of degree N in $[-1,1]$ as follows [4, 8]

$$C_j(x) = \frac{L_{N+1}(x)}{L'_{N+1,x}(x_j)(x - x_j)},\ x \in [-1,1] \tag{3.5}$$

where the subscript x denotes x differentiation and x_j are the zeros of $L_{N+1}(x)$. We have

$$C_j(x_i) = \delta_{ji} = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases}$$

3.3 Function approximation

We change the variable $t = \dfrac{x+1}{2}$ to obtain cardinal functions basis in the interval $[0,1]$, then the shifted Legendre cardinal functions are defined on the interval $[0,1]$ as follows:

$$C_i^\star(t) = C_i(2t-1), \quad i = 1, \ldots N+1. \tag{3.6}$$

Theorem 3.2. Any function $f(t)$ mean square integrable on $[0,1]$ can be expanded by elements of shifted cardinal Legendre function as follow

$$f(t) \simeq \sum_{j=1}^{N+1} u_j C_j^\star(t) = U^T \Phi_N(t), \tag{3.7}$$

where $u_j = f(t_j)$, $t_j = \dfrac{x_j+1}{2}$, $j = 1, \ldots N+1$ are the shifted points of x_j, $U = (u_1, u_2, \ldots, u_{N+1})^T$ and $\Phi_N(t) = (C_1^\star, C_2^\star, \ldots C_{N+1}^\star)^T$.

Proof . Let $f(t) \simeq \sum_{j=1}^{N+1} u_j C_j^\star(t)$, then $f(t_i) \simeq \sum_{j=1}^{N+1} u_j C_j^\star(t_i) = \sum_{j=1}^{N+1} u_j \delta_{ji}$. Then $u_i = f(t_i)$. $\square$

Theorem 3.3. Any function $g(t,s)$ mean square integrable on $[0,1]\times[0,1]$ can be approximated by cardinal Legendre functions as follow

$$f(t,s) \simeq \sum_{j=1}^{N+1}\sum_{i=1}^{N+1} f(t_i,s_j)C_i^{\star}(t)C_j^{\star}(t) = \Phi_N(t)^T K_1 \Phi_N(s), \tag{3.8}$$

where $K_{1,(i,j)} = f(t_i,t_j)$.

Proof . We can proof this theorem by the similar way as the proof of theorem ((3.2)). □

3.4 Operational matrices of integration

Let $\Phi_N(t) = \left(C_1^{\star}, C_2^{\star}, \dots C_{N+1}^{\star}\right)^T$, then

Lemma 3.4. We have

$$\int_0^t \Phi_N(s)ds = A^{-1}Q\Phi_N(t). \tag{3.9}$$

where the $(N+1)\times(N+1)$ matrix A is called the transform matrix (or Vandermonde's matrix) and is given by

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ t_1 & t_2 & \dots & t_{N+1} \\ t_1^2 & t_2^2 & \dots & t_{N+1}^2 \\ \vdots & \vdots & \vdots & \\ t_1^{N-1} & t_2^{N-1} & \dots & t_{N+1}^{N-1} \\ t_1^N & t_2^N & \dots & t_{N+1}^N \end{pmatrix}$$

and

$$\mathbf{Q} = \begin{pmatrix} t_1 & t_2 & \dots & t_{N+1} \\ \frac{t_1^2}{2} & \frac{t_2^2}{2} & \dots & \frac{t_{N+1}^2}{2} \\ \vdots & \vdots & \vdots & \\ \frac{t_1^{N-1}}{N-1} & \frac{t_2^{N-1}}{N-1} & \dots & \frac{t_{N+1}^{N-1}}{N-1} \\ \frac{t_1^{N}}{N} & \frac{t_2^{N}}{N} & \dots & \frac{t_{N+1}^{N}}{N} \\ \frac{t_1^{N+1}}{N+1} & \frac{t_2^{N+1}}{N+1} & \dots & \frac{t_{N+1}^{N+1}}{N+1} \end{pmatrix}$$

Proof . Let $\psi_i(t) = t^{i-1}$ for $i = 1,\dots N+1$, by expanding $\psi_i(t)$ in $(N+1)$ terms of the shifted Legendre cardinal functions, we obtain $\psi_i(t) = \sum_{j=1}^{N+1} \psi_i(t_j)C_j^{\star}(t)$, $i = 1,2,\dots N+1$. Then

$$\begin{pmatrix} \psi_1(t) \\ \psi_2(t) \\ \vdots \\ \psi_{N+1}(t) \end{pmatrix} = A \begin{pmatrix} C_1^{\star}(t) \\ C_2^{\star}(t) \\ \vdots \\ C_{N+1}^{\star}(t) \end{pmatrix} = A\Phi_N(t).$$

Since the matrix A is invertible then $\Phi_N(t) = A^{-1}\Psi_N(t)$, where

$$\Psi_N(t) = \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \\ \vdots \\ \psi_{N+1}(t) \end{pmatrix}.$$

Hence

$$\begin{aligned}\int_0^t \Phi_N(s)ds &= \int_0^t A^{-1}\Psi_N(s)ds\\ &= A^{-1}\int_0^t \Psi_N(s)ds = A^{-1}\begin{pmatrix} t\\ \frac{t^2}{2}\\ \vdots\\ \frac{t^{N+1}}{N+1}\end{pmatrix}\end{aligned}$$

Now, let $g_i(t) = \frac{t^i}{i}$, $i = 1, 2, \ldots, N+1$, we have $g_i(t) = \sum_{j=1}^{N+1} g_i(t_j)C_j^\star(t) = Q\Phi_N(t)$. Then

$$\int_0^t \Phi_N(s)ds = A^{-1}Q\Phi_N(t).$$

□

3.5 Stochastic operational matrices of integration

In this subsection, we give stochastic operational matrix of integration with repect to Brownian motion. We have

$$\begin{aligned}\int_0^t \Phi_N(s)dB(s) &= \int_0^t A^{-1}\Psi_N(s)dB(s) = A^{-1}\int_0^t \Psi_N(s)dB(s)\\ &= A^{-1}\left[\int_0^t dB(s), \int_0^t sdB(s), ..., \int_0^t s^N dB(s)\right]^T\end{aligned}$$

we apply Itô formula, we get

$$\begin{pmatrix} \int_0^t dB(s)\\ \int_0^t sdB(s)\\ \int_0^t s^2 dB(s)\\ \vdots\\ \int_0^t s^N dB(s)\end{pmatrix} = B(t)\Psi_N(t) - \begin{pmatrix} 0\\ \int_0^t B(s)ds\\ 2\int_0^t sB(s)ds\\ \vdots\\ N\int_0^t s^{N-1}B(s)ds\end{pmatrix} = A_N(t) = (a_i)_{i=0,\ldots,N}$$

where $a_i = t^iB(t) - i\int_0^t s^{i-1}B(s)ds, \quad i = 0, ..., N$. For the integral $\int_0^t s^{i-1}B(s)ds$, we can use Simpson rule as follow

$$\int_0^t s^{i-1}B(s)ds \simeq \frac{t}{6}\left(0^{i-1}B(0) + 4(\frac{t}{2})^{i-1}B(\frac{t}{2}) + t^{i-1}B(t)\right), \quad i = 1, 2, \ldots N,$$

so

$$\begin{aligned}a_i &= t^iB(t) - i\frac{t}{6}\left(4(\frac{t}{2})^{i-1}B(\frac{t}{2}) + t^{i-1}B(t)\right) = \left((1-\frac{i}{6})B(t) - \frac{i}{3\times 2^{i-2}}B(t/2)\right)t^i, \quad i = 1, 2, \ldots, N\\ a_i &= B(t) \text{ for } i = 0.\end{aligned}$$

Also we approximate $B(t)$ and $B(\frac{t}{2})$ for $0 \leq t \leq 1$ by $B(0.5)$ and $B(0.25)$, then we obtain

$$A^{-1}A_N(t) = A^{-1}\begin{pmatrix} B(0.5) & 0 & 0 & ... & 0 \\ 0 & \frac{5}{6}B(0.5)-\frac{2}{3}B(0.25) & 0 & ... & 0 \\ \vdots & \vdots & \vdots & ... & \vdots \\ \vdots & \vdots & \vdots & ... & \vdots \\ \vdots & \vdots & \vdots & ... & (1-\frac{N}{6})B(0.5)-\frac{N}{3\times 2^{N-2}}B(0.25) \end{pmatrix}\begin{pmatrix} 1 \\ t \\ t^2 \\ . \\ . \\ . \\ . \\ t^N \end{pmatrix}$$

Then

$$A^{-1}A_N(t) = A^{-1}A_s\Psi_N(t) = A^{-1}A_sA\Phi_N(t) = P_s\Phi_N(t), \tag{3.10}$$

where

$$A_s = \begin{pmatrix} B(0.5) & 0 & 0 & ... & 0 \\ 0 & \frac{5}{6}B(0.5)-\frac{2}{3}B(0.25) & 0 & ... & 0 \\ \vdots & \vdots & \vdots & ... & \vdots \\ \vdots & \vdots & \vdots & ... & \vdots \\ \vdots & \vdots & \vdots & ... & (1-\frac{N}{6})B(0.5)-\frac{N}{3\times 2^{N-2}}B(0.25) \end{pmatrix}$$

and $P_s = A^{-1}A_sA$ is $(N+1)\times(N+1)$ stochastic operational matrix. Then

$$\int_0^t \Phi_N(t)dB(t) \simeq P_s\Phi_N(t). \tag{3.11}$$

4 Solving stochastic integral equation

We approximate equation ((1.1)) as follows

$$z_1(t) = a(t, X(t)),\ z_2(t) = b(t, X(t)), \quad t \in [0,1]. \tag{4.1}$$

By using equation ((1.1)) and ((4.1)), we have

$$\begin{cases} z_1(t) = & a\Big(t, X_0 + \int_0^t z_1(s)ds + \int_0^t z_2(s)dB(s)\Big), \\ z_2(t) = & b(t, X_0 + \int_0^t z_1(s)ds + \int_0^t z_2(s)dB(s)). \end{cases} \tag{4.2}$$

By expanding $z_1(t)$ and $z_2(t)$ by elements of cardinal functions, we get

$$z_1(t) = U_1^T\Phi_N(t), \quad z_2(t) = U_2^T\Phi_N(t). \tag{4.3}$$

By substituting equation (4.3)) in ((4.2)), we obtain

$$\begin{cases} z_1(t) = & a\Big(t, X_0 + \int_0^t U_1^T\Phi_N(s)ds + \int_0^t U_2^T\Phi_N(s)dB(s)\Big), \\ z_2(t) = & b\Big(t, X_0 + \int_0^t U_1^T\Phi_N(s)ds + \int_0^t U_2^T\Phi_N(s)dB(s)\Big). \end{cases} \tag{4.4}$$

which is equivalent to

$$\begin{cases} z_1(t) = & a\Big(t, X_0 + U_1^T\int_0^t \Phi_N(s)ds + U_2^T\int_0^t \Phi_N(s)dB(s)\Big), \\ z_2(t) = & b\Big(t, X_0 + U_1^T\int_0^t \Phi_N(s)ds + U_2^T\int_0^t \Phi_N(s)dB(s)\Big). \end{cases} \tag{4.5}$$

By using equation ((3.9)) and ((3.11)), we get

$$\begin{cases} U_1^T\Phi_N(t) = & a\Big(t, X_0 + U_1^TA^{-1}Q\Phi_N(t) + U_2^TP_s\Phi_N(t)\Big), \\ U_2^T\Phi_N(t) = & b\Big(t, X_0 + U_1^TA^{-1}Q\Phi_N(t) + U_2^TP_s\Phi_N(t)\Big). \end{cases} \tag{4.6}$$

We collocate ((4.6)) at shifted points t_j, $j = 1, 2, \ldots N+1$, we have

$$\begin{cases} U_1^Te_j^N = & a\Big(t_j, X_0 + U_1^TA^{-1}Qe_j^N + U_2^TP_se_j^N\Big), \\ U_2^Te_j^N = & b\Big(t_j, X_0 + U_1^TA^{-1}Qe_j^N + U_2^TP_se_j^N\Big), \end{cases} \tag{4.7}$$

where e_j^N denotes the column of ordre j of identity matrix I of order $N+1$. The system (4.7)) can be solved for the unknown U_1 and U_2 with Matlab software packages or by the Newton's iterative method. By determining U_1 and U_2, we can determine the approximate solution of $X(t)$ as follow

$$X_N(x) = X_0 + U_1^T A^{-1} Q \Phi_N(t) + U_2^T P_s \Phi_N(t). \tag{4.8}$$

5 Convergence analysis

In this section, we investigate the convergence and error analysis of the proposed method in the Sobolev space.

Definition 5.1. [5] The Sobolev space $H_w^m(a,b)$ is defined as follow

$$H_w^m(a,b) = \left\{ u \in L_w^2(a,b),\ u^{(j)}(t) \in L_w^2(a,b),\ j = 0,1,\dots m \right\}, \tag{5.1}$$

where w be a weight function and $m \geq 0$ be an integer.

Remark 1. 1. The Sobolev space $H_w^m(a,b)$ is endowed with the following weighted inner product

$$\langle u(t), v(t) \rangle_{m,w} = \sum_{i=1}^{m} \int_a^b u^{(j)} v^{(j)} w(t) dt. \tag{5.2}$$

The space $H_w^m(a,b)$ is a Hilbert space with the following norm

$$||u(t)||_{H_w^m(a,b)} = \left(\sum_{i=1}^{m} ||u^{(j)}||_{L_w^2(a,b)} \right)^{1/2}. \tag{5.3}$$

2. The sobolev space $H_w^m(a,b)$ satisfy $H_w^{m+1}(a,b) \subset H_w^m(a,b) \subset H_w^{m-1}(a,b) \subset \dots H_w^0(a,b) = L_w^2(a,b)$ and $C^m([a,b]) \subset H_w^m(a,b)$.

Lemma 5.2. [5] Let $u \in H_w^m(-1,1)$, $w(t) = 1$ and $I_N u = \sum_{j=1}^{N+1} u(x_j) C_j(x)$ be the Legendre interpolant of $u(t)$, where $C_j(x)$ are defined in ((3.5)) and x_j are the zeros of $L_{N+1}(x)$. Then, the truncated error $u - I_N u$ satisfies

$$||u - I_N u||_{L_w^2(-1,1)} \leq \hat{C}_m N^{-m} \left(\sum_{j=\min(m,N)}^{m} ||u^{(j)}||_{L_w^2(-1,1)} \right)^{1/2}. \tag{5.4}$$

where $\hat{C}_m$ is a positive constant independent of N and dependent on m. Moreover, in the maximum norm, it yields

$$||u - I_N u||_{L_w^\infty(-1,1)} \leq \hat{C}_m N^{1/2-m} \left(\sum_{j=\min(m,N)}^{m} ||u^{(j)}||_{L_w^2(-1,1)} \right)^{1/2}. \tag{5.5}$$

where $\hat{C}_m$ is a positive constant independent of N and dependent on m, and $||u||_{L_w^\infty(-1,1)} = \sup_{-1 \leq t \leq 1} |u(t)|$.

Theorem 5.3. Let $u \in H_{w^\star}^m(0,1)$, $w^\star(t) = 1$ and $I_N^\star u = \sum_{j=1}^{N+1} u_j C_j^\star(t)$, $u_j = u(t_j)$ be the Legendre interpolant of $u(t)$, where $C_j^\star(t)$ are defined in ((3.6)) and $t_j = \dfrac{x_j + 1}{2}$, $j = 1, \dots N+1$ are the shifted points of x_j. Then, the truncated error $u - I_N^\star u$ satisfies

$$||u - I_N^\star u||_{L_{w^\star}^2(0,1)} \leq \hat{C}_m N^{-m} \left(\sum_{j=\min(m,N)}^{m} (1/2)^{2j}\, ||u^{(j)}||_{L_{w^\star}^2(0,1)} \right)^{1/2}. \tag{5.6}$$

where $\hat{C}_m$ is a positive constant independent of N and dependent on m. Moreover, in the maximum norm, it yields

$$||u - I_N u||_{L^\infty(0,1)} \leq\leq \hat{C}_m N^{1/2-m} \sqrt{2} \left(\sum_{j=\min(m,N)}^{m} (1/2)^{2j}\, ||u^{(j)}||_{L_{w^\star}^2(0,1)} \right)^{1/2}. \tag{5.7}$$

where $\hat{C}_m$ is a positive constant independent of N and dependent on m, and $||u||_{L^\infty(0,1)} = \sup_{0 \leq t \leq 1} |u(t)|$.

Proof . The proof proceeds in a same manner as the one of Theorem (5.4) in [9]. □

Theorem 5.4. Suppose $X(t) \in H_w^m(0,1)$ and $X_N(t)$ be the exact and approximate solutions of equation ((1.1)), respectively, furthermore, we suppose that

(H1) $|a(t,X_1(t)) - a(t,X_2(t))| + |b(t,X_1(t)) - b(t,X_2(t))| \le L|X_1 - X_2|$, (Lipschitz condition),

(H2) $|a(t,X(t))| + |b(t,X(t))| \le L(1+|X|)$, (Linear growth condition),
where $t \in [0,1]$, $X_1, X_2 \in \mathbb{R}$ and L_i are positive constants for $i = 1,2$.

(H3) $E|X_0|^2 < \infty$.

Then $X_n(t)$ converges to $X(t)$ in L^2.

Proof . Let $e_N(t) = X(t) - X_N(t)$ be an error function between approximate solution $X_N(t)$ and exact solution $X(t)$. Then, we have

$$X(t) - X_N(t) = \int_0^t (z_1(s) - \bar{z}_1(s))ds + \int_0^t (z_2(s) - \bar{z}_2(s))dB(s), \tag{5.8}$$

where $z_i(t)$, $i = 1,2$ are given by $z_1(t) = a(t,X(t))$, $z_2(t) = b(t,X(t))$. Let $\bar{z}_i(t)$, $i = 1,2$ are the approximation by shifted cardinal Legendre functions of $z_i(t)$,

$$\bar{z}_1(t) = \text{app}_N(a(t,X_N(t)),\ \bar{z}_2(t) = \text{app}_N(b(t,X_N(t)) \text{ and } z_1^N(t) = a(t,X_N(t)),\ z_2^N(t) = b(t,X_Nt)).$$

We have

$$\begin{aligned} e_N(t) = & \int_0^t (z_1(s) - \bar{z}_1(s))ds + \int_0^t (z_2(s) - \bar{z}_2(s))dB(s) \\ E|e_N(t)|^2 = & E\Big(\Big| \int_0^t (z_1(s) - \bar{z}_1(s))ds + \int_0^t (z_2(s) - \bar{z}_2(s))dB(s) \Big| \Big)^2, \end{aligned}$$

using the inequality $(b+c)^2 \le 2(b^2+c^2)$, we obtain

$$E|e_N(t)|^2 \le 2E\Big|\int_0^t (z_1(s) - \bar{z}_1(s))ds\Big|^2 + 2E\Big|\int_0^t (z_2(s) - \bar{z}_2(s))dB(s)\Big|^2$$

by using the Itô isometry and Schwartz inequality, we have

$$E|e_N(t)|^2 \le 2E\Big(\int_0^t |z_1(s) - \bar{z}_1(s)|^2 ds\Big) + 2E\Big(\int_0^t |z_2(s) - \bar{z}_2(s)|^2 ds\Big),$$

$$2E\Big(\int_0^t |z_1(s) - \bar{z}_1(s)|^2 ds\Big) \le 4E\Big(\int_0^t |z_1(s) - z_1^N(s)|^2 ds\Big) + 4E\Big(\int_0^t |z_1^N(s) - \bar{z}_1(s)|^2 ds\Big),$$

and

$$2E\Big(\int_0^t |z_2(s) - \bar{z}_2(s)|^2 ds\Big) \le 4E\Big(\int_0^t |z_2(s) - z_2^N(s)|^2 ds\Big) + 4E\Big(\int_0^t |z_2^N(s) - \bar{z}_2(s)|^2 ds\Big).$$

By considering theorem (5.3), there exists $\alpha_i(m,N)$, $i = 1,2$ such that

$$E||z_i^N(s) - \bar{z}_i(s)||^2 \le \big(\alpha_i(m,N)\big)^2,\ i = 1,2.$$

where $\alpha_i(m,N) = \hat{C}_m N^{-m}\Big(\sum_{j=\min(m,N)}^{m} (1/2)^{2j}\, ||(z_i^N)^{(j)}||_{L^2_{w^\star}(0,1)}\Big)^{1/2}$, $i = 1,2$. Then

$$E|e_n(t)|^2 \le 4\big(\alpha_1(m,N) + \alpha_2(m,N)\big)^2 + 4\Big(\int_0^t E|z_1(s) - z_1^n(s)|^2 ds + \int_0^t E|z_2(s) - z_2^n(s)|^2 ds\Big).$$

Moreover, using Lipschitz condition, one has

$$E|e_n(t)|^2 \le 4\big(\alpha_1(m,N) + \alpha_2(m,N)\big)^2 + 8L\int_0^t E|e_n(s)|^2 ds. \tag{5.9}$$

Hence by Gronwall inequality, we get

$$E|e_N(t)|^2 \longrightarrow 0, \text{ as } N \longrightarrow \infty.$$

□

Remark 2. From lemma ((5.2)), the error is sufficiently small if m is sufficiently large.

6 Numerical examples

To demonstrate the accuracy and effectiveness of the method proposed herein, we have applied it to several examples. These examples are solved in different references, so the numerical results obtained here can be compared with those of other numerical methods. In order to analyze the error of the method we introduce the absolute error between exact and approximate solutions, with M simulations, $e_N(t) = |X(t) - X_N(t)|$.

Example 6.1. Let given the deterministic Riccati differential equation

$$u'(t) + u^2(t) - 1 = 0,\ u(0) = 0. \tag{6.1}$$

The exact solution is given by $u(t) = \dfrac{\exp(2t) - 1}{\exp(2t) + 1}$. The numerical results of this example are given in table (1).

Table 1: The absolute errors obtained by the proposed method with different values of N for Example (6.1)

t	$N = 6$	$N = 10$	$N = 12$
0.0	6.4731 E-6	5.1826 E-9	7.0595 E -11
0.1	1.9335 E-6	1.4375 E-9	1.9858 E-11
0.2	2.0340 E-6	1.1345 E-9	6.6977 E-11
0.3	3.8270 E-7	1.2643 E-9	5.6513 E-11
0.4	1.9180 E-6	1.0563 E-9	2.2130 E-10
0.5	3.9244 E-7	7.6294 E-8	1.9696 E-6
0.6	1.8670 E-6	7.6311 E-10	3.9650 E-9
0.9	3.7044 E-6	1.0957 E-9	1.1698 E-8
0.8	2.0978 E-6	5.4205 E-10	2.7887 E-8
0.9	1.4228 E-6	2.4433 E-9	1.4775 E-6
1.0	6.4731 E-6	5.1724 E-9	1.0582 E-9

Example 6.2. Let us consider the problem

$$X(t) = X_0 + \int_0^t a^2 \cos(X(s)) \sin^3(X(s))ds - a \int_0^t \sin^2(X(s))dB(s),\ t \in [0, 1]. \tag{6.2}$$

The exact solution is $X(t) = arccot(aB(s) + \cot(X_0))$. The computed errors for $N = 5$, $a = 1/8$ and different values of X_0 are summarized in table (2).

Table 2: The absolute errors obtained by the proposed method with different values of X_0 with $M = 500$ simultations for Example (6.2)

t	$X0 = 0.01$	$X0 = \pi/32$	$X0 = 0.001$	$X0 = 1$
0	4.0171 E-6	3.8327 E -4	4.0202 E-8	6.2593 E-2
0.1	1.6608 E-5	1.5645 E-3	1.6642 E-7	5.9772 E-2
0.2	1.3697 E-4	1.4837 E-2	1.3541 E-6	1.1500 E-2
0.3	1.8395 E-5	1.7325 E-3	1.8434 E-7	3.2472 E-2
0.4	1.4249 E-5	1.3701 E-3	1.4249 E-7	1.2364 E-3
0.5	1.9835 E-5	1.8680 E-3	1.9877 E-7	6.1292 E-2
0.6	1.8980 E-4	2.1690 E-2	1.8676 E-6	2.8464 E-2
0.7	3.9812 E-5	3.2924 E-3	3.9711 E-7	4.1968 E-2
0.8	6.7643 E-5	6.1056 E-3	6.8096 E-7	4.9793 E-3
0.9	6.4465 E-6	6.2461 E-4	6.4410 E-8	1.7478 E-2
1.0	6.4384 E-5	6.4939 E-3	6.4077 E-7	1.7478 E-2

Example 6.3. Consider the deterministic Volterra integral equation as follows [10]

$$-\frac{1}{15}\Big(- 8\exp(2t) + 6\sin(t) + 3\cos(t) + 5\exp(-t)\Big) - \int_0^t \big(\exp(s-t) + \sin(t-s)X(s)\big)ds,$$

where the exact solution is $X(t) = \exp(2t)$. The numerical results are summarized in table (3), figure (1) and figure (2).

Table 3: The absolute errors obtained by the proposed method with different values of N for Example (6.3)

t	$N = 4$	$N = 10$
0	1.6414 E-2	1.9052 E-8
0.2	6.4196 E-3	5.5029 E -9
0.4	6.8821 E-3	1.1685 E-10
0.6	2.6189 E-4	6.5420 E-9
0.8	1.1788 E-2	4.9335 E-9
1	8.4551 E-2	1.7840 E-7

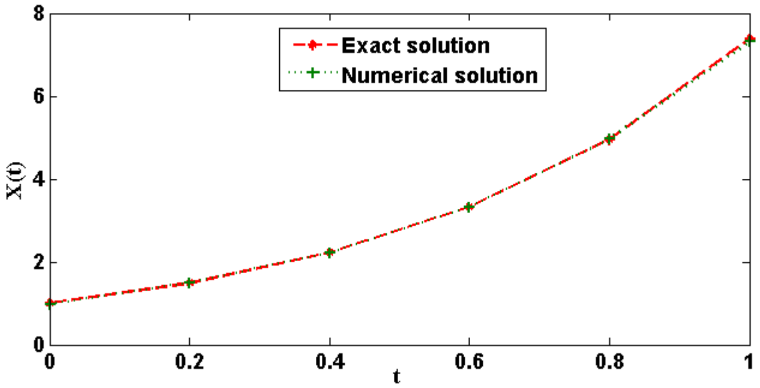

Figure 1: Exact and approximate solutions for $N = 4$ for example (6.3).

Example 6.4. Consider the linear Volterra integral equation

$$X(t) = \frac{1}{12} + \int_0^t \cos(s)X(s)ds + \int_0^t \sin(s)X(s)dB(s),\ s,t \in [0,1). \tag{6.3}$$

The exact solution is

$$X(t) = \frac{1}{12}\exp(-\frac{t}{4} + \sin(t) + \frac{\sin(2t)}{8} + \int_0^t \sin(s)dB(s)), \quad s,t \in [0,1).$$

In this example, we take $X_0 = \frac{1}{12}$, $n = 5$, $n = 7$ and $n = 9$. The results are summarized in table (4).

Example 6.5. (The basic Black-Scholes model) Let given the following linear stochastic equation

$$dX(t) = \lambda X(t)dt + \mu X(t)dW(t),\ X(0) = X_0,\ t \in [0,1], \tag{6.4}$$

where the exact solution is given by $X(t) = exp((\lambda - \frac{1}{2}\mu^2)t + \mu W(t))$. The results obtained for $\lambda = -100$, $\mu = 1$, $N = 9$ and $M = 10000$ simulations of this example are given in Table (5).

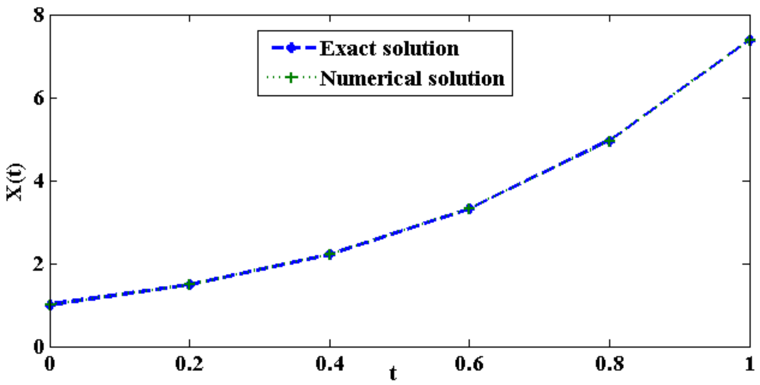

Figure 2: Exact and approximate solutions for $N = 10$ for example (6.3).

Table 4: Computed errors of cardinal shifted Legendre solution of Example (6.4)

t	$n=5$	$n=7$	$n=9$	$\hat{m}=32$ [6]	$\hat{m}=128$[6]
0	4.8461 E-4	1.0924 E-5	1.9905 E-6		
0.1	8.7041 E-3	1.3050 E-3	8.4205 E-4	0.00027710	0.00020525
0.2	8.9302 E-3	1.6426 E-3	1.2920 E-3		
0.3	3.9782 E-2	6.6196 E-4	1.3555 E-3	0.00030417	0.00045023
0.4	1.2240 E-2	1.8642 E-3	1.2598 E-3		
0.5	4.6941 E-2	5.7119 E-3	1.5698 E-3	0.06034923	0.12302136
0.6	1.5917 E-2	9.8883 E-3	3.2920 E-3		
0.7	3.1020 E-2	1.2572 E-2	7.8045 E-3	0.00676411	0.00800211
0.8	1.3880 E-2	1.1550 E-2	1.6411 E-2		
0.9	1.1846 E-2	5.3969 E-3	2.9506 E-2	0.01404822	0.01578822

Table 5: Computed errors for Example (6.5).

t	$X_0=0.001$	$X_0=0.01$	$X_0=0.1$	$X_0=1$
0	1.0968 E-5	4.1376 E-3	3.1817 E-2	5.0676 E-1
0.1	3.4040 E-4	1.3261 E-3	2.0642 E-2	2.1272 E-2
0.2	1.1517 E-4	7.7968 E-4	1.0664 E-2	2.0320 E-2
0.3	7.8544 E-7	4.3569 E-4	5.0918 E-3	1.6041 E-2
0.4	3.2718 E-5	3.5683 E-4	4.7638 E-3	8.3613 E-3
0.5	8.3004 E-5	3.6748 E-4	5.6970 E-3	2.7126 E-3
0.6	7.2813 E-5	2.9280 E-4	4.7074 E-3	1.5862 E-2
0.7	3.3842 E-5	2.1419 E-4	3.0152 E-3	5.6668 E-3
0.8	2.2559 E-6	1.7920 E-4	1.9934 E-3	1.0030 E-2
0.9	3.1680 E-6	1.3367 E-4	1.4717 E-3	7.4090 E-3

7 Conclusion

A new numerical technique is constructed for solving numerically different kind of deterministic and stochastic integral and integro-differential equations which can not be solved analytically. The proposed approach is based on cardinal Legendre functions where the collocation points are the zeros of shifted Legendre $L_{N+1}(x)$ polynomials. The deterministic and stochastic operational matrices of these orthogonal functions have been obtained in order to reduce our problem to a system of algebraic equations. Some illustrative test problems are given to show the efficiency and accuracy of the proposed technique.The results of the present method have been compared with analytical solutions and with others techniques. The numerical tests of the proposed method were in a good agreement with the exact solutions, so this approach can be applied to solve some stochastic problems such that stochastic population growth, stochastic Volterra's population model, stochastic pendulum problem ... etc. The proposed technique can be also used to solve a class of variable-order optimal control problems in the Caputo sense and other types of fractional differential equations. Our aim is that this survey paper will stimulate further interest in the area of optimal control computation and also for stochastic integro and partial differential equations. There are still many possibilities for future research.

References

[1] A. Alipanah, *Spectral Methods using cardinal functions*, PHD Thesis in applied Mathematic, AmirKabir University of Technology, 2006.

[2] M. Asgari, E. Hashemizadeh, M. Khodabin and K. Maleknedjad, *Numerical solution of nonlinear stochastic integral equation by stochastic operational matrix based on Bernstein polynomials*, Bull. Math. Soc. Sci. Math. Rouman. Tome. **57** (2014), 3–12.

[3] W.F Blyth, R.L May and P. Widyaningsih, *Volterra integral equations Solved in Fredholm form using Walsh functions*, ANZIAM. J. **45** (2004), 269–282.

[4] J.P. Boyd, *Chebychev and Fourier spectral methods*, Dover Publications, Inc., 2000.

[5] C. Canuto, M. Hussaini, A. Quarteroni and T. Zang, *Spectral methods in fluid dynamics*, Springer, Berlin, 1988.

[6] F. Mohamadi, *A wavelet-based computational method for solving stochastic Itô-Volterra integral equations*, J. Comput. Phys. **298** (2015), 254–265.

[7] M.E.A. El-Mikkawy and G.S. Cheon, *Combinatorial and hypergeometric identities via the Legendre polynomials- A computational approach*, App. Math. Comput. **166** (2005), 181–195.

[8] D. Funaro, *Polynomial approximation of differential equations*, Springer Verlag, New York, 1992.

[9] M.H. Heydari, M.R. Mahmoudi, A. Shakiba and Z. Avazzadeh, *Chebyshev cardinal wavelets and their application in solving nonlinear stochastic differential equations with fractional Brownian motion*, Commun. Nonlinear Sci. Numer. Simul. **64** (2018), 98—121.

[10] M. Heydari, Z. Avazzadeh and G.B. Loghmani, *Chebychev cardinal functions for solving Volterra-Fredholm integro differential equations using operational matrices*, Iran. J. Sci. Technol. **36** (2012), no.1, 13–24.

[11] M. Ghasemi and C.M.T. Kajani, *Application of Hes homotopy perturbation method to nonlinear integro-differential equations :Wavelet- Galerkin method and homotopy perturbation method*, Appl. Math. Comput. **18** (2007), no. 1, 450–455.

[12] A. Gil, J. Segura and N.M. Temme, Numerical Methods for Special Functions, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2007.

[13] Gulsu, M. and Sezer, M. The approximate solution of high-order linear difference equation with variable coefficients in terms of Taylor polynomials , App. Math. Comput. 168(2005) 76-88.

[14] M.M. Kader, N.H. Sweilam and W.Y. Kota, *Cardinal functions for Legendre pseudo-spectral method for solving the integro-differential equations*, J. Egypt. Math. Soc. **22** (2014), 511–516.

[15] P.E. Kloeden and E. Platen, *Numerical solution of stochastic differential equations*, Springer, Berlin, 1992.

[16] K. Parand and M. Delkhosh, *Operational matrix to solve nonlinear Riccati differential equations of arbitrary order*, St. Pertersburg Polytech. Univ. J. Phys. Math. **3** (2017), 242–254.

[17] M. Lakestani and M. Deghan, *The use of Chebychev cardinal functions for the solution of of a partial differential equation with an unknown time-independent coefficient subjectto an extra measurement*, J. Comput. Appl. Math. **235** (2010), no. 3, 669–678.

[18] M. Lakestani and M. Deghan, *Numerical solution of fourth-order integro-differential equations using Chebychev cardinal functions*, Int. J. Comput. Math. **87** (2010), no. 6, 1389–1394.

[19] Y. Mahmoudi, *Wavelet Galerkin method for numerical solution of nonlinear integral equation*, App. Math. Comput. **167** (2005), no. 2, 1119–1129.

[20] K. Maleknejed, R. Mollapourasl and M. Alizadeh, *Numerical solution of Volterra type integral equation of the first kind with wavelet basis*, Appl. Math. Comput. **194** (2007), no. 2, 400–405.

[21] K. Maleknejad, M. Khodabin and Rostami, *Numerical method for solving $m-$ dimensional stochastic Itô Volterra integral equations by stochastic operational matrix based on block-pulse functions*, Comput. Math. Appl. **63** (2012), 133–143.

[22] X. Mao, *Approximate solutions for a class of stochastic evolution equations with variable delays-part II*, Numer. Funct. Anal. Optim. **15** (1994), no. 1-2, 65–76.

[23] G.N. Milstein, *Numerical integration of stochastic differential equations*, Math. Appl. Kluwer, Dordrecht, 1995.

[24] F. Mirzaee and E. Hadadiyan, *A collocation technique for solving nonlinear stochastic Itô-Volterra integral equation*, Appl. Math. Comput. **247** (2014), 1011–1020.

[25] F. Mirazee, S. Alipour and N. Samadyar, *Numerical solution based on hybrid of block-pulse and parabolic function for solving a system of nonlinear stochastic Itô-Volterra integral equations of fractional order*, J. Comput. Appl. Math. **349** (2019), 157–171.

[26] F. Mirazee and N. Samadyar, *On the numerical solution of stochastic quadratic integral equations via operational matrix method*, Math. Meth. Appl. Sci. **41** (2018), no. 12, 4465–4479.

[27] F. Mirazee and N. Samadyar, *Application of hat basis functions for solving two-dimensional stochastic fractional integral equations*, Comput. Appl. Math. **37** (2018), no. 4, 4899–4916.

[28] F. Mirazee and S. Alipour, *Approximation solution of nonlinear quadratic integral equations of fractional order via piecewise linear functions*, J. Comput. Appl. Math. **331** (2018), 217–227.

[29] F. Mirazee and A. Hamzeh, *A computational method for solving nonlinear stochastic Volterra integral equations*, J. Comput. Appl. Math. **306** (2016), 166–178.

[30] F. Mirazee and N. Samadyar, *Numerical solutions based on two- dimensional orthonormal Bernstein polynomials for solving some classes of two-dimensional nonlinear integral equations of fractional order*, Appl. Math. Comput. **344** (2019), 191–203.

[31] F. Mirazee and N. Samadyar, *Using radial basis functions to solve two dimansional linear stochastic integral equations on non-rectangular domains*, Engin. Anal. Bound. Elem. **92** (2018), 180–195.

[32] F. Mirazee and N. Samadyar, *Numerical solution of nonlinear stochastic Itô-Volterra integral equations driven by fractional Brownian motion*, Math. Meth. Appl. Sci. **41** (2018), no. 4, 1410–1423.

[33] F. Mirazee, N. Samadyar and S.F. Hoseini, *Euler polynomial solutions of nonlinear stochastic Itô-Volterra integral equations*, J. Comput. Appl. Math. **330** (2018), 574–585.

[34] M.H. Heydari, *A new direct method based on the Chebyshev cardinal functions for variable-order fractional optimal control problems*, J. Franklin Inst. **355** (2018), 4970—4995.

[35] M.H. Reihani and Z. Abadi, *Rationalized Haar functions method for solving Fredholm and Volterra integral equations*, J. Comput. Appl. Math. **200** (2007), 12–20.

[36] X. Shang and D. Ha, *Numerical solution of Fredholm integral equations of the first kind by using linear Legendre multi-wavelets*, Appl. Math. Comput. **191** (2007), no. 2, 440–444.

[37] A.M. Wazwaz, *A first in integral equations*, New Jersey, World Scientific, 1997.

[38] S. Yousefi and M. Razzaghi, *Legendre wavelets method for the nonlinear Volterra-Fredholm integral equations*, Math. Comput. Simul. **70** (2005), no. 1, 1–8.

[39] S.A. Yousefi, A. Lotfi and M. Dehghan, *He's variational iteration method for solving nonlinear mixed Volterra-Fredholm integral equations*, Comput. Math. Appl. **58** (2009), no. 11-12, 2172–2176.