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# Fixed points and stability of new approximation algorithms for contractive-type operators in normed linear spaces

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**Abstract**

In this paper, new iterative schemes called DI-iterative scheme, Chugh-DI iterative scheme and IH-iterative scheme are introduced and studied. In addition, convergence and stability results were obtained without necessarily imposing sum conditions on the iteration parameters, which, among other things, increase the bulkiness and complexity of computations as was the case for several works studied so far in the literature.

Keywords: Strong convergence, DI-iterative scheme, Chugh-DI iterative scheme, IH-iterative scheme, Stability, Contractive operator, fixed point, Normed linear space.
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## 1 Introduction

Let $(Y, \rho)$ be a complete metric space and $\Gamma : Y \longrightarrow Y$ a selfmap of $Y$. Suppose that $F_\Gamma = \{q \in Y : \Gamma q = q\}$ is the set of fixed points of $\Gamma$.

In the last few years, many interesting iterative schemes for which the fixed points of operators could be approximated have been developed and used in literature, see for example, [2], [9], [12]-[23], [24]-[30] and the references therein for more details. Following the Kirk's remarkable iterative algorithm of 1971, the iterative schemes below have been studied extensively by different researchers:

Let $X$ be a normed linear space and $\Gamma : X \longrightarrow X$ be a self-map on $X$.

$(I)$ For arbitrarily $y_0 \in X$, define the sequence $\{y_n\}_{n=0}^{\infty}$ iteratively as follows:

$$y_{n+1} = \sum_{j=0}^{\ell} \alpha_j \Gamma^j y_n, \sum_{j=0}^{\ell} \alpha_j = 1, n \geq 0. \tag{1.1}$$

The iteration method defined by (1.1) is due to Kirk [15].

$(II)$ In [20], Olatinwo presented the Foowing iterative schemes:

(a) for an arbitrary point $y_0 \in X$ and for $\alpha_{n,j} \geq 0, \alpha_{n,0} \neq 0, \alpha_{n,j} \in [0,1]$ *and* $\ell$ as a fixed integer, define the sequence $\{y_n\}_{n=0}^{\infty}$ by

$$y_{n+1} = \sum_{j=0}^{\ell} \alpha_{n,j} \Gamma^j y_n, \sum_{j=0}^{\ell} \alpha_{n,j} = 1, n \geq 0 \tag{1.2}$$

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(b) for an arbitrary point $y_0 \in X$ and for $\ell \geq m, \alpha_{n,j}\,\beta_{n,j} \geq 0, \alpha_{n,0}, \beta_{n,0} \neq 0, \alpha_{n,j}\beta_{n,j} \in [0,1]$ *and* $\ell, m$ as fixed integers, define the sequence $\{y_n\}_{n=0}^{\infty}$ by

$$\begin{aligned} y_{n+1} &= \alpha_{n,0}y_n + \sum_{j=0}^{\ell}\alpha_{n,j}\Gamma^j z_n, \sum_{j=0}^{\ell}\alpha_{n,j} = 1; \\ z_n &= \sum_{j=0}^{m}\beta_{n,j}\Gamma^j y_n, \sum_{j=0}^{\ell}\beta_{n,j} = 1, n \geq 0, \end{aligned} \tag{1.3}$$

and called them Kirk-Mann and Kirk-Ishikawa iterative scheme respectively.

(III) In 2012, Chugh and Kumar [8] presented the following Kirk-Noor-type iterative scheme: for an arbitrary point $y_0 \in X$ and for $\ell \geq m \geq p, \alpha_{n,j}, \gamma_{n,k}, \beta_{n,i} \geq 0, \gamma_{n,0}, \alpha_{n,0}, \beta_{n,0} \neq 0, \alpha_{n,j}, \gamma_{n,k}, \beta_{n,i} \in [0,1]$ *and* $\ell, m, p$ as fixed integers, define the sequence $\{y_n\}_{n=0}^{\infty}$ by

$$\begin{aligned} y_{n+1} &= \gamma_{n,0}y_n + \sum_{k=1}^{\ell}\gamma_{n,k}\Gamma^k z_n, \sum_{k=0}^{\ell}\gamma_{n,k} = 1; \\ z_n &= \alpha_{n,0}y_n + \sum_{j=1}^{m}\alpha_{n,j}\Gamma^j t_n, \sum_{j=0}^{m}\alpha_{n,j} = 1; \\ t_n &= \sum_{i=0}^{p}\beta_{n,i}\Gamma^i y_n, \sum_{i=0}^{p}\beta_{n,i} = 1, n \geq 0, \end{aligned} \tag{1.4}$$

(IV) Also, in 2012, Hussain, Chugh, Kummar and Ratig [10] introduced the following iterative schemes in the sense of Kirk [15]:

(i) for an arbitrary point $y_0 \in X$ and for $\ell \geq m \geq p, \alpha_{n,j}, \gamma_{n,k}, \beta_{n,i} \geq 0, \gamma_{n,0}, \alpha_{n,0}, \beta_{n,0} \neq 0, \alpha_{n,j}, \gamma_{n,k}, \beta_{n,i} \in [0,1]$ *and* $\ell, m, p$ as fixed integers, define the sequence $\{y_n\}_{n=0}^{\infty}$ by

$$\begin{aligned} y_{n+1} &= \gamma_{n,0}z_n + \sum_{k=1}^{\ell}\gamma_{n,k}\Gamma^k z_n, \sum_{k=0}^{\ell}\gamma_{n,k} = 1; \\ z_n &= \alpha_{n,0}t_n + \sum_{j=1}^{m}\alpha_{n,j}\Gamma^j t_n, \sum_{j=0}^{m}\alpha_{n,j} = 1; \\ t_n &= \sum_{i=0}^{p}\beta_{n,i}\Gamma^i y_n, \sum_{i=0}^{p}\beta_{n,i} = 1, n \geq 0, \end{aligned} \tag{1.5}$$

(ii) for an arbitrary point $y_0 \in X$, retaining the conditions in (i), define the sequence $\{y_n\}_{n=0}^{\infty}$ by

$$\begin{aligned} y_{n+1} &= \gamma_{n,0}z_n + \sum_{k=1}^{\ell}\gamma_{n,k}\Gamma^k z_n, \sum_{k=0}^{\ell}\gamma_{n,k} = 1; \\ z_n &= \alpha_{n,0}\Gamma t_n + \sum_{j=1}^{m}\alpha_{n,j}\Gamma^j t_n, \sum_{j=0}^{m}\alpha_{n,j} = 1; \\ t_n &= \sum_{i=0}^{p}\beta_{n,i}\Gamma^i y_n, \sum_{i=0}^{p}\beta_{n,i} = 1, n \geq 0, \end{aligned} \tag{1.6}$$

They called the iterative schemes defined by (1.5) and (1.6) Kirk-SP and Kirk-CR iterative schemes, respectively.

In practical sense, convergence of the various iterative schemes studied above is not enough to ascertain their usability, the stability of the schemes is quite priceless. Ostrowski [22] was the first researcher to perform an experiment on this using Banach contractive conditions. Thereafter, other researchers have investigated and developed this subject extensively. Some recent works in this direction could be seen in [2, 4, 5, 7, 8, 11, 13, 21, 18, 22, 19, 29, 28] and the references therein.

**Remark 1.1.** All the iterative schemes for approximating common fixed points of finite families of mappings studied above requires the condition that, for each $n$, $\sum_{k=0}^{\ell}\gamma_{n,k} = 1, \sum_{j=0}^{m}\alpha_{n,j} = 1$ *and* $\sum_{i=0}^{p}\beta_{n,i} = 1$ (see, for example, (1.1) to (1.6)) on the control parameters $\{\{\gamma_{n,k}\}_{n=1}^{\infty}\}_{k=1}^{\ell}, \{\{\alpha_{n,j}\}_{n=1}^{\infty}\}_{j=1}^{m}$ *and* $\{\{\beta_{n,i}\}_{n=1}^{\infty}\}_{i=1}^{p}$, respectively. However, in real life applications, if $\ell, m$ *and* $p$ are very large, it would be very difficult or almost impossible to generate a family of such control parameters. Again, the computational cost of generating such a family of control parameters is quite enormous and also takes a very long process. Consequently, the following question becomes necessary:

**Question 1.1.** Is it possible to construct alternative iterative schemes that would address the problems generated by the sum conditions $\left(\sum_{k=0}^{\ell}\gamma_{n,k} = 1, \sum_{j=0}^{m}\alpha_{n,j} = 1 \ and \ \sum_{i=0}^{p}\beta_{n,i} = 1\right)$ imposed on the control parameters $\{\{\gamma_{n,k}\}_{n=1}^{\infty}\}_{k=1}^{\ell}, \{\{\alpha_{n,j}\}_{n=1}^{\infty}\}_{j=1}^{m} \ and \ \{\{\beta_{n,i}\}_{n=1}^{\infty}\}_{i=1}^{p}$, respectively while maintaining the convergence and stability results of the papers studied?

Following the same argument as in [14] regarding the linear combination of the products of countably finite family of control parameters and the problems identified in each of the iterative schemes studied, the aim of this paper is to provide an affirmative answer to Question 1.1.

## 2 Preliminary

The following definitions, lemmas and propositions will be needed to prove our main results.

**Definition 2.1.** (see [22]) Let (Y, d) be a metric space and let $\Gamma : Y \longrightarrow Y$ be a self-map of $Y$. Let $\{x_n\}_{n=0}^{\infty} \subseteq Y$ be a sequence generated by an iteration scheme

$$x_{n+1} = g(\Gamma, x_n), \tag{2.1}$$

where $x_0 \in Y$ is the initial approximation and $g$ is some function. Suppose that $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point $q$ of $\Gamma$. Let $\{t_n\}_{n=0}^{\infty} \subseteq Y$ be an arbitrary sequence and set $\epsilon_n = d(t_n, g(\Gamma, t_n)), n = 1, 2, \cdots$ Then, the iteration scheme (2.1) is called $\Gamma$-stable if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\to\infty} y_n = q$.

Note that in practice, the sequence $\{t_n\}_{n=0}^{\infty}$ could be obtained in the following manner: let $x_0 \in Y$. Set $x_{n+1} = g(\Gamma, x_n)$ and let $t_0 = x_0$. Now, $x_1 = g(\Gamma, x_0)$ because of rounding in the function $\Gamma$, and a new value $t_1$ (approximately equal to $x_1$) might be calculated to give $t_2$, an approximate value of $g(\Gamma, t_1)$. The procedure is continued to yield the sequence $\{t_n\}_{n=0}^{\infty}$, an approximate sequence of $\{x_n\}_{n=0}^{\infty}$.

**Lemma 2.2.** (see, e.g., [2]) Let $\{\tau_n\}_{n=0}^{\infty}$ be a sequence of positive numbers such that $\tau_n \to 0 \ as \ n \to \infty$. For $0 \le \delta < 1$, let $\{w_n\}_{n=0}^{\infty}$ be a sequence of positive numbers satisfying $w_{n+1} \le \delta w_n + \tau_n, n = 0, 1, 2, \cdots$ Then, $w_n \to 0 \ as \ n \to \infty$.

**Lemma 2.3.** (see, e.g., [20]) Let (Y, $\|\ .\|$) be a normed space, the self-map $\Gamma : Y \longrightarrow Y$ satisfies (1.13) and $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a monotone increasing subadditive function such that $\psi(0) = 0, \psi(Mt) = M\psi(t), M \ge 0, t \in \mathbb{R}^+$. Then, $\forall i \in \mathbb{N} \ and \ \forall s, t \in Y$, we have

$$\|\Gamma^j s - \Gamma^j t\| \le \rho^j \|s - t\| + \sum_{i=0}^{j} \binom{j}{i} \rho^{j-1} \phi(\|s - \Gamma s\|). \tag{2.2}$$

**Proposition 2.4.** (see, e.g., [14]) Let $\{\alpha_i\}_{i=1}^{\infty} \subseteq \mathbb{N}$ be a countable subset of the set of real numbers $\mathbb{R}$, where $k$ is a fixed nonnegative integer and $\mathbb{N}$ is any integer with $k + 1 \le N$. Then, the following holds:

$$\alpha_k + \sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1} (1 - \alpha_j) + \prod_{j=k}^{N} (1 - \alpha_j) = 1. \tag{2.3}$$

**Proposition 2.5.** (see, e.g., [14]) Let $t, u$ and $v$ be arbitrary elements of a real Hilbert space $H$. Let $k$ be any fixed nonnegetive integer and $N \in \mathbb{N}$ be such that $k + 1 \le N$. Let $\{v_i\}_{i=1}^{N-1} \subseteq H$ and $\{\alpha_i\}_{i=1}^{N} \subseteq [0, 1]$ be a countable finite subset of $H$ and $\mathbb{R}$, respectively. Define

$$y = \alpha_k + \sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1} (1 - \alpha_j) v_{i-1} + \prod_{j=k}^{N} (1 - \alpha_j) v.$$

Then,

$$\begin{aligned}\|y-u\|^2 &= \alpha_k\|t-u\|^2+\sum_{i=k+1}^{N}\alpha_i\prod_{j=k}^{i-1}(1-\alpha_j)\|v_{i-1}-u\|^2+\prod_{j=k}^{N}(1-\alpha_j)\|v-u\|^2\\ &\quad -\alpha_k\Big[\sum_{i=k+1}^{N}\alpha_i\prod_{j=k}^{i-1}(1-\alpha_j)\|t-v_{i-1}\|^2+\prod_{j=k}^{i-1}(1-\alpha_j)\|t-v\|^2\Big]\\ &\quad -(1-\alpha_k)\Big[\sum_{i=k+1}^{N}\alpha_i\prod_{j=k}^{i-1}(1-\alpha_j)\|v_{i-1}-(\alpha_{i+1}+w_{i+1})\|^2+\alpha_N\prod_{j=k}^{i-1}(1-\alpha_j)\|v-v_{N-1}\|^2\Big],\end{aligned}$$

where $w_k=\sum_{i=k+1}^{N}\alpha_i\prod_{j=k}^{i-1}(1-\alpha_j)v_{i-1}+\prod_{j=k}^{i-1}(1-\alpha_j)v, k=1,2,\cdots,N$ and $w_n=(1-c_n)v$.

## 3 Main Results I

Let $H$ be a Hilbert space and let $\Gamma: H\longrightarrow H$ be a self-map of H. For arbitrary $y_0\in Y$, we define the following iteratively algorithms for the sequence $\{y_n\}_{n=0}^{\infty}$:

$$\begin{cases} y_{n+1}=\delta_{n,1}t_n+\sum_{\ell=2}^{a}\delta_{n,k}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1}t_n+\prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^a t_n;\\ t_n=\gamma_{n,1}\omega_n+\sum_{t=2}^{b}\gamma_{n,t}\prod_{s=1}^{t-1}(1-\gamma_{n,s})\Gamma^{t-1}\omega_n+\prod_{s=1}^{b}(1-\gamma_{n,s})\Gamma^b\omega_n;\\ \omega_n=\sum_{r=1}^{c}\tau_{n,r}\prod_{i=1}^{r-1}(1-\tau_{n,i})\Gamma^{r-1}y_n+\prod_{i=1}^{c}(1-\tau_{n,i})\Gamma^c y_n, n\geq 0,1,2,..,\end{cases} \tag{3.1}$$

$$\begin{cases} y_{n+1}=\delta_{n,1}t_n+\sum_{\ell=2}^{a}\delta_{n,k}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1}t_n+\prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^a t_n;\\ t_n=\gamma_{n,1}\Gamma y_n+\sum_{t=2}^{b}\gamma_{n,t}\prod_{s=1}^{t-1}(1-\gamma_{n,s})\Gamma^{t-1}\omega_n+\prod_{s=1}^{b}(1-\gamma_{n,s})\Gamma^b\omega_n;\\ \omega_n=\sum_{r=1}^{c}\tau_{n,r}\prod_{i=1}^{r-1}(1-\tau_{n,i})\Gamma^{r-1}y_n+\prod_{i=1}^{c}(1-\tau_{n,i})\Gamma^c y_n, n\geq 0,1,2,..,\end{cases} \tag{3.2}$$

and

$$\begin{cases} y_{n+1}=\delta_{n,1}y_n+\sum_{\ell=2}^{a}\delta_{n,k}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1}t_n+\prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^a t_n;\\ t_n=\gamma_{n,1}y_n+\sum_{t=2}^{b}\gamma_{n,t}\prod_{s=1}^{t-1}(1-\gamma_{n,s})\Gamma^{t-1}\omega_n+\prod_{s=1}^{b}(1-\gamma_{n,s})\Gamma^b\omega_n;\\ \omega_n=\sum_{r=1}^{c}\tau_{n,r}\prod_{i=1}^{r-1}(1-\tau_{n,i})\Gamma^{r-1}y_n+\prod_{i=1}^{c}(1-\tau_{n,i})\Gamma^c y_n, n\geq 0,1,2,..,\end{cases} \tag{3.3}$$

where $\{\{\delta_{n,k}\}_{n=0}^{\infty}\}_{k=1}^{a}, \{\{\gamma_{n,t}\}_{n=0}^{\infty}\}_{t=1}^{b}, \{\{\tau_{n,r}\}_{n=0}^{\infty}\}_{r=1}^{c}$ are countable finite family of real sequences in $[0,1]$ and $a,b,c\in\mathbb{N}$. We shall call the iterative schemes defined by (3.1), (3.2) and (3.3) the DI-iterative scheme, Chugh-DI iterative scheme and IH-iterative scheme respectively.

**Remark 3.1.** If $a=b=c=2, \delta_{n,1}=\delta_n, \delta_{n,2}=\alpha_n, \gamma_{n,1}=\gamma_n, \gamma_{n,2}=\xi_n, \tau_{n,1}=\tau_n, \tau_{n,2}=\rho_n, \Gamma^1=I$ (where $I$ is an identity mapping) and $\Gamma^2=\Gamma$, we obtain the following iteration algorithms:

(*a*) From (3.1), we have

$$\begin{cases} y_{n+1}=\delta_n t_n+(1-\delta_n)[\alpha_n t_n+(1-\alpha_n)\Gamma t_n]\\ t_n=\gamma_n\omega_n+(1-\gamma_n)[\xi_n\omega_n+(1-\xi_n)\Gamma\omega_n]\\ \omega_n=\tau_n y_n+(1-\tau_n)[\rho_n y_n+(1-\rho_n)\Gamma y_n].\end{cases} \tag{3.4}$$

The following well known iteration schemes are consequences of (3.4):

(*i*) if $\alpha_n=\xi_n=\rho_n=0$ in (3.4), we have

$$\begin{cases} y_{n+1}=\delta_n t_n+(1-\delta_n)\Gamma t_n\\ t_n=\gamma_n\omega_n+(1-\gamma_n)\Gamma\omega_n\\ \omega_n=\tau_n y_n+(1-\tau_n)\Gamma y_n,\end{cases} \tag{3.5}$$

and is called SP-iteration scheme.

(*ii*) if $\tau_n=1$ in (3.5), we get

$$\begin{aligned} y_{n+1} &= \delta_n t_n+(1-\delta_n)\Gamma t_n\\ t_n &= \gamma_n\omega_n+(1-\gamma_n)\Gamma\omega_n \end{aligned} \tag{3.6}$$

(b) From (3.2), we have

$$\begin{cases} y_{n+1} = \delta_n t_n + (1-\delta_n)[\alpha_n t_n + (1-\alpha_n)\Gamma t_n]r \\ t_n = \gamma_n \Gamma\omega_n + (1-\gamma_n)[\xi_n\omega_n + (1-\xi_n)\Gamma\omega_n] \\ \omega_n = \tau_n y_n + (1-\tau_n)[\rho_n y_n + (1-\rho_n)\Gamma y_n]. \end{cases} \tag{3.7}$$

Again the following well known iteration scheme is a consequence of (3.7):

(i) if $\alpha_n = \xi_n = \rho_n = 0$ in (3.7), we have

$$\begin{cases} y_{n+1} = \delta_n t_n + (1-\delta_n)\Gamma t_n \\ t_n = \gamma_n \Gamma\omega_n + (1-\gamma_n)\Gamma\omega_n \\ \omega_n = \tau_n y_n + (1-\tau_n)\Gamma y_n, \end{cases} \tag{3.8}$$

and is called CR-iteration scheme and

(c) From (3.3), we have

$$\begin{cases} y_{n+1} = \delta_n y_n + (1-\delta_n)[\alpha_n t_n + (1-\alpha_n)\Gamma t_n] \\ t_n = \gamma_n y_n + (1-\gamma_n)[\xi_n\omega_n + (1-\xi_n)\Gamma\omega_n] \\ \omega_n = \tau_n y_n + (1-\tau_n)[\rho_n y_n + (1-\rho_n)\Gamma y_n], \end{cases} \tag{3.9}$$

so that, if $\alpha_n = \xi_n = \rho_n = 0$ in (3.9), the following well known (Noor) iteration scheme is obtained

$$\begin{cases} y_{n+1} = \delta_n y_n + (1-\delta_n)\Gamma t_n \\ t_n = \gamma_n y_n + (1-\gamma_n)\Gamma\omega_n \\ \omega_n = \tau_n y_n + (1-\tau_n)\Gamma y_n. \end{cases} \tag{3.10}$$

**Theorem 3.2.** Let $H$ be a Hilbert space, $\Gamma : H \longrightarrow H$ be a self-map of $H$ satisfying the contractive condition

$$\|\Gamma^s\xi - \Gamma^s\omega\| \le \nu^s\|\xi - \omega\| + \sum_{j=0}^{s}\binom{s}{j}\rho^{s-j}\phi(\|\xi - \Gamma\xi\|), \tag{3.11}$$

where $\xi, \omega \in H, 0 \le \nu^s < 1$, and let $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a subadditive monotone increasing function with $\phi(0) = 0$ and $\phi(Mt) = M\phi(t), M \ge 0, t \in \mathbb{R}^+$. For arbitrary $y_0 \in H$, let $\{y_n\}_{n=0}^{\infty}$ be the $DI$-iteration scheme defined by (3.1). Then,

(i) $\Gamma$ defined by (3.11) has a fixed point $q$;

(ii) the $DI$-iteration scheme converges strongly to $q \in \Gamma$.

**Proof .** First, we argue that $\Gamma$ satisfying contractive condition (3.11) has a unique fixed point. Assume for contradiction that there exists two points $q_1, q_2 \in F(\Gamma)$ with $q_1 \ne q_2$. Then, we have

$$\begin{aligned} 0 < \|q_1 - q_2\| = \|\Gamma^s q_1 - \Gamma^s q_2\| &\le \nu^s\|q_1 - q_2\| + \sum_{s=1}^{j}\binom{j}{s}\nu^{j-s}\phi(\|q\sqrt{1} - \Gamma q_1\|) \\ &= \nu^s\|q_1 - q_2\| + \sum_{s=1}^{j}\binom{j}{s}\nu^{j-s}\phi(0). \end{aligned}$$

Then, $(1-\nu^s)\|q_1 - q_2\| \le 0$. Using the fact that $\nu^s \in [0, 1)$, we get $0 < 1 - \nu^s$ and $\|q_1 - q_2\| \le 0$. Since the norm is a nonnegative function, we get $\|q_1 - q_2\| = 0; q_1 = q_2 = q(say)$. Therefore, $\Gamma$ has a unique fixed point.

Again, we show that the sequence $\{y_n\}_{n=0}^{\infty}$ generated by (3.1) converges strongly to $q \in F(\Gamma)$. By (3.1) and Proposition 2.5, with $y_{n+1} = y, t_n = t, q = u, k = 1$ *and* $y_{n,\mathbb{N}} \in \Gamma^{k-1}t_n$, we have the following estimates:

$$\|y_{n+1} - q\|^2 \le \delta_{n,1}\|t_n - q\|^2 + \sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})\|\Gamma^{\ell-1}t_n - \Gamma^{\ell-1}q\|^2 + \prod_{p=1}^{a}(1-\delta_{n,p})\|\Gamma^a t_n - \Gamma^a q\|^2. \tag{3.12}$$

But from (3.11), with $\xi = t_n$, we have

$$\begin{aligned}\|\Gamma^{\ell-1}t_n - \Gamma^{\ell-1}q\| &\leq \nu^s\|t_n - q\| + \sum_{j=1}^{s}\binom{s}{j}\nu^{s-j}\phi(\|q-\Gamma q\|) \\ &= \nu^s\|t_n - q\|. \end{aligned} \tag{3.13}$$

Using (3.12) and (3.13), we get

$$\|y_{n+1} - q\|^2 = \Big[\delta_{n,1} + \sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})(\nu^s)^2 + \prod_{p=1}^{a}(1-\delta_{n,p})(\nu^s)^2\Big]\|t_n - q\|^2. \tag{3.14}$$

Also,

$$\|t_n - q\|^2 = \gamma_{n,1}\|\omega_n - q\|^2 + \sum_{t=2}^{b}\gamma_{n,t}\prod_{s=1}^{t-1}(1-\gamma_{n,s})\|\Gamma^{t-1}\omega_n - \Gamma^{t-1}q\|^2 + \prod_{s=1}^{b}(1-\gamma_{n,s})\|\Gamma^b\omega_n - \Gamma^b q\|^2. \tag{3.15}$$

But from (3.11), with $\xi = \omega_n$, we have

$$\begin{aligned}\|\Gamma^{t-1}\omega_n - \Gamma^{t-1}q\| &\leq \nu^s\|\omega_n - q\| + \sum_{j=1}^{s}\binom{s}{j}\nu^{s-j}\phi(\|q-\Gamma q\|) \\ &= \nu^s\|\omega_n - q\|. \end{aligned} \tag{3.16}$$

From (3.15) and (3.16), we obtain

$$\|t_n - q\|^2 = \Big[\gamma_{n,1} + \sum_{t=2}^{b}\gamma_{n,t}\prod_{s=1}^{t-1}(1-\gamma_{n,s})(\nu^s)^2 + \prod_{s=1}^{b}(1-\gamma_{n,s})(\nu^s)^2\Big]\|\omega_n - q\|^2. \tag{3.17}$$

Moreover,

$$\|\omega_n - q\|^2 = \tau_{n,1}\|y_n - q\|^2 + \sum_{r=2}^{b}\tau_{n,r}\prod_{i=1}^{r-1}(1-\tau_{n,i})\|\Gamma^{r-1}y_n - \Gamma^{r-1}q\|^2 + \prod_{i=1}^{c}(1-\tau_{n,i})\|\Gamma^c y_n - \Gamma^c q\|^2. \tag{3.18}$$

Using (3.11), with $\xi = y_n$, and following the same approach as was used to obtain (3.17), we get

$$\|\omega_n - q\|^2 = \Big[\tau_{n,1} + \sum_{r=2}^{c}\tau_{n,r}\prod_{i=1}^{r-1}(1-\tau_{n,i})(\nu^s)^2 + \prod_{i=1}^{c}(1-\tau_{n,i})(\nu^s)^2\Big]\|y_n - q\|^2. \tag{3.19}$$

Now, (3.12), (3.15) and (3.19) imply that

$$\begin{aligned}\|y_{n+1} - q\|^2 = {} & \Big(\delta_{n,1} + \sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})(\nu^s)^2 + \prod_{p=1}^{a}(1-\delta_{n,p})(\nu^s)^2\Big)\Big(\gamma_{n,1} + \sum_{t=2}^{b}\gamma_{n,t}\prod_{s=1}^{t-1}(1-\gamma_{n,s})(\nu^s)^2 \\ & + \prod_{s=1}^{b}(1-\gamma_{n,s})(\nu^s)^2\Big)\Big(\tau_{n,1} + \sum_{r=2}^{c}\tau_{n,r}\prod_{i=1}^{r-1}(1-\tau_{n,i})(\nu^s)^2 + \prod_{i=1}^{c}(1-\tau_{n,i})(\nu^s)^2\Big)\|y_n - q\|^2. \end{aligned} \tag{3.20}$$

Using Lemma 2.3 and the fact that $\nu^s \in [0, 1)$, we have

$$A < B = 1, \tag{3.21}$$

where

$$A = \Big(\delta_{n,1} + \sum_{\ell=2}^{a} \delta_{n,\ell} \prod_{p=1}^{\ell-1}(1-\delta_{n,p})(\nu^s)^2 + \prod_{p=1}^{a}(1-\delta_{n,p})(\nu^s)^2\Big)\Big(\gamma_{n,1} + \sum_{t=2}^{b} \gamma_{n,t} \prod_{s=1}^{t-1}(1-\gamma_{n,s})(\nu^s)^2 + \prod_{s=1}^{b}(1-\gamma_{n,s})(\nu^s)^2\Big)\Big(\tau_{n,1} + \sum_{r=2}^{c} \tau_{n,r} \prod_{i=1}^{r-1}(1-\tau_{n,i})(\nu^s)^2 + \prod_{i=1}^{c}(1-\tau_{n,i})(\nu^s)^2\Big)$$

and

$$B = \Big(\delta_{n,1} + \sum_{\ell=2}^{a} \delta_{n,\ell} \prod_{p=1}^{\ell-1}(1-\delta_{n,p}) + \prod_{p=1}^{a}(1-\delta_{n,p})\Big)\Big(\gamma_{n,1} + \sum_{t=2}^{b} \gamma_{n,t} \prod_{s=1}^{t-1}(1-\gamma_{n,s}) + \prod_{s=1}^{b}(1-\gamma_{n,s})\Big)\Big(\tau_{n,1} + \sum_{r=2}^{c} \tau_{n,r} \prod_{i=1}^{r-1}(1-\tau_{n,i}) + \prod_{i=1}^{c}(1-\tau_{n,i})\Big).$$

By Lemma 2.2 and (3.21), we obtain from (3.20) that $y_n \to q \; as \; n \to \infty$; that is, DI-iteration scheme converges strongly to $q \in F(\Gamma)$. □

**Theorem 3.3.** Let $H$ be a Hilbert space, $\Gamma : H \longrightarrow H$ be a self-map of $H$ satisfying the contractive condition

$$\|\Gamma^s\xi - \Gamma^s\omega\| \le \nu^s\|\xi - \omega\| + \sum_{j=0}^{s} \binom{s}{j} \rho^{s-j}\phi(\|\xi - \Gamma\xi\|), \tag{3.22}$$

where $\xi, \omega \in H, 0 \le \nu^s < 1$, and let $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a subadditive monotone increasing function with $\phi(0) = 0$ and $\phi(Mt) = M\phi(t), M \ge 0, t \in \mathbb{R}^+$. For arbitrary $y_0 \in H$, let $\{y_n\}_{n=0}^{\infty}$ be the Chugh-$DI$iteration scheme defined by (3.2). Then,

(*i*) $\Gamma$ defined by (3.22) has a fixed point $q$;
(*ii*) The Chugh-$DI$ iteration scheme converges strongly to $q \in \Gamma$.

**Proof .** Assume for contradiction that there exists two points $q_1, q_2 \in F(\Gamma)$ with $q_1 \neq q_2$. Then, following the same technique as in (*i*) of Theorem 3.2, we have $q_1 = q_2 = q( \; say \;)$. Hence, $q$ is the unique fixed point of $\Gamma$. Next, we prove that $x_n \to q \; as \; n \to \infty$. Now, by (3.2), Proposition 2.4 and (3.22), we get

$$\|y_{n+1} - q\|^2 = \Big[\delta_{n,1} + \sum_{\ell=2}^{a} \delta_{n,\ell} \prod_{p=1}^{\ell-1}(1-\delta_{n,p})(\nu^s)^2 + \prod_{p=1}^{a}(1-\delta_{n,p})(\nu^s)^2\Big]\|t_n - q\|^2. \tag{3.23}$$

Again, from (3.2), (3.22) and Proposition 2.5, with $k = 1, t_n = y_n, \Gamma y_n = t, y_{n,\mathbb{N}} \in \Gamma^{j-1}$, , we get

$$\begin{aligned}\|t_n - q\|^2 &= \gamma_{n,1}\|\Gamma y_n - q\|^2 + \sum_{t=2}^{b} \gamma_{n,t} \prod_{s=1}^{t-1}(1-\gamma_{n,s})\|\Gamma^{t-1}\omega_n - \Gamma^{t-1}q\|^2 + \prod_{s=1}^{b}(1-\gamma_{n,s})\|\Gamma^b\omega_n - \Gamma^b q\|^2 \\ &\le \nu\gamma_{n,1}\|y_n - q\|^2 + \sum_{t=2}^{b} \gamma_{n,t} \prod_{s=1}^{t-1}(1-\gamma_{n,s})\|\Gamma^{t-1}\omega_n - \Gamma^{t-1}q\|^2 + \prod_{s=1}^{b}(1-\gamma_{n,s})\|\Gamma^b\omega_n - \Gamma^b q\|^2,\end{aligned}$$

which by (3.16) yields

$$\|t_n - q\|^2 = \nu\gamma_{n,1}\|y_n - q\|^2 + \Big[\sum_{t=2}^{b} \gamma_{n,t} \prod_{s=1}^{t-1}(1-\gamma_{n,s})(\nu^s)^2 + \prod_{s=1}^{b}(1-\gamma_{n,s})(\nu^s)^2\Big]\|\omega_n - q\|^2. \tag{3.24}$$

Furthermore, using the method as was used to obtain (3.19) in Theorem 3.2, we obtain

$$\|\omega_n - q\|^2 = \Big[\tau_{n,1} + \sum_{r=2}^{c} \tau_{n,r} \prod_{i=1}^{r-1}(1-\tau_{n,i})(\nu^s)^2 + \prod_{i=1}^{c}(1-\tau_{n,i})(\nu^s)^2\Big]\|y_n - q\|^2. \tag{3.25}$$

(3.23), (3.24) and (3.25) yield

$$\|y_{n+1}-q\|^2 =\Big(\delta_{n,1}+\sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})(\nu^s)^2+\prod_{p=1}^{a}(1-\delta_{n,p})(\nu^s)^2\Big)\times\Big\{\nu\gamma_{n,1}+\Big(\sum_{t=2}^{b}\gamma_{n,t}\prod_{s=1}^{t-1}(1-\gamma_{n,s})(\nu^s)^2 + \prod_{s=1}^{b}(1-\gamma_{n,s})(\nu^s)^2\Big)\Big(\tau_{n,1}+\sum_{r=2}^{c}\tau_{n,r}\prod_{i=1}^{r-1}(1-\tau_{n,i})(\nu^s)^2+\prod_{i=1}^{c}(1-\tau_{n,i})(\nu^s)^2\Big)\Big\}\|y_n-q\|^2. \tag{3.26}$$

Using Proposition 2.4 and the fact that $\nu^s \in [0,1)$, we have

$$A^\star < B^\star = 1, \tag{3.27}$$

where

$$\begin{aligned} A^\star &= \Big(\delta_{n,1}+\sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})(\nu^s)^2+\prod_{p=1}^{a}(1-\delta_{n,p})(\nu^s)^2\Big)\\ &\quad\times\Big\{\nu\gamma_{n,1}+\Big(\sum_{t=2}^{b}\gamma_{n,t}\prod_{s=1}^{t-1}(1-\gamma_{n,s})(\nu^s)^2+\prod_{s=1}^{b}(1-\gamma_{n,s})(\nu^s)^2\Big)\Big(\tau_{n,1}\\ &\quad+\sum_{r=2}^{c}\tau_{n,r}\prod_{i=1}^{r-1}(1-\tau_{n,i})(\nu^s)^2+\prod_{i=1}^{c}(1-\tau_{n,i})(\nu^s)^2\Big)\Big\} \end{aligned}$$

and

$$\begin{aligned} B^\star &= \Big(\delta_{n,1}+\sum_{k=2}^{a}\delta_{n,k}\prod_{p=1}^{k-1}(1-\delta_{n,p})+\prod_{p=1}^{a}(1-\delta_{n,p})\Big)\Big(\gamma_{n,1}+\sum_{t=2}^{b}\gamma_{n,t}\prod_{s=1}^{t-1}(1-\gamma_{n,s})\\ &\quad+\prod_{s=1}^{b}(1-\gamma_{n,s})\Big)\Big(\tau_{n,1}+\sum_{r=2}^{c}\tau_{n,r}\prod_{i=1}^{r-1}(1-\tau_{n,i})+\prod_{i=1}^{c}(1-\tau_{n,i})\Big) \end{aligned}$$

By Lemma 2.2 and (3.27), it follows from (3.26) that $y_n \to q \ as \ n \to \infty$; that is, Chugh-DI iteration scheme converges strongly to $q \in F(\Gamma)$. □

**Theorem 3.4.** Let $H$ be a normed space, $\Gamma : H \longrightarrow H$ be a self-map of $H$ satisfying the contractive condition

$$\|\Gamma^s\xi-\Gamma^s\omega\| \le \nu^s\|\xi-\omega\|+\sum_{j=0}^{s}\binom{s}{j}\rho^{s-j}\phi(\|\xi-\Gamma\xi\|), \tag{3.28}$$

where $\xi,\omega \in H, 0 \le \nu^s < 1$, and let $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a subadditive monotone increasing function with $\phi(0) = 0$ and $\phi(Mt) = M\phi(t), M \ge 0, t \in \mathbb{R}^+$. For arbitrary $y_0 \in H$, let $\{y_n\}_{n=0}^\infty$ be the $IH$-iteration scheme defined by (3.3). Then,

(*i*) $\Gamma$ defined by (3.33) has a unique fixed point $q$;
(*ii*) The $IH$-iteration scheme converges strongly to $q \in \Gamma$.

**Proof .** (*i*) The result follows immediately as in the proof of (*i*) in Theoerem 3.2.
(*ii*) To show that the sequence $\{y_n\}_{n=0}^\infty$ generated by (3.3) converges strongly to a point $q \in F(\Gamma)$, we shall use (3.3), Proposition 2.5, with $y_{n+1} = y, y_n = t, q = u, k = 1, y_{n,\mathbb{N}} \in \Gamma^{k-1}t_n$ and then estimate as follows:

$$\|y_{n+1}-q\|^2 \le \delta_{n,1}\|y_n-q\|^2+\sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})\|\Gamma^{\ell-1}t_n-\Gamma^{\ell-1}q\|^2+\prod_{p=1}^{a}(1-\delta_{n,p})\|\Gamma^a t_n-\Gamma^a q\|^2,$$

which on the application of (3.16) yields

$$\|y_{n+1}-q\|^2 \le \delta_{n,1}\|y_n-q\|^2+\Big(\sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})(\nu^s)^2+\prod_{p=1}^{a}(1-\delta_{n,p})(\nu^s)^2\Big)\|t_n-q\|^2. \tag{3.29}$$

Since

$$\|t_n - q\|^2 \leq \gamma_{n,1}\|y_n - q\|^2 + \sum_{t=2}^{b}\gamma_{n,t}\prod_{s=1}^{t-1}(1-\gamma_{n,s})\|\Gamma^{t-1}\omega_n - \Gamma^{t-1}q\|^2 + \prod_{s=1}^{b}(1-\gamma_{n,s})\|\Gamma^b\omega_n - \Gamma^b q\|^2$$

we obtain using (3.16) that

$$\|t_n - q\|^2 \leq \gamma_{n,1}\|y_n - q\|^2 + \Big[\sum_{t=2}^{b}\gamma_{n,t}\prod_{s=1}^{t-1}(1-\gamma_{n,s})(\nu^s)^2 + \prod_{s=1}^{c}(1-\gamma_{n,s})(\nu^s)^2\Big]\|\omega_n - q\|^2. \tag{3.30}$$

Again, since by (3.19)

$$\|\omega_n - q\|^2 = \Big[\tau_{n,1} + \sum_{r=2}^{c}\tau_{n,r}\prod_{i=1}^{r-1}(1-\tau_{n,i})(\nu^s)^2 + \prod_{i=1}^{c}(1-\tau_{n,i})(\nu^s)^2\Big]\|y_n - q\|^2,$$

it follows (from (3.30)) that

$$\begin{aligned}\|t_n - q\|^2 \leq\ & \Big[\gamma_{n,1} + \Big(\sum_{t=2}^{b}\gamma_{n,t}\prod_{s=1}^{t-1}(1-\gamma_{n,s})(\nu^s)^2 + \prod_{s=1}^{c}(1-\gamma_{n,s})(\nu^s)^2\Big)\Big(\tau_{n,1} \\ & + \sum_{r=2}^{c}\tau_{n,r}\prod_{i=1}^{r-1}(1-\tau_{n,i})(\nu^s)^2 + \prod_{i=1}^{c}(1-\tau_{n,i})(\nu^s)^2\Big)\Big]\|y_n - q\|^2.\end{aligned} \tag{3.31}$$

(3.29) and (3.31) and Proposition 2.3 (with $k = 1$) imply

$$\begin{aligned}\|y_{n+1} - q\|^2 \leq\ & \Big\{\delta_{n,1} + \Big(\sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})(\nu^s)^2 + \prod_{p=1}^{a}(1-\delta_{n,p})(\nu^s)^2\Big) \\ & \times\Big[\gamma_{n,1} + \Big(\sum_{t=2}^{b}\gamma_{n,t}\prod_{s=1}^{t-1}(1-\gamma_{n,s})(\nu^s)^2 + \prod_{s=1}^{c}(1-\gamma_{n,s})(\nu^s)^2\Big)\Big(\tau_{n,1} \\ & + \sum_{r=2}^{c}\tau_{n,r}\prod_{i=1}^{r-1}(1-\tau_{n,i})(\nu^s)^2 + \prod_{i=1}^{c}(1-\tau_{n,i})(\nu^s)^2\Big)\Big]\Big\}\|y_n - q\|^2 \\ <\ & \Big\{\delta_{n,1} + \Big(\sum_{k=2}^{a}\delta_{n,k}\prod_{p=1}^{k-1}(1-\delta_{n,p}) + \prod_{p=1}^{a}(1-\delta_{n,p})\Big)\Big[\gamma_{n,1} + \Big(\sum_{t=2}^{b}\gamma_{n,t}\prod_{s=1}^{t-1}(1-\gamma_{n,s}) \\ & + \prod_{s=1}^{c}(1-\gamma_{n,s})\Big)\Big(\tau_{n,1} + \sum_{r=2}^{c}\tau_{n,r}\prod_{i=1}^{r-1}(1-\tau_{n,i}) + \prod_{i=1}^{c}(1-\tau_{n,i})\Big)\Big]\Big\} \times \|y_n - q\|^2 \\ =\ & \|y_n - q\|^2.\end{aligned} \tag{3.32}$$

By applying Lemma 2.2, we obtain from (3.32) that $y_n \to q\ as\ n \to \infty$; that is, the IH-iteration scheme strongly converges to $q \in F(\Gamma)$. □

If $c = 0$ in (3.3), then we have the following corollary:

**Corollary 3.5.** Let $H$ be a Hilbert space, $\Gamma : H \longrightarrow H$ be a self-map of $H$ satisfying the contractive condition

$$\|\Gamma^s\xi - \Gamma^s\omega\| \leq \nu^s\|\xi - \omega\| + \sum_{j=0}^{s}\binom{s}{j}\rho^{s-j}\phi(\|\xi - \Gamma\xi\|), \tag{3.33}$$

where $\xi, \omega \in H, 0 \leq \nu^s < 1$, and let $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a subadditive monotone increasing function with $\phi(0) = 0$ and $\phi(Mt) = M\phi(t), M \geq 0, t \in \mathbb{R}^+$. For arbitrary $y_0 \in H$, let $\{y_n\}_{n=0}^{\infty}$ be the enhanced-Kirk-Ishikawa-iteration scheme defined by

$$\begin{cases} y_{n+1} = \delta_{n,1}y_n + \sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1}t_n + \prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^a t_n; \\ t_n = \sum_{t=t}^{b}\gamma_{n,t}\prod_{s=1}^{t-1}(1-\gamma_{n,s})\Gamma^{t-1}\omega_n + \prod_{s=1}(1-\gamma_{n,s})\Gamma^b\omega_n, n \geq 0, 1, 2, .., . \end{cases} \tag{3.34}$$

Then,

(*i*) $\Gamma$ defined by (3.33) has a unique fixed point $q$;

(*ii*) The sequence defined by (3.34) converges strongly to $q \in \Gamma$.

If $b = c = 0$, then we have the following corollary:

**Corollary 3.6.** Let $H$ be a HiLbert space, $\Gamma : H \longrightarrow H$ be a self-map of $H$ satisfying the contractive condition

$$\|\Gamma^s\xi - \Gamma^s\omega\| \leq \nu^s\|\xi - \omega\| + \sum_{j=0}^{s}\binom{s}{j}\rho^{s-j}\phi(\|\xi - \Gamma\xi\|), \tag{3.35}$$

where $\xi, \omega \in H, 0 \leq \nu^s < 1$, and let $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a subadditive monotone increasing function with $\phi(0) = 0$ and $\phi(Mt) = M\phi(t), M \geq 0, t \in \mathbb{R}^+$. For arbitrary $y_0 \in H$, let $\{y_n\}_{n=0}^{\infty}$ be the enhanced-Kirk-Mann-iteration scheme defined by

$$y_{n+1} = \sum_{\ell=1}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1 - \delta_{n,p})\Gamma^{k-1}t_n + \prod_{p=1}^{a}(1 - \delta_{n,p})\Gamma^a t_n, n \geq 0, 1, 2, .., \tag{3.36}$$

Then,

(*i*) $\Gamma$ defined by (3.35) has a unique fixed point $q$;

(*ii*) The sequence defined by (3.36) converges strongly to $q \in \Gamma$.

## 4 Main Results II

In this section, we consider the stability results for the DI-iterative scheme, Chugh-DI iterative scheme and IH-iterative scheme defined by (3.1), (3.2) and (3.3) for mappings satisfying (2.2), respectively. The stabilities of (3.5), (3.8) and (3.10) follows immediately as corollaries.

**Theorem 4.1.** Let $H$ be a Hilbert space, $\Gamma : H \longrightarrow H$ be a self-map of $H$ satisfying the contractive condition

$$\|\Gamma^s\xi - \Gamma^s\omega\| \leq \nu^s\|\xi - \omega\| + \sum_{j=0}^{s}\binom{s}{j}\rho^{s-j}\phi(\|\xi - \Gamma\xi\|), \tag{4.1}$$

where $\xi, \omega \in H, 0 \leq \nu^s < 1$, and let $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ retains its usual meaning with $\phi(0) = 0$ and $\phi(Mt) = M\phi(t), M \geq 0, t \in \mathbb{R}^+$. For arbitrary $y_0 \in H$, let $\{y_n\}_{n=0}^{\infty}$ be the DI-iterative scheme defined by (4.1). Suppose $F(\Gamma) \neq \emptyset$ and $q \in F(\Gamma)$. Then, the DI-iterative scheme is $\Gamma$-stable.

**Proof .** We want to prove that the DI-iterative scheme is $\Gamma$-stable. Let $\{f_n\}_{n=0}^{\infty} \subseteq H$ be an arbitrary sequence and set

$$\epsilon_n = \|f_{n+1} - \delta_{n,1}g_n - \sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1 - \delta_{n,p})\Gamma^{\ell-1}g_n - \prod_{p=1}^{a}(1 - \delta_{n,p})\Gamma^a g_n\|^2, \tag{4.2}$$

where

$$g_n = \gamma_n u_n + \sum_{t=2}^{b}\gamma_{n,j}\prod_{s=1}^{t-1}(1 - \gamma_{n,s})\Gamma^{t-1}u_n + \prod_{s=1}^{b}(1 - \gamma_{n,p})\Gamma^b u_n \tag{4.3}$$

and

$$u_n = \sum_{r=1}^{c}\tau_{n,r}\prod_{i=1}^{r-1}(1 - \tau_{n,i})\Gamma^{r-1}f_n + \prod_{i=1}^{c}(1 - \tau_{n,i})\Gamma^c f_n. \tag{4.4}$$

Suppose $\epsilon_n \to 0 \ as \ n \to \infty$. Then, we prove that $f_n \to q \ as \ n \to \infty$. Now, using Proposition 2.5 with $u = q, g_n = t, k = 1, \Gamma^{\ell-1}g_n = v_{j-1} \ and \ \Gamma^a g_n = v$, we get

$$\begin{aligned}\|f_{n+1} - q\|^2 &= \|\delta_{n,1}g_n + \sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1 - \delta_{n,p})\Gamma^{\ell-1}g_n + \prod_{p=1}^{a}(1 - \delta_{n,p})\Gamma^a g_n - q - [\delta_{n,1}g_n \\ &\quad + \sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1 - \delta_{n,p})\Gamma^{\ell-1}g_n + \prod_{p=1}^{a}(1 - \delta_{n,p})\Gamma^a g_n - f_{n+1}]\|^2\end{aligned}$$

$$
\begin{aligned}
&\leq \left\| -[\delta_{n,1}g_n + \sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1}g_n + \prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^{a}g_n - f_{n+1}]\right\|^2 \\
&\quad +\left\|\delta_{n,1}g_n + \sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1}g_n + \prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^{a}g_n - q\right\|^2 \\
&= \left\|f_{n+1} - \delta_{n,1}g_n - \sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1}g_n - \prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^{a}g_n]\right\|^2 \\
&\quad +\left\|\delta_{n,1}g_n + \sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1}g_n + \prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^{a}g_n - q\right\|^2 \\
&= \epsilon_n + \left\|\delta_{n,1}g_n + \sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1}g_n + \prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^{a}g_n - q\right\|^2 (\text{ by } (4.2)) \\
&\leq \epsilon_n + \delta_{n,1}\|g_n - q\|^2 + \sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})\|\Gamma^{\ell-1}g_n - \Gamma^{\ell-1}q\|^2 + \prod_{p=1}^{a}(1-\delta_{n,p})\|\Gamma^{a}g_n - \Gamma^{a}q\|^2. \qquad (4.5)
\end{aligned}
$$

But from (3.11), with $\xi = g_n$, we have

$$
\begin{aligned}
\|\Gamma^{\ell-1}g_n - \Gamma^{\ell-1}q\| &\leq \nu^s\|g_n - q\| + \sum_{j=1}^{s}\binom{s}{j}\nu^{s-j}\phi(\|q - \Gamma q\|) \\
&= \nu^s\|g_n - q\|. \qquad (4.6)
\end{aligned}
$$

Using (4.5) and (4.6), we get

$$
\|f_{n+1} - q\|^2 = \epsilon_n + \left[\delta_{n,1} + \sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})(\nu^s)^2 + \prod_{p=1}^{a}(1-\delta_{n,p})(\nu^s)^2\right]\|g_n - q\|^2. \qquad (4.7)
$$

Also,

$$
\|g_n - q\|^2 = \gamma_{n,1}\|u_n - q\|^2 + \sum_{t=2}^{b}\gamma_{n,t}\prod_{s=1}^{t-1}(1-\gamma_{n,s})\|\Gamma^{t-1}u_n - \Gamma^{t-1}q\|^2 + \prod_{s=1}^{b}(1-\gamma_{n,s})\|\Gamma^{b}u_n - \Gamma^{b}q\|^2. \qquad (4.8)
$$

Again, from (3.11), with $\xi = u_n$, we have

$$
\begin{aligned}
\|\Gamma^{t-1}u_n - \Gamma^{t-1}q\| &\leq \nu^s\|u_n - q\| + \sum_{j=1}^{s}\binom{s}{j}\nu^{s-j}\phi(\|q - \Gamma q\|) \\
&= \nu^s\|u_n - q\|. \qquad (4.9)
\end{aligned}
$$

(4.8) and (4.9) imply

$$
\|g_n - q\|^2 = \left[\gamma_{n,1} + \sum_{t=2}^{b}\gamma_{n,t}\prod_{s=1}^{t-1}(1-\gamma_{n,s})(\nu^s)^2 + \prod_{s=1}^{b}(1-\gamma_{n,s})(\nu^s)^2\right]\|u_n - q\|^2. \qquad (4.10)
$$

Moreover,

$$
\|u_n - q\|^2 = \tau_{n,1}\|f_n - q\|^2 + \sum_{r=2}^{b}\tau_{n,r}\prod_{i=1}^{r-1}(1-\tau_{n,i})\|\Gamma^{r-1}f_n - \Gamma^{r-1}q\|^2 + \prod_{i=1}^{c}(1-\tau_{n,i})\|\Gamma^{c}f_n - \Gamma^{c}q\|^2. \qquad (4.11)
$$

Using (3.11), with $\xi = y_n$, and following the same approach as in (4.10), we get

$$\|u_n - q\|^2 = \Big[\tau_{n,1} + \sum_{r=2}^{c} \tau_{n,r} \prod_{i=1}^{r-1}(1-\tau_{n,i})(\nu^s)^2 + \prod_{i=1}^{c}(1-\tau_{n,i})(\nu^s)^2\Big]\|f_n - q\|^2. \tag{4.12}$$

(4.7), (4.10) and (4.12) imply that

$$\begin{aligned}\|f_{n+1} - q\|^2 &= \epsilon_n + \Big(\delta_{n,1} + \sum_{\ell=2}^{a} \delta_{n,\ell} \prod_{p=1}^{\ell-1}(1-\delta_{n,p})(\nu^s)^2 + \prod_{p=1}^{a}(1-\delta_{n,p})(\nu^s)^2\Big)\Big(\gamma_{n,1} + \sum_{t=2}^{b} \gamma_{n,t} \prod_{s=1}^{t-1}(1-\gamma_{n,s})(\nu^s)^2 \\ &\quad + \prod_{s=1}^{b}(1-\gamma_{n,s})(\nu^s)^2\Big)\Big(\tau_{n,1} + \sum_{r=2}^{c} \tau_{n,r} \prod_{i=1}^{r-1}(1-\tau_{n,i})(\nu^s)^2 + \prod_{i=1}^{c}(1-\tau_{n,i})(\nu^s)^2\Big)\|f_n - q\|^2. \end{aligned} \tag{4.13}$$

Using Proposition 2.4 and the fact that $\nu^s \in [0, 1)$, we have

$$A^\star < B^\star = 1, \tag{4.14}$$

where

$$\begin{aligned}A^\star &= \Big(\delta_{n,1} + \sum_{\ell=2}^{a} \delta_{n,\ell} \prod_{p=1}^{\ell-1}(1-\delta_{n,p})(\nu^s)^2 + \prod_{p=1}^{a}(1-\delta_{n,p})(\nu^s)^2\Big)\Big(\gamma_{n,1} + \sum_{t=2}^{b} \gamma_{n,t} \prod_{s=1}^{t-1}(1-\gamma_{n,s})(\nu^s)^2 \\ &\quad + \prod_{s=1}^{b}(1-\gamma_{n,s})(\nu^s)^2\Big)\Big(\tau_{n,1} + \sum_{r=2}^{c} \tau_{n,r} \prod_{i=1}^{r-1}(1-\tau_{n,i})(\nu^s)^2 + \prod_{i=1}^{c}(1-\tau_{n,i})(\nu^s)^2\Big)\end{aligned}$$

and

$$\begin{aligned}B^\star &= \Big(\delta_{n,1} + \sum_{\ell=2}^{a} \delta_{n,\ell} \prod_{p=1}^{\ell-1}(1-\delta_{n,p}) + \prod_{p=1}^{a}(1-\delta_{n,p})\Big)\Big(\gamma_{n,1} + \sum_{t=2}^{b} \gamma_{n,t} \prod_{s=1}^{t-1}(1-\gamma_{n,s}) + \prod_{s=1}^{b}(1-\gamma_{n,s})\Big) \\ &\quad \Big(\tau_{n,1} + \sum_{r=2}^{c} \tau_{n,r} \prod_{i=1}^{r-1}(1-\tau_{n,i}) + \prod_{i=1}^{c}(1-\tau_{n,i})\Big).\end{aligned}$$

By Lemma 2.2, (4.14) and the fact that $\epsilon_n \to 0$ *as* $n \to \infty$, we obtain from (4.13) that $f_n \to q$ *as* $n \to \infty$ as required. On the other hand, suppose $f_n \to q$ *as* $n \to \infty$. Then, we show that $\epsilon_n \to 0$ *as* $n \to \infty$. Indeed, from (4.2) and Proposition 2.5 with $u = q, g_n = t, k = 1, \Gamma^{\ell-1} g_n = v_{j-1}$ *and* $\Gamma^a g_n = v$, we get

$$\begin{aligned}\epsilon_n &= \|f_{n+1} - q - \Big[\delta_{n,1} g_n + \sum_{\ell=2}^{a} \delta_{n,\ell} \prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1} g_n + \prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^a g_n - q\Big]\|^2 \\ &\le \|f_{n+1} - q\|^2 + \|\delta_{n,1} g_n + \sum_{\ell=2}^{a} \delta_{n,\ell} \prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1} g_n + \prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^a g_n - q\|^2 \\ &\le \|f_{n+1} - q\|^2 + \delta_{n,1}\|g_n - q\|^2 + \sum_{\ell=2}^{a} \delta_{n,\ell} \prod_{p=1}^{\ell-1}(1-\delta_{n,p})\|\Gamma^{\ell-1} g_n - q\|^2 + \prod_{p=1}^{a}(1-\delta_{n,p})\|\Gamma^a g_n - q\|^2 \end{aligned} \tag{4.15}$$

From (4.6) and (4.15), we obtain

$$\epsilon_n \le \|f_{n+1} - q\|^2 + \Big[\delta_{n,1} + \sum_{\ell=2}^{a} \delta_{n,\ell} \prod_{p=1}^{\ell-1}(1-\delta_{n,p})(\nu^s)^2 + \prod_{p=1}^{a}(1-\delta_{n,p})(\nu^s)^2\Big]\|g_n - q\|^2,$$

and from (4.10) and (4.12), we get

$$\begin{aligned}\epsilon_n \quad \leq \quad & \|f_{n+1}-q\|^2 + \Big(\delta_{n,1} + \sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})(\nu^s)^2 + \prod_{p=1}^{a}(1-\delta_{n,p})(\nu^s)^2\Big)\Big(\gamma_{n,1} + \sum_{t=2}^{b}\gamma_{n,t}\prod_{s=1}^{t-1}(1-\gamma_{n,s})(\nu^s)^2 \\ & + \prod_{s=1}^{b}(1-\gamma_{n,s})(\nu^s)^2\Big)\Big(\tau_{n,1} + \sum_{r=2}^{c}\tau_{n,r}\prod_{i=1}^{r-1}(1-\tau_{n,i})(\nu^s)^2 + \prod_{i=1}^{c}(1-\tau_{n,i})(\nu^s)^2\Big)\|f_n-q\|^2. \end{aligned} \tag{4.16}$$

Since $f_n \to 0$ *as* $n \to \infty$, it follows from (4.16) that $\epsilon_n \to 0$ *as* $n \to \infty$, which completes the proof. □

**Theorem 4.2.** Let $H$ be a Hilbert space, $\Gamma : H \longrightarrow H$ be a self-map of $Z$ satisfying the contractive condition

$$\|\Gamma^s\xi - \Gamma^s\omega\| \leq \nu^s\|\xi-\omega\| + \sum_{j=0}^{s}\binom{s}{j}\rho^{s-j}\phi(\|\xi-\Gamma\xi\|), \tag{4.17}$$

where $\xi,\omega \in H, 0 \leq \nu^s < 1$, and let $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ retains its usual meaning with $\phi(0) = 0$ and $\phi(Mt) = M\phi(t), M \geq 0, t \in \mathbb{R}^+$. For arbitrary $y_0 \in H$, let $\{y_n\}_{n=0}^{\infty}$ be the Chugh-DI iterative scheme defined by (4.17). Suppose $F(\Gamma) \neq \emptyset$ and $q \in F(\Gamma)$. Then, the Chugh-DI iterative scheme is $\Gamma$-stable.

**Proof .** Let $\{f_n\}_{n=0}^{\infty} \subseteq H$ be an arbitrary sequence and set

$$\epsilon_n = \|f_{n+1} - \delta_{n,1}g_n - \sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1}g_n - \prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^a g_n\|^2, \tag{4.18}$$

where

$$g_n = \gamma_n\Gamma u_n + \sum_{t=2}^{b}\gamma_{n,j}\prod_{s=1}^{t-1}(1-\gamma_{n,s})\Gamma^{t-1}u_n + \prod_{s=1}^{b}(1-\gamma_{n,p})\Gamma^b u_n \tag{4.19}$$

and

$$u_n = \sum_{r=1}^{c}\tau_{n,r}\prod_{i=1}^{r-1}(1-\tau_{n,i})\Gamma^{r-1}f_n + \prod_{i=1}^{c}(1-\tau_{n,i})\Gamma^c f_n. \tag{4.20}$$

We want to prove that the Chugh-DI iterative scheme is $\Gamma$-stable. Now, suppose $\epsilon_n \to 0$ *as* $n \to \infty$. Then, we prove that $f_n \to q$ *as* $n \to \infty$. Using Proposition 2.4 with $u = q, g_n = t, k = 1, \Gamma^{\ell-1}g_n = v_{j-1}$ *and* $\Gamma^a g_n = v$, we get

$$\begin{aligned}\|f_{n+1}-q\|^2 =& \|\delta_{n,1}g_n + \sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1}g_n + \prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^a g_n - q - [\delta_{n,1}g_n \\ & + \sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1}g_n + \prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^a g_n - f_{n+1}]\|^2 \\ \leq& \| - [\delta_{n,1}g_n + \sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1}g_n + \prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^a g_n - f_{n+1}]\|^2 \\ & + \|\delta_{n,1}g_n + \sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1}g_n + \prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^a g_n - q\|^2 \\ =& \|f_{n+1} - \delta_{n,1}g_n - \sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1}g_n - \prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^a g_n]\|^2 \\ & + \|\delta_{n,1}g_n + \sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1}g_n + \prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^a g_n - q\|^2 \\ =& \epsilon_n + \|\delta_{n,1}g_n + \sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1}g_n + \prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^a g_n - q\|^2 (\ by\ (4.2)) \\ \leq& \epsilon_n + \delta_{n,1}\|g_n-q\|^2 + \sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})\|\Gamma^{\ell-1}g_n - \Gamma^{\ell-1}q\|^2 + \prod_{p=1}^{a}(1-\delta_{n,p})\|\Gamma^a g_n - \Gamma^a q\|^2. \end{aligned} \tag{4.21}$$

By (3.11), with $\xi = g_n$, we have

$$\begin{aligned} \|\Gamma^{\ell-1}g_n - \Gamma^{\ell-1}q\| &\leq \nu^s\|g_n - q\| + \sum_{j=1}^{s}\binom{s}{j}\nu^{s-j}\phi(\|q-\Gamma q\|) \\ &= \nu^s\|g_n - q\|. \end{aligned} \tag{4.22}$$

Using (4.21) and (4.22), we get

$$\|f_{n+1}-q\|^2 = \epsilon_n + \Big[\delta_{n,1} + \sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})(\nu^s)^2 + \prod_{p=1}^{a}(1-\delta_{n,p})(\nu^s)^2\Big]\|g_n - q\|^2. \tag{4.23}$$

Now, from(4.17), (4.19) and Proposition 2.4, with $u = q, f_n = t, k = 1, \Gamma^{\ell-1}f_n = v_{j-1}$ *and* $\Gamma^a f_n = v$, , we get

$$\begin{aligned} \|g_n - q\|^2 &= \gamma_{n,1}\|\Gamma f_n - q\|^2 + \sum_{t=2}^{b}\gamma_{n,t}\prod_{s=1}^{t-1}(1-\gamma_{n,s})\|\Gamma^{t-1}u_n - \Gamma^{t-1}q\|^2 + \prod_{s=1}^{b}(1-\gamma_{n,s})\|\Gamma^b u_n - \Gamma^b q\|^2 \\ &\leq \nu\gamma_{n,1}\|f_n - q\|^2 + \sum_{t=2}^{b}\gamma_{n,t}\prod_{s=1}^{t-1}(1-\gamma_{n,s})\|\Gamma^{t-1}u_n - \Gamma^{t-1}q\|^2 + \prod_{s=1}^{b}(1-\gamma_{n,s})\|\Gamma^b u_n - \Gamma^b q\|^2, \end{aligned}$$

which by (3.16), with $\omega_n = u_n$, yields

$$\|g_n - q\|^2 = \nu\gamma_{n,1}\|f_n - q\|^2 + \Big[\sum_{t=2}^{b}\gamma_{n,t}\prod_{s=1}^{t-1}(1-\gamma_{n,s})(\nu^s)^2 + \prod_{s=1}^{b}(1-\gamma_{n,s})(\nu^s)^2\Big]\|u_n - q\|^2. \tag{4.24}$$

Furthermore, using the same approach as in (3.25) of Theorem 3.2 with $\omega_n = u_n$, we obtain

$$\|u_n - q\|^2 = \Big[\tau_{n,1} + \sum_{r=2}^{c}\tau_{n,r}\prod_{i=1}^{r-1}(1-\tau_{n,i})(\nu^s)^2 + \prod_{i=1}^{c}(1-\tau_{n,i})(\nu^s)^2\Big]\|f_n - q\|^2. \tag{4.25}$$

(4.23), (4.24) and (4.25) yield

$$\begin{aligned} \|f_{n+1}-q\|^2 &= \epsilon_n + \Big(\delta_{n,1} + \sum_{k=2}^{a}\delta_{n,k}\prod_{p=1}^{k-1}(1-\delta_{n,p})(\nu^s)^2 + \prod_{p=1}^{a}(1-\delta_{n,p})(\nu^s)^2\Big) \\ &\quad\times\Big\{\nu\gamma_{n,1} + \Big(\sum_{t=2}^{b}\gamma_{n,t}\prod_{s=1}^{t-1}(1-\gamma_{n,s})(\nu^s)^2 + \prod_{s=1}^{b}(1-\gamma_{n,s})(\nu^s)^2\Big)\Big(\tau_{n,1} + \sum_{r=2}^{c}\tau_{n,r}\prod_{i=1}^{r-1}(1-\tau_{n,i})(\nu^s)^2 \\ &\quad+\prod_{i=1}^{c}(1-\tau_{n,i})(\nu^s)^2\Big)\Big\}\|f_n - q\|^2. \end{aligned} \tag{4.26}$$

Using Proposition 2.4 and the fact that $\nu^s \in [0,1)$, we have

$$C^\star < D^\star = 1, \tag{4.27}$$

where

$$\begin{aligned} C^\star &= \Big(\delta_{n,1} + \sum_{k=2}^{a}\delta_{n,k}\prod_{p=1}^{k-1}(1-\delta_{n,p})(\nu^s)^2 + \prod_{p=1}^{a}(1-\delta_{n,p})(\nu^s)^2\Big) \\ &\quad\times\Big\{\nu\gamma_{n,1} + \Big(\sum_{t=2}^{b}\gamma_{n,t}\prod_{s=1}^{t-1}(1-\gamma_{n,s})(\nu^s)^2 + \prod_{s=1}^{b}(1-\gamma_{n,s})(\nu^s)^2\Big)\Big(\tau_{n,1} \\ &\quad+\sum_{r=2}^{c}\tau_{n,r}\prod_{i=1}^{r-1}(1-\tau_{n,i})(\nu^s)^2 + \prod_{i=1}^{c}(1-\tau_{n,i})(\nu^s)^2\Big)\Big\} \end{aligned}$$

and

$$D^\star = \Big(\delta_{n,1}+\sum_{k=2}^{a}\delta_{n,k}\prod_{p=1}^{k-1}(1-\delta_{n,p})+\prod_{p=1}^{a}(1-\delta_{n,p})\Big)\Big(\gamma_{n,1}+\sum_{t=2}^{b}\gamma_{n,t}\prod_{s=1}^{t-1}(1-\gamma_{n,s}) + \prod_{s=1}^{b}(1-\gamma_{n,s})\Big)\Big(\tau_{n,1}+\sum_{r=2}^{c}\tau_{n,r}\prod_{i=1}^{r-1}(1-\tau_{n,i})+\prod_{i=1}^{c}(1-\tau_{n,i})\Big)$$

By Lemma 2.2, (4.27) and the fact that $\epsilon_n \to 0$ *as* $n\to\infty$, it follows from (4.26) that $f_n \to q$ *as* $n \to \infty$ as required.

Conversely, suppose $f_n \to q$ *as* $n\to\infty$. Then, we show that $\epsilon_n \to 0$ *as* $n \to \infty$. Indeed, from (4.18) and Proposition 2.4 with $u = q, g_n = t, k = 1, \Gamma^{\ell-1}g_n = v_{j-1}$ *and* $\Gamma^a g_n = v$, we get

$$\begin{aligned}\epsilon_n &= \|f_{n+1}-q-\Big[\delta_{n,1}g_n+\sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1}g_n+\prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^a g_n - q\Big]\|^2\\ &\le \|f_{n+1}-q\|^2+\|\delta_{n,1}g_n+\sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1}g_n+\prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^a g_n-q\|^2\\ &\le \|f_{n+1}-q\|^2+\delta_{n,1}\|g_n-q\|^2+\sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})\|\Gamma^{\ell-1}g_n-q\|^2+\prod_{p=1}^{a}(1-\delta_{n,p})\|\Gamma^a g_n-q\|^2. \qquad (4.28)\end{aligned}$$

Since, from (4.22) and (4.28),

$$\epsilon_n \le \|f_{n+1}-q\|^2+\Big(\delta_{n,1}+\sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})+\prod_{p=1}^{a}(1-\delta_{n,p})\Big)\|g_n-q\|^2,$$

it follows from (4.24) and (4.25) that

$$\begin{aligned}\epsilon_n &\le \|f_{n+1}-q\|^2+\Big(\delta_{n,1}+\sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})+\prod_{p=1}^{a}(1-\delta_{n,p})\Big)\Big\{\nu\gamma_{n,1}\\ &\quad+\Big(\sum_{t=2}^{b}\gamma_{n,t}\prod_{s=1}^{t-1}(1-\gamma_{n,s})(\nu^s)^2+\prod_{s=1}^{b}(1-\gamma_{n,s})(\nu^s)^2\Big)\Big(\tau_{n,1}+\sum_{r=2}^{c}\tau_{n,r}\prod_{i=1}^{r-1}(1-\tau_{n,i})(\nu^s)^2\\ &\quad+\prod_{i=1}^{c}(1-\tau_{n,i})(\nu^s)^2\Big)\Big\}\|f_n-q\|^2. \qquad (4.29)\end{aligned}$$

Since $\nu\in[0,1)$ and $f_n \to 0$ *as* $n\to\infty$, it follows from (4.29) that $\epsilon_n \to 0$ *as* $n\to\infty$, which completes the proof. □

**Theorem 4.3.** Let $H$ be a Hilbert space, $\Gamma : H \longrightarrow H$ be a self-map of $H$ satisfying the contractive condition

$$\|\Gamma^s\xi-\Gamma^s\omega\| \le \nu^s\|\xi-\omega\|+\sum_{j=0}^{s}\binom{s}{j}\rho^{s-j}\phi(\|\xi-\Gamma\xi\|), \qquad (4.30)$$

where $\xi,\omega\in H, 0\le\nu^s<1$, and let $\phi:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ retains its usual meaning with $\phi(0)=0$ and $\phi(Mt)=M\phi(t), M\ge 0, t\in\mathbb{R}^+$. For arbitrary $y_0\in H$, let $\{y_n\}_{n=0}^{\infty}$ be the IH-iterative scheme defined by (4.30). Suppose $F(\Gamma)\neq\emptyset$ and $q\in F(\Gamma)$. Then, the IH-iterative scheme is $\Gamma$-stable.

**Proof .** Let $\{f_n\}_{n=0}^{\infty}\subseteq H$ be an arbitrary sequence and set

$$\epsilon_n = \|f_{n+1}-\delta_{n,1}f_n-\sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1}g_n-\prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^a g_n\|^2, \qquad (4.31)$$

where

$$g_n=\gamma_n f_n+\sum_{t=2}^{b}\gamma_{n,j}\prod_{s=1}^{t-1}(1-\gamma_{n,s})\Gamma^{t-1}u_n+\prod_{s=1}^{b}(1-\gamma_{n,p})\Gamma^b u_n \qquad (4.32)$$

and

$$u_n = \sum_{r=1}^{c} \tau_{n,r} \prod_{i=1}^{r-1} (1-\tau_{n,i})\Gamma^{r-1} f_n + \prod_{i=1}^{c}(1-\tau_{n,i})\Gamma^c f_n. \tag{4.33}$$

We want to prove that the IH-iterative scheme is Γ-stable. Suppose $\epsilon_n \to 0$ *as* $n \to \infty$. Then, we prove that $f_n \to q$ *as* $n \to \infty$. Now, using Proposition 2.4 with $u = q, f_n = t, k = 1, \Gamma^{\ell-1} g_n = v_{j-1}$ *and* $\Gamma^a g_n = v$, we get

$$\begin{aligned}
\|f_{n+1} - q\|^2 =& \|\delta_{n,1} f_n + \sum_{\ell=2}^{a} \delta_{n,\ell} \prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1} g_n + \prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^a g_n - q - [\delta_{n,1} f_n \\
& + \sum_{\ell=2}^{a} \delta_{n,\ell} \prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1} g_n + \prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^a g_n - f_{n+1}]\|^2 \\
\leq& \| - [\delta_{n,1} f_n + \sum_{\ell=2}^{a} \delta_{n,\ell} \prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1} g_n + \prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^a g_n - f_{n+1}]\|^2 \\
& + \|\delta_{n,1} f_n + \sum_{\ell=2}^{a} \delta_{n,\ell} \prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1} g_n + \prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^a g_n - q\|^2 \\
=& \|f_{n+1} - \delta_{n,1} f_n - \sum_{\ell=2}^{a} \delta_{n,\ell} \prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1} g_n - \prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^a g_n]\|^2 \\
& + \|\delta_{n,1} f_n + \sum_{\ell=2}^{a} \delta_{n,\ell} \prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1} g_n + \prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^a g_n - q\|^2 \\
=& \epsilon_n + \|\delta_{n,1} f_n + \sum_{\ell=2}^{a} \delta_{n,\ell} \prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1} g_n + \prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^a g_n - q\|^2 (\textit{ by } (4.2)) \\
\leq& \epsilon_n + \delta_{n,1}\|f_n - q\|^2 + \sum_{\ell=2}^{a} \delta_{n,\ell} \prod_{p=1}^{\ell-1}(1-\delta_{n,p})\|\Gamma^{\ell-1} g_n - \Gamma^{\ell-1} q\|^2 + \prod_{p=1}^{a}(1-\delta_{n,p})\|\Gamma^a g_n - \Gamma^a q\|^2.
\end{aligned} \tag{4.34}$$

Since from (4.30), with $\xi = g_n$,

$$\begin{aligned}
\|\Gamma^{\ell-1} g_n - \Gamma^{\ell-1} q\| &\leq \nu^s \|g_n - q\| + \sum_{j=1}^{s} \binom{s}{j} \nu^{s-j} \phi(\|q - \Gamma q\|) \\
&= \nu^s \|g_n - q\|,
\end{aligned} \tag{4.35}$$

it follows from (4.34) that

$$\|f_{n+1} - q\|^2 \leq \epsilon_n + \delta_{n,1}\|f_n - q\|^2 + \Big[\sum_{\ell=2}^{a} \delta_{n,\ell} \prod_{p=1}^{\ell-1}(1-\delta_{n,p})(\nu^s)^2 + \prod_{p=1}^{a}(1-\delta_{n,p})(\nu^s)^2\Big] \|g_n - q\|^2. \tag{4.36}$$

Also, since

$$\begin{aligned}
\|g_n - q\|^2 \leq& \gamma_{n,1}\|f_n - q\|^2 + \sum_{t=2}^{b} \gamma_{n,t} \prod_{s=1}^{t-1}(1-\gamma_{n,s})\|\Gamma^{t-1} u_n - \Gamma^{t-1} q\|^2 \\
& + \prod_{s=1}^{b}(1-\gamma_{n,s})\|\Gamma^b u_n - \Gamma^b q\|^2 \quad \text{(by (4.32) and Proposition 2.4 )}
\end{aligned}$$

we obtain, using the same approach as in (4.35), that

$$\|g_n - q\|^2 \leq \gamma_{n,1}\|f_n - q\|^2 + \Big[\sum_{t=2}^{b} \gamma_{n,t} \prod_{s=1}^{t-1}(1-\gamma_{n,s})(\nu^s)^2 + \prod_{s=1}^{c}(1-\gamma_{n,s})(\nu^s)^2\Big] \|u_n - q\|^2. \tag{4.37}$$

Similarly, using (4.33) and Proposition 2.4, we obtain

$$\|u_n - q\|^2 = \Big[\tau_{n,1} + \sum_{r=2}^{c}\tau_{n,r}\prod_{i=1}^{r-1}(1-\tau_{n,i})(\nu^s)^2 + \prod_{i=1}^{c}(1-\tau_{n,i})(\nu^s)^2\Big]\|f_n - q\|^2 \tag{4.38}$$

$(4.36), (4.37)$ *and* $(4.38)$ imply

$$\begin{aligned}
\|f_{n+1} - q\|^2 &\leq \epsilon_n + \Big\{\delta_{n,1} + \Big(\sum_{k=2}^{a}\delta_{n,k}\prod_{p=1}^{k-1}(1-\delta_{n,p})(\nu^s)^2 + \prod_{p=1}^{a}(1-\delta_{n,p})(\nu^s)^2\Big) \\
&\quad \times\Big[\gamma_{n,1} + \Big(\sum_{t=2}^{b}\gamma_{n,t}\prod_{s=1}^{t-1}(1-\gamma_{n,s})(\nu^s)^2 + \prod_{s=1}^{c}(1-\gamma_{n,s})(\nu^s)^2\Big)\Big(\tau_{n,1} \\
&\quad + \sum_{r=2}^{c}\tau_{n,r}\prod_{i=1}^{r-1}(1-\tau_{n,i})(\nu^s)^2 + \prod_{i=1}^{c}(1-\tau_{n,i})(\nu^s)^2\Big)\Big]\Big\}\|f_n - q\|^2 \\
&< \epsilon_n + \Big\{\delta_{n,1} + \Big(\sum_{k=2}^{a}\delta_{n,k}\prod_{p=1}^{k-1}(1-\delta_{n,p}) + \prod_{p=1}^{a}(1-\delta_{n,p})\Big)\Big[\gamma_{n,1} + \Big(\sum_{t=2}^{b}\gamma_{n,t}\prod_{s=1}^{t-1}(1-\gamma_{n,s}) \\
&\quad + \prod_{s=1}^{c}(1-\gamma_{n,s})\Big)\Big(\tau_{n,1} + \sum_{r=2}^{c}\tau_{n,r}\prod_{i=1}^{r-1}(1-\tau_{n,i}) + \prod_{i=1}^{c}(1-\tau_{n,i})\Big)\Big]\Big\} \times \|f_n - q\|^2 \\
&= \epsilon_n + \|f_n - q\|^2.
\end{aligned} \tag{4.39}$$

Since $\epsilon_n \to 0$ *as* $n \to \infty$, we obtain from Lemma 2.2 and (4.39) that $f_n \to q$ *as* $n \to \infty$.

Conversely, suppose $f_n \to q$ *as* $n \to \infty$. Then, we show that $\epsilon_n \to 0$ *as* $n \to \infty$. lndeed, from (4.18) and Proposition 2.5 with $u = q, g_n = t, k = 1, \Gamma^{\ell-1}g_n = v_{j-1}$ *and* $\Gamma^a g_n = v$, we get

$$\begin{aligned}
\epsilon_n &= \|f_{n+1} - q - \Big[\delta_{n,1}f_n + \sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1}g_n + \prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^a g_n - q\Big]\|^2 \\
&\leq \|f_{n+1} - q\|^2 + \|\delta_{n,1}f_n + \sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})\Gamma^{\ell-1}g_n + \prod_{p=1}^{a}(1-\delta_{n,p})\Gamma^a g_n - q\|^2 \\
&\leq \|f_{n+1} - q\|^2 + \delta_{n,1}\|f_n - q\|^2 + \sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})\|\Gamma^{\ell-1}g_n - q\|^2 + \prod_{p=1}^{a}(1-\delta_{n,p})\|\Gamma^a g_n - q\|^2.
\end{aligned} \tag{4.40}$$

Since, from (4.35) and (4.40),

$$\epsilon_n \leq \|f_{n+1} - q\|^2 + +\delta_{n,1}\|f_n - q\|^2 + \Big(\sum_{\ell=2}^{a}\delta_{n,\ell}\prod_{p=1}^{\ell-1}(1-\delta_{n,p})(\nu^s)^2 + \prod_{p=1}^{a}(1-\delta_{n,p})(\nu^s)^2\Big)\|g_n - q\|^2,$$

it follows from (4.37), (4.38) and the fact that $\nu \in [0,1)$ that

$$\begin{aligned}
\epsilon_n \leq &\|f_{n+1} - q\|^2 + \Big\{\delta_{n,1} + \Big(\sum_{k=2}^{a}\delta_{n,k}\prod_{p=1}^{k-1}(1-\delta_{n,p}) + \prod_{p=1}^{a}(1-\delta_{n,p})\Big)\Big[\gamma_{n,1} + \Big(\sum_{t=2}^{b}\gamma_{n,t}\prod_{s=1}^{t-1}(1-\gamma_{n,s}) + \prod_{s=1}^{c}(1-\gamma_{n,s})\Big) \\
&\Big(\tau_{n,1} + \sum_{r=2}^{c}\tau_{n,r}\prod_{i=1}^{r-1}(1-\tau_{n,i}) + \prod_{i=1}^{c}(1-\tau_{n,i})\Big)\Big]\Big\} \times \|f_n - q\|^2.
\end{aligned} \tag{4.41}$$

Since $f_n \to 0$ *as* $n \to \infty$, it follows from (4.41) that $\epsilon_n \to 0$ *as* $n \to \infty$, which completes the proof. □

## Conclusion

An affirmative answer has been provided for Question 1.1. The results obtained in this paper improve the corresponding results in [8, 10, 15, 20] and several others currently announced in literature.

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