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Cofinitely $\bigoplus D_j$-supplemented modules

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Abstract

For any R-module W, $D_j(W)$ presented as the total of all J-small sub-modules. If A and B are sub-module of W, we say A is $\bigoplus$ D_j'supplement of B in W if $W = A + B = A \bigoplus \acute{A}$, for $\acute{A} \hookrightarrow W$, and $A \bigcap B \ll_j D_j(A)$. If every sub-module has $\bigoplus$ D_j-supplemented, then W is $\bigoplus$ D_j-supplemented A sub-module A of W. If a sentence is conclusive, it is said to be cofinite i.e., $\frac{W}{A}$ is finitely generated. Also we introduce cofinite $\bigoplus$ D_j-supplemented if every cofinite sub-module of W has $\bigoplus$ D_j-supplemented.

Keywords: Cofinite, finitely generated, module, supplemented module
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1 Introduction

Assume that R is a ring with identity and all modules are unitary left R-modules. Suppose that A is a micromodule inside the R-module W [2, 3]. If $A + N = W$ for any sub-module N of W, then $W = N$ is a sub-module of W [4, 6]. Let A and N be sub-modules of a module W, A is a supplement of N in W if the impact on property is modest $W = A + N$, equivalently if $W - A + N$ and $\ln N \ll L$, if every sub-module of a module W has a supplement in W, then W is called a supplemented module [1, 5, 7]. As a generalization of a small sub-module A. Kabban and Khalid in [4] introduced J-small sub-modules. A sub-module A of W is called J-small sub-module of W written as $A \ll_j W$ if whenever $W = A + N$ with $J\left(\frac{W}{N}\right) = \frac{W}{N}$ implies that $W = N$ [4]. It is known that $J(W)$ is the sum of all small sub-modules of W. Abdlkareem and Khalid in [4] introduced $Rad_j(W)$ as the sum of all J-small sub-modules of W, for short we refer to $D_j(W)$ instead of $Rad_j(W)$. In this paper we introduce $\bigoplus D_j$-supplemented module. Let B and N sub-modules of a module W, B is called $\bigoplus$ D_j-supplemented If B is a straight-forward summation of W and there exists a sub-module of N in W, with $W = B + N$, and $B \cap N \ll_j D_j(B)$, so W is called $\bigoplus D_j$-supplemented module if each sub-module of a program is tested of W has $\bigoplus D_j$-supplement in W. Sub module A of a larger module; A cofinite sub module of W is referred to as W. if $\frac{W}{A}$ is finitely created. We define cofinite $\bigoplus D_j$-supplemented as follows; a module W is called a cofinite $\bigoplus D_j$-supplemented module if for every coifinite sub-module has $\bigoplus D_j$-supplement. Clearly every $\bigoplus D_j$-supplemented module is a cofinite $\bigoplus D_j$-supplemented module. In this work, main properties of these concepts were proved.

2 $\bigoplus D_j$-supplemented modules

This section is devoted to introduce $\bigoplus D_j$-supplemented module.

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Definition 2.1. Let W be an R-module and let A, B is a sub-module of W. We say N is a $\bigoplus D_j$-supplement of A in W if $W = A + B$, B directly summarizes of W and $A \cap B \ll_j D_j(B)$. Then W is said to be $\bigoplus D_j$-supplemented, if every sub-module of W has $\bigoplus D_j$-supplement.

Example 2.2.

1. It is obvious that each semi-simple module is $\bigoplus D_j$-supplemented. In particular Z_6 as Z_module is $\bigoplus D_j$-supplemented.
2. consider Z_8 as Z_module, $D_j(Z_8) =< \underline{2} >$ Notice that $< \underline{2} > +Z_8 = Z_8$ and $< \underline{2} > \cap Z_8 =< \underline{2} > \not\ll D_J(Z_8) =< \underline{2} >$, so Z_8 as Z-module is not $\bigoplus D_j$-supplemented module.

Remember that a sub module N of W is said to be completely invariant if and only if for every f End (W), $f(N) \leq N$ and W If every sub-module of W is entirely invariant, then W is considered a duo module [2].

Theorem 2.3. Let W be a $\bigoplus D_j$-supplemented, let A an invariant submodule of the W, then $\frac{W}{A}$ is $\bigoplus D_j$-supplemented.

Proof . An invariant submodule of L may be defined of W and let $\frac{N}{A}$ any sub-module of $\frac{M}{A}$ since W is $\bigoplus D_j$-supplemented therefore there is a direct summand B of W that is equal to the sum of W and B. $W = N + B$ and $N \cap B \ll_j D_j(B)$ with $W = B \bigoplus B'$ for $B' \subseteq W$. Now $\frac{W}{A} = \frac{N+B}{A} = \frac{N}{A} + \frac{B+A}{A}$, $\frac{N}{A} \cap \frac{B+A}{A} = \frac{N\cap(B+A)}{A} = \frac{(N\cap B)+A}{A}$, but $N \cap B \ll_j D_j(B)$, then $\frac{(N\cap B)+A}{A} \ll_j \frac{D_j(B)+A}{A} \subseteq D_j\left(\frac{B+A}{A}\right)$ by [3], since $W = B \bigoplus B'$ and L is fully invariant, then $\frac{W}{A} = \frac{B \bigoplus B'}{A} = \frac{B+A}{A} \bigoplus \frac{B' \bigoplus A}{A}$ by [2]. so $\frac{B+A}{A}$ is a 'supplement of $\frac{N+A}{A}$ and $\frac{B+A}{A}$ is a direct summation of $\frac{W}{A}$. Therefore $\frac{W}{A}$ is $\bigoplus D_j$-supplemented. □

Theorem 2.4. Let $W = W_1 \bigoplus W_2$ be a duo module, than W_1 and W_2 are $\bigoplus D_j$-supplemented modules if and only if W is $\bigoplus D_j$-'supplemented [1].

Proof . $\Rightarrow$) Suppose that W_1 and W_2 $\bigoplus D_j$-supplemented, and let A sub-module of W since $W \cap A = A = (W_1 \cap A) \bigoplus (W_2 \cap A)$ take $W_1 \cap A = A \subseteq W_1$, and $W_2 \cap A = A_2 \subseteq W_2$, so there exists $B_1 \subseteq_{\bigoplus} W_1$, and there exists $B_2 \subseteq_{\bigoplus} W_2$ Such that $W_1 = B_1 + A_1$ and $W_2 = B_2 + A_2$, $B_1 \cap A_1 \ll_j D_j(B_1)$ and $B_2 \cap A_2 \ll_j D_j(B_2)$. $W_1 = B_1 \bigoplus B_1'$ and $W_2 = B_2 \bigoplus B_2'$ but $W = W_1 \bigoplus W_2$, thus $W = B_1 + A_1 \bigoplus B_2 + A_2$

$W = (B_1+B_2)+(A_1+A_2)$. $W = (B_1+B_2)+A$ and $(B_1+B_2)\cap A = B_1\cap A+B_2\cap A = B_1\cap(A\cap W_1)+B_2\cap(A\cap W_2) \subseteq D_j(B_1) + D_j(B_2) \ll_j D_j(B_1 + B_2)$, now $W = W_1 \bigoplus W_2 = B_1 \bigoplus B_1' \bigoplus B_2 \bigoplus B_2' = (B_1 \bigoplus B_2) \bigoplus (B_1' \bigoplus B_2')$ but $W = W_1 \bigoplus W_2$, then $W = B_1 \bigoplus B_1' \bigoplus B_2 \bigoplus B_2' = B_1 \bigoplus B_2 \bigoplus B_1' \bigoplus B_2'$ $\Leftarrow$) let $A_1 \underline{\hookrightarrow} W_1$, then $A_1 \underline{\hookrightarrow} W$, so there exists $B_1 \hookrightarrow_{\bigoplus} W$ such that $W = A_1 + B_1 W = B_1 \bigoplus B_1'$ and $A_1 \cap B_1 \ll_j D_j(B_1)$ thus $W_1 = W \cap W_1 = (A_1 + B_1) \cap W_1 = A_1 + (B_1 \cap W_1)$ by modular law, $W_1 = W \cap W_1 = (B_1 \bigoplus B_1') \cap W_1 = (B_1 \cap W_1) \bigoplus (B_1' \cap W_1)$ so $B_1 \cap W_1 \hookrightarrow_{\bigoplus} W_1$ Now $A_1 \cap (B_1 \cap W_1) = A_1 \cap B_1 \ll_j D_j(B_1)$ and $D_j(B_1 \cap W_1) \underline{\hookrightarrow} D_j(B_1) \subseteq D_j(W_1)$

$D_j(B_1\cap W_1) \hookrightarrow_{\bigoplus} D_j(W_1)$ so $A_1\cap B_1 \ll_j D_j(B_1\cap W_1)$ by [4] similarly one can show that W_2 is $\bigoplus D_j$-supplemented. □

Corollary 2.5. Let $W = W_1 \bigoplus W_2 \bigoplus ...\ ... \bigoplus W_n$ be a duo module, then W_1, $W_2, ...\ ...$ and W_n are $\bigoplus$ D_j-supplemented if and only if M is $\bigoplus$ D_j-supplemented.

Proposition 2.6. Let $W = W_1 \bigoplus W_2$ be a two-parter Sub-modules B and A are part of W_1. If B is $\bigoplus$ D_j-supplement on A in M, then $B \bigoplus W_2$ is $\bigoplus$ D_j-supplement of A in W.

Proof . Since B is $\bigoplus$ D_j-supplement of A in W_1, $W_1 = B + A$ and $B \cap A \ll_j D_j(B) W_1 = B \bigoplus B'$ for $B' \subseteq W_1$ since $W = W_1 \bigoplus W_2$, then $W = (B + A) \bigoplus W_2$, hence $W = A + (B \bigoplus W_2)$ and $(B \bigoplus W_2) \cap A = (B \bigoplus W_2) \cap W_1 \cap A = B \cap A \ll_j D_j(B)$ and since $B \subseteq B \bigoplus W_2$, then $D_j(B) \subseteq D_j(B \bigoplus W_2)$ and $W = W_1 \bigoplus W_2 = (B \bigoplus B') \bigoplus W_2 = B \bigoplus W_2 \bigoplus B'$, therefore $B \bigoplus W_2$ is $\bigoplus$ D_j-supplement of A in W. □

Proposition 2.7. Let any module W has sub-modules B and A, if B is $\bigoplus$ D_j-supplement of A in W, Similarly for each completely invariant sub-module N of W such that $N \subseteq A$, $\frac{B+N}{N}$ is $\bigoplus$ D_j-supplement of $\frac{A}{N}$ in $\frac{W}{N}$ [6].

Proof . Let B be a $\bigoplus$ D_j-supplement of A in W, then $W = A + B$ but B is the direct sum of W, then $W = B \bigoplus B'$, $B' \subseteq W$ and $A \cap B \ll_j D_j(B)$

Now $\frac{M}{N} = \frac{A+B}{N} = \frac{A}{N} + \frac{B+N}{N}$. $\frac{A}{N} \cap \frac{B+N}{N} \ll_j D_j\left(\frac{B+N}{N}\right)$, $\frac{A}{N} \cap \frac{B+N}{N} = \frac{A\cap(B+N)}{N} = \frac{(A\cap B)+N}{N}$ (by modular law) since $A \cap B \ll_j D_j(B)$, then $\frac{(A\cap B)+N}{N} \subseteq \frac{D_j(B)+N}{N}$ and $\frac{D_j(B)+N}{N} \ll_j D_j\left(\frac{B+N}{N}\right)$ by [3]. Now $W = B \bigoplus B'$ and N is a fully invariant of, then $\frac{W}{N} = \frac{B+N}{N} \bigoplus \frac{B'+N}{N}$ by [2]; therefore $\frac{B+N}{N}$ is $\bigoplus D_j$-'supplement of $\frac{A}{N}$ in $\frac{W}{N}$. □

Proposition 2.8. Let W be any R-module. If A has $\bigoplus D_j$-supplement in W, then $\frac{A}{N}$ has $\bigoplus D_j$-supplement in $\frac{W}{N}$ where N is a totally invariant of W in such a way that $N \subseteq A$.

Proof . Since A has $\bigoplus D_j$-supplement in W, there is a straight summation B of W such that $B + A = W$ and, $B \cap A \ll_j D_j(B)$ with $W = B \bigoplus B'$, $B' \subseteq W$ for $B' \leq W$. Now $\frac{W}{N} = \frac{A}{N} + \frac{B+N}{N}$, $\frac{A}{N} \cap \frac{B+N}{N} = \frac{A \cap (B+N)}{N} = \frac{(B \cap A)+N}{N}$ (by modular law). Since $B \cap A \ll_j D_j(B)$, we have $\frac{(B \cap A)+N}{N} \subseteq \frac{D_j(B)+N}{N}$ and $\frac{D_j(B)+N}{N} \ll_j D_j\left(\frac{B+N}{N}\right)$, by [4], therefore $\frac{B+N}{N} \bigoplus D_j$-supplement of $\frac{A}{N}$ in $\frac{W}{N}$. $\square$

Proposition 2.9. Let A, B be sub-modules of a module W and Let B be a $\bigoplus D_j$-supplement of A in W if $C \ll_j W$ and $D_j(B)$ direct summand of B then B is $\bigoplus D_j$-supplement of $A + C$ [7].

Proof . Let B be a $\bigoplus D_j$-supplement of A in W, then $W = B + A$, and B is a direct summand of W with $B \cap A \ll_j D_j(B)$ with $W = B \bigoplus B'$ for $B' \subseteq W$ so $B + (A + C) = W$. To show $B \cap (A + C) \ll_j D_j(B)$ let $B \cap (A + C) + N = B$ with $J\left(\frac{B}{N}\right) = \frac{B}{N}$ so $W = B \cap (A + C) + N + (A + C) = N + (A + C) = (A + N) + C$ since $\frac{W}{A+X} = \frac{B+(A+C)+N}{A+N} = \frac{B+(A+N)}{(A+N)} \cong \frac{B}{B+(A+N)} = \frac{B}{N+(A \cap B)}$ (second isomorphism and modular law are used to figure out how to make things that look like each other) and $J\left(\frac{B}{N}\right) = \frac{B}{N}$ we get $J\left(\frac{B}{N+(A \cap B)}\right) = \frac{B}{N+(A \cap B)}$ by [4]. Hence $J\left(\frac{W}{A+N}\right) = \frac{W}{A+N}$, but $C \ll_j W$, then $W = A + N$ so $B = B \cap W = B \cap (A + N) = B \cap A + N$ (by modular law) but $B \cap A \ll_j D_j(B) \subseteq B$, then by [4] $B \cap A \ll_j B$ and since $J\left(\frac{B}{N}\right) = \frac{B}{N}$, therefore $N = B$. So $B \cap (A + C) \ll_j B$ so $B \cap (A + C) \subseteq D_j(B)$ and $D_j(B) \subset_{\bigoplus} B$ so by [4], $B \cap (A + C) \ll_j D_j(B)$. $\square$

Theorem 2.10. Let $W = W_1 \bigoplus W_2$ is an R-module. Then W_2 is $\bigoplus D_j$-supplemented module if and only if a direct summand B of W exists, $B \subseteq W_2$, $W = B + A$ and $B \cap A \ll_j D_j(B)$ for every sub-module $\frac{A}{W_1}$ of $\frac{W}{W_1}$.

Proof . Let $\frac{A}{W_1}$ be any sub-module of $\frac{W}{W_1}$ Since $A \cap W_2 \subseteq W_2$ and W_2 is $\bigoplus D_j$-supplemented module, there exists a direct summand B of W_2 such that $W_2 = (A \cap W_2) + B$ and $A \cap W_2 \cap B = A \cap B \ll_j D_j(B)$ with $W_2 = B \bigoplus B'$ for $B' \subseteq W_2$. Now $W = W_1 + W_2 = W_1 + (A \cap W_2) + B \subseteq W_1 + A + B$ but, $W_1 \subseteq A$; therefore $W = B + A$. So we get the result. Conversely let A be a sub-module of W_2. Consider the sub-module $\frac{A \bigoplus W_1}{W_1}$ of $\frac{W}{W_1}$ By our assumption there exists a straight summand B of W such that $B \subseteq W_2$, $W = (A + W_1) + B$ and $(A + W_1) \cap B \ll_j D_j(B)$, $W = B \bigoplus B'$ for $B' \subseteq W$. Since $W_2 = W_2 \cap W = W_2 \cap [(A + W_1) + B] = B + [(A + W_1) \cap W_2] = B + A + (W_1 \cap W_2) = B + A$ by (modular law) since $\cap A \subseteq (A + W_1) \cap B \ll_j D_j(B)$, then $B \cap A \ll_j D_j(B)$ and $W_2 \cap W = W_2 \cap (B \bigoplus B') = B \bigoplus W_2 \cap B'$ by (modular law). $W = B \bigoplus B' W_2 = W \cap W_2 = (B \bigoplus B') \cap W_2 = B \bigoplus (B' \cap W_2)$ (by modular law), so $B \hookrightarrow_{\bigoplus} W_2$. Therefore W_2 is $\bigoplus D_j$-supplemented module. $\square$

3 Cofinitely $\bigoplus D_j$-supplemented modules

A sub-module N of W is called cofinite if $\frac{W}{N}$ is finitely generated. In this section we introduce Cofinitely $\bigoplus D_j$-supplemented modules.

Definition 3.1. Let W be any R-module and let N a sub-module of W. We say that W is a cofinitely $\bigoplus D_j$-supplemented module if for each cofinite sub-module N of W there exists a direct summand B of W such that $W = N + B$ and $N \cap B \subseteq D_j(B)$.

Theorem 3.2. Let W be a cofinite $\bigoplus D_j$-supplemented module and let A be a fully invariant sub-module of W, then $\frac{W}{A}$ is cofinite $\bigoplus D_j$-supplemented module.

Proof . Let A be a fully invariant sub-module of W and $\frac{B}{A}$ any cofinite sub-module of $\frac{W}{A}$, then $\frac{W}{B} \cong \frac{\frac{W}{A}}{\frac{B}{A}}$. Therefore $\frac{W}{B}$ is finitely generated, hence B cofinite sub-module of W. Since W is cofinite $\bigoplus D_j$-supplemented module, there is a direct summand N of W such that $W = B + N$, $B \cap N \ll_j D_j(N) W = N \bigoplus N'$, $N' \subseteq W$ Now $\frac{W}{A} = \frac{B+N}{A} = \frac{B}{A} + \frac{N+A}{A}$ and $\frac{B}{A} \cap \frac{N+A}{A} \ll_j D_j\left(\frac{N+A}{A}\right) \frac{B}{A} \cap \frac{N+A}{A} = \frac{B \cap (N+A)}{A} = \frac{(B \cap N)+A}{A}$ by (modular law) since $B \cap N \ll_j D_j(N)$, then $\frac{(B \cap N)+A}{A} \subseteq \frac{D_J(N)+A}{A}$ and $\frac{D_J(N)+A}{A} \ll_j D_j\left(\frac{N+A}{A}\right)$ by [3] and $W = N \bigoplus N'$, then $\frac{W}{A} = \frac{N \bigoplus N'}{A} = \frac{N+A}{A} \bigoplus \frac{N'+A}{A}$ thus $\frac{W}{A}$ is cofinite $\bigoplus D_j$-supplemented module. $\square$

Theorem 3.3. Let W be a cofinite $\bigoplus D_j$-supplemented modules and let $N \subseteq W$ if for every direct summand K of $W \frac{N+K}{N}$ is direct summand of $\frac{W}{N}$, then $\frac{W}{N}$ cofinite $\bigoplus D_j$-supplemented module.

Proof . For $N \subseteq A$, let $\frac{A}{N}$ be a cofinite sub-module of $\frac{W}{N}$, then A is a cofinite sub-module of W since W is cofinite $\bigoplus D_j$-supplemented modules so there exists direct summand B of W such that $W = A + B = B \bigoplus B'$ and $A \cap B \ll_j D_j(B)$ for some $B' \subseteq W$ Now $\frac{W}{N} = \frac{A}{N} + \frac{B+N}{N}$ by hypothesis $\frac{B+N}{N}$ is direct summand of $\frac{W}{N}$, Note that $\frac{A}{N} \cap \frac{B+N}{N} = \frac{A \cap B+N}{N} = \frac{N+(B\cap A)}{N}$ since $A \cap B \ll_j D_j(B)$, then $\frac{((B\cap A)+N)}{N} \ll_j D_j\left(\frac{B+N}{N}\right)$, therefore $\frac{W}{N}$ is a cofinite $\bigoplus D_j$-supplemented modules. □

Theorem 3.4. Let W be a module and B be a completely invariant sub-module of W if W is a cofinitely invariant module. $\bigoplus D_j$-supplemented module, then $\frac{W}{B}$ is a cofinitely $\bigoplus D_j$-supplemented module moreover if B is cofinitely direct summand of W, then B is also cofinitely $\bigoplus D_j$-supplemented module.

Proof . Suppose that W is cofinitely $\bigoplus D_j$-supplemented then by Theorem 2.3, $\frac{W}{B}$ is a cofinitely $\bigoplus D_j$-supplemented direct summary of $\frac{W}{B}$. Therefore $\frac{W}{B}$ is a cofinitely $\bigoplus D_j$-supplemented. Now suppose that B is a cofinitely direct summand of W, so there is a sub-module $B' \subseteq W$ such that $W = B \bigoplus B'$ and B' finitely generated because $\frac{W}{B}$ is finitely generated. Let C be a cofinite sub-module of B, thus C is a cofinite sub-module of W, but W cofinitely $\bigoplus D_j$-supplemented sub-module, so there exists $A,\ A'W$ such that $W = A \bigoplus A',\ W = C + A,\ C \cap A \ll_j D_j(A)$. $B = C + (B \cap A),\ B = (B \cap A) \bigoplus (B \cap A')$ and $C \cap (B \cap A) = C \cap A \ll_j D_j(A)$. □

Definition 3.5. An R-module W is called a cofinitely $\bigoplus D_j$-supplemented module if every cofinite sub-module of W has a D_j-supplement that is a direct summand of W. Clearly Z_6 as a Z-module is a cofinitely $\bigoplus D_j$-supplemented but Z_4 as Z-module is not.

Proposition 3.6. Let $W = W_1 \bigoplus W_2$ be a duo module if W_1 and W_2 are cofinite $\bigoplus D_j$-supplemented, then W is a Cofinite $\bigoplus D_j$-supplemented.

Proof . We have $W = W_1 \bigoplus W_2$. Let A be a cofinite sub-module of W, then $A = A \cap W = A \cap W_1 \bigoplus A \cap W_2$, so $\frac{W}{A} \simeq \frac{W_1}{W_1 \cap A} \bigoplus \frac{W_2}{W_2 \cap A}$ and $\frac{W_1}{W_1 \cap A}$ and $\frac{W_2}{W_2 \cap A}$ are finitely generated, thus $W_1 \cap A$ is cofinite in W_1 and $W_2 \cap A$ is cofinite in $W_2 \cap A$, but W_1 and W_2 are cofinite $\bigoplus D_j$-supplemented, so there exists $B_1,\ B_2 K_1 \subset_{\bigoplus} W_1$ and $B_2 \subset_{\bigoplus} W_2$ such that $W_1 = A \cap W_1 + B_1,\ A \cap W_1 \cap B_1 \ll_j D_j(W_1)$ with $W_1 = B_1 \bigoplus B$ for $B \leq W_1$ and $W_2 = A \cap W_2 + B_2,\ A \cap W_2 \cap B_2 \ll_j D_j(W_2)$ and $W_2 = B_2 \bigoplus B_2'$, Now $W = W_1 \bigoplus W_2 = A \cap W_1 + B_1 + A \cap W_2 + B_2 = A \cap W_1 + A \cap W_2 + B_1 + B_2 = A + (B_1 + B_2),\ A \cap (B_1 + B_2) = (A \cap W_1 \cap B_1) + (A \cap W_2 \cap B_2) \ll_j D_j(W_1) + D_j(W_2) \leq D_j(W_1 + W_2) W = W_1 \bigoplus W_2 = B_1 \bigoplus B_2' \bigoplus B_2 \bigoplus B_2' = (B_1 \bigoplus B_2) \bigoplus (B_2 \bigoplus B_2')$. □

Corollary 3.7. Any direct sum of cofinitely $\bigoplus D_j$-supplemented modules is cofinitely $\bigoplus D_j$-supplemented.

Proof . Let $B,\ A$ and N be sub-modules of W if $(B \cap A) = (N + B) \cap (N + A)$, or $N \cap (B + A) = (N \cap B) + (N \cap A)$, then W is called distributive [2]. □

Theorem 3.8. Let W be a cofinite $\bigoplus D_j$-supplemented distributive module, then $\frac{W}{N}$ is cofinite $\bigoplus D_j$-supplemented module for every sub-module N of W.

Proof . Let A be a direct summand of W, then $W = A \bigoplus A'$ for some sub-module A' of W. Now , $\frac{W}{N} = \left[\frac{A+N}{N}\right] + \left[\frac{A'+N}{N}\right]$ and $N = N + (A \cap A') = (N + A) \cap (N + A')$ by distribute of W this implies that $\frac{W}{N} = \frac{A+N}{N} \bigoplus \frac{A'+N}{N}$, so $\frac{W}{N}$ is cofinite $\bigoplus D_j$-supplemented module. □

Corollary 3.9. Let W be a cofinite $\bigoplus D_j$-supplemented distributive module, as a result, every direct sum of W is a cofinite $\bigoplus D_j$-supplemented module.

Assuming that N and B are direct summands of W, then the sum of $N + B$ will likewise be W [1]. This is known as the Summand Sum Property (SSP).

Theorem 3.10. Let W be a cofinite $\bigoplus D_j$-supplemented module with the (ssp), if W is cofinite, then every direct sum of W is $\bigoplus D_j$-supplemented module.

Proof . Since N provide a brief summary of W, let $W = N \bigoplus N'$ for some $N' \subseteq W$. Assume that A is a direct sum of W, since W has (ssp) $A+N'$ is direct sum of W. Let $W = (A+N') \bigoplus B$ for some $B \subseteq W$, then $\frac{W}{N'} = \frac{A+N'}{N'} \bigoplus \frac{(K+N')}{N'}$, therefore $\frac{W}{N'} \simeq N$ is cofinite $\bigoplus D_j$-supplemented. □

Recall the property (D_3), means that if W_1 and W_2 are direct summands of W and $W = W_1 + W_2$, then $W_1 \cap W_2$ is also a straight-forward summation of W [2].

Proposition 3.11. Let W be a cofinite $\bigoplus D_j$-supplemented a module with (D_3), then every cofinite straight-forward summation of W is cofinite $\bigoplus D_j$-supplemented.

Proof . Since N is a cofinite straight-forward summation of W, $W = N \bigoplus \acute{N}$ for $\acute{N} \underline{\hookrightarrow} W$. So $\acute{N}$ is created indefinitely, since A is a cofinite sub-module of N, we have $\frac{W}{A} = \frac{N \bigoplus \acute{N}}{A} = \frac{N}{A} \bigoplus \frac{\acute{N}+A}{A} \frac{\frac{W}{A}}{\frac{N}{A}} \simeq \frac{W}{A}$ which is finitely generated. Thus $\frac{\acute{N}+A}{A}$ is finitely generated and hence $\frac{W}{A}$ is finitely generated. So A has a cofinite of W submodule, thus $\exists B \hookrightarrow_{\bigoplus} W$ such that $W = A + B = B \bigoplus B'$ for $B' \underline{\hookrightarrow} W$ and $A \cap B \ll_j D_j(B)$ hence $N = N \cap W = N \cap (A+B) = A + (N \cap B)$, by $(D_3) N \cap B \hookrightarrow_{\bigoplus} W$ and hence $N \cap B \hookrightarrow_{\bigoplus} N$ and $\cap N \cap B = A \cap B \ll_j D_j(B \cap N)$. □

References

[1] J.L. Garcia, *Properties of direct summands of modules*, Commun. Algebra **17** (1989), no. 1, 73–92.

[2] A. Haramanci, D. Keskin and P.F. Smith, *On $\bigoplus$-supplemented Modules*, Acta Math. Hungar. **83** (1999), no. 1, 161–169.

[3] A.A. Hussein and W. Khalid, *$\bigoplus -RadJ$-supplemented modules*, J. Phys.: Conf. Ser. IOP Publishing, **1818** (2021), no. 1, p. 012203.

[4] A. Kabban and W. Khalid, *On Jacobson-small sub-modules*, Iraqi J. Sci. **60** (2019), no. 7, 1584–1591.

[5] F. Kasch, *Modules and rings*, Academic press, 1982.

[6] M.T. Koşan, *Generalized cofinitely semiperfect modules*, Int. Electr. J. Algebra **5** (2009), no. 5, 58–69.

[7] R. Wisbauer, *Foundations of module and ring theory*, Gordon and Breach, Philadelphia, 1991.