Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 237–251
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2020.21296.2244

Coefficient estimates for subclasses of analytic functions related to Bernoulli's lemniscate and an application of Poisson distribution series

Gangadharan Murugusundaramoorthy[a,*], Teodor Bulboacă[b]

[a]Department of Mathematics, SAS, Vellore Institute of Technology, deemed to be University, Vellore-632014, India

[b]Faculty of Mathematics and Computer Science, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania

(Communicated by Ali Jabbari)

Abstract

Using the q-calculus operator we defined a new subclass of analytic functions $\mathcal{M}_q(\vartheta,\Phi)$ defined in the open unit disk $\Delta=\{z\in\mathbb{C}:|z|<1\}$ related with Bernoulli's lemniscate and obtained certain coefficient estimates, Fekete-Szegő inequality results for $f\in\mathcal{M}_q(\vartheta,\Phi)$. As a special case of our result, we obtain Fekete-Szegő inequality for a class of functions defined through Poisson distribution and further with the help of MAPLE™ software we find Hankel determinant inequality for $f\in\mathcal{M}_q(\vartheta,\Phi)$. Our investigation generalises some previous results obtained in different articles.

Keywords: Analytic functions, differential subordination, Fekete-Szegö problem, $q-$calculus operator, Bernoulli's lemniscate.
2020 MSC: Primary 30C45; Secondary 30C80

1 Introduction

We denote by $\mathcal{H}(\Delta)$ the class of functions which are analytic in the open unit disk $\Delta:=\{z\in\mathbb{C}:|z|<1\}$, and let $\mathcal{A}$ be the subclass of $\mathcal{H}(\Delta)$ consisting of the functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n\,z^n,\ z\in\Delta. \tag{1.1}$$

Let $\mathcal{P}$ be the well-known class of *Carathéodory functions*, that is $p\in\mathcal{H}(\Delta)$ with the power series expansion

$$p(z)=1+p_1z+p_2z^2+\ldots,\ z\in\Delta, \tag{1.2}$$

and $\operatorname{Re}p(z)>0$ for all $z\in\Delta$.

*Corresponding author
Email addresses: gmsmoorthy@yahoo.com (Gangadharan Murugusundaramoorthy), bulboaca@math.ubbcluj.ro (Teodor Bulboacă)

Received: July 2020 *Accepted:* September 2020

For two functions $f, g \in \mathcal{H}(\Delta)$, the function f is called to be *subordinate* to the function g, written $f(z) \prec g(z)$, if there exists a function $\psi \in \mathcal{H}(\Delta)$ with $|\psi(z)| < 1$, $z \in \Delta$, and $\psi(0) = 0$, such that $f(z) = g(\psi(z))$ for all $z \in \Delta$. In particular, if g is univalent in Δ then the following equivalence relationship holds true:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

Let $h_j(z) = \sum\limits_{n=0}^{\infty} a_{n,j}\, z^n$ $(j = 1, 2)$ which are analytic in Δ, then the well-known *Hadamard (or convolution) product* of h_1 and h_2 is given by

$$(h_1 * h_2)(z) := \sum_{n=0}^{\infty} a_{n,1}\, a_{n,2}\, z^n,\ z \in \Delta.$$

Quantum calculus (*q-calculus* and *h-calculus*) is common classical calculus without the notion of limits. Here, h represents the constant of Planck, while q represents quantum. Due to its application in a variety of branches such as physics, mathematics, the area of *q-calculus* has gained great importance for researchers. The first study on *q-calculus* was systematically established by Jackson [5], that is, he was the first to expand *q-integral* and *q-derivative*. Now, we give some concept details of *q-calculus* which are used in the paper.

The Jackson's *q-derivative* $(0 < q < 1)$ of a function $f \in \mathcal{A}$ is expressed by

$$\mathfrak{D}_q f(z) := \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z}, & \text{if} \quad z \neq 0, \\ f'(0), & \text{if} \quad z = 0, \end{cases}$$

and $\mathcal{D}_q^2 f = \mathfrak{D}_q(\mathfrak{D}_q f)$. Thus, from the above definition we deduce that

$$\mathfrak{D}_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1},$$

where

$$[n]_q := \frac{1-q^n}{1-q}.$$

If $q \to 1^-$, we get $[n]_q \to n$. For the function $h(z) = z^n$, we get $\mathfrak{D}_q h(z) = \mathfrak{D}_q z^n = \dfrac{1-q^n}{1-q} z^{n-1} = [n]_q z^{n-1}$ and $\lim\limits_{q \to 1^-} \mathfrak{D}_q h(z) = \lim\limits_{q \to 1^-} \left([n]_q z^{n-1}\right) = n z^{n-1} = h'(z)$, where h' is the ordinary derivative. The q-derivative operator $\mathfrak{D}_q$ was presumably first applied by Ismail et al. [4] to study a q-extension of the class $\mathcal{S}_q$ of starlike functions in Δ. For more details on study of the q-calculus and the fractional q-calculus in Geometric Function Theory of Complex Analysis one can refere the recent article by Srivastava[24].

In [3], Fekete and Szegő obtained estimates of the functional $\left|a_3 - \mu a_2^2\right|$ for μ is real. That is, if $f \in \mathcal{A}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 4\mu - 3, & \text{if} \quad \mu \geq 1, \\ 1 + 2\exp\left(\dfrac{-2\mu}{1-\mu}\right), & \text{if} \quad 0 \leq \mu \leq 1, \\ 3 - 4\mu, & \text{if} \quad \mu \leq 0. \end{cases}$$

Furthermore, Keogh and Merkes [6] derived sharp estimates for $|a_3 - \mu a_2^2|$ when f is close-to-convex, starlike, and convex in Δ.

In our paper we have defined a new subclass of $\mathcal{A}$ using the concept of subordination and the linear operator $\mathfrak{D}_q$ as below:

Definition 1.1. For $0 < q < 1$ and $0 \leq \vartheta \leq 1$, we let $\mathcal{M}_q(\vartheta, \Phi)$ denotes the subclass of $\mathcal{A}$, members of which are of the form (1.1) and satisfy the condition

$$\left| \left(\frac{z\mathfrak{D}_q f(z)}{(1-\vartheta) f(z) + \vartheta z} \right)^2 - \frac{1}{1-q} \right| < \frac{1}{1-q},\ z \in \Delta,$$

or equivalently

$$\frac{z\mathfrak{D}_q f(z)}{(1-\vartheta)f(z)+\vartheta z} \prec \sqrt{\frac{2(1+z)}{2+(1-q)z}} = \Phi(z), \tag{1.3}$$

with $\left.\sqrt{\dfrac{2(1+z)}{2+(1-q)z}}\,\right|_{z=0} = 1.$

By fixing $\vartheta = 1$ we deduce a new class $\mathcal{R}_q(\Phi)$ as defined below:

Definition 1.2. Let $0 < q < 1$ and $\vartheta = 1$, $\mathcal{M}_q(1,\Phi) =: \mathcal{R}_q(\Phi)$ denotes the subclass of $\mathcal{A}$, members of which are of the form (1.1) and satisfy the subordination condition

$$\mathfrak{D}_q f(z) \prec \sqrt{\frac{2(1+z)}{2+(1-q)z}}, \quad \text{with} \quad \left.\sqrt{\frac{2(1+z)}{2+(1-q)z}}\,\right|_{z=0} = 1$$

or equivalently

$$\left|(\mathfrak{D}_q f(z))^2 - \frac{1}{1-q}\right| < \frac{1}{1-q},\ z \in \Delta.$$

Further, by taking $\vartheta = 0$ we get $\mathcal{M}_q(0,\Phi) =: \mathcal{S}_q(\Phi)$ [15]:

Definition 1.3. Let $0 < q < 1$ and $\vartheta = 0$, $\mathcal{M}_q(0,\Phi) =: \mathcal{S}_q(\Phi)$ denotes the subclass of $\mathcal{A}$, members of which are of the form (1.1) and satisfy the subordination condition

$$\frac{z\mathfrak{D}_q f(z)}{f(z)} \prec \sqrt{\frac{2(1+z)}{2+(1-q)z}},$$

or equivalently

$$\left|\left(\frac{z\mathfrak{D}_q f(z)}{f(z)}\right)^2 - \frac{1}{1-q}\right| < \frac{1}{1-q},\ z \in \Delta,$$

with $\left.\sqrt{\dfrac{2(1+z)}{2+(1-q)z}}\,\right|_{z=0} = 1$, where $0 < q < 1$.

Remark 1.4. (i) If $q \to 1^-$, we let $\mathcal{M}_q(\vartheta,\Phi) =: \mathcal{S}(\vartheta,\Phi)$, that is a function $f \in \mathcal{S}(\vartheta,\Phi)$ if it satisfies the subordination condition

$$\frac{zf'(z)}{(1-\vartheta)f(z)+\vartheta z} \prec \sqrt{1+z}, \quad \text{with} \quad \left.\sqrt{1+z}\,\right|_{z=0} = 1,$$

or equivalently

$$\left|\left(\frac{zf'(z)}{(1-\vartheta)f(z)+\vartheta z}\right)^2 - 1\right| < 1,\ z \in \Delta.$$

(ii) Remark that the subclass

$$\mathcal{S}(0,\phi) = \mathcal{SL}^* := \left\{ f \in \mathcal{A} : \left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| < 1,\ z \in \Delta \right\},$$

was introduced and studied by Sokól and Stankiewicz [23], and the subclass

$$\mathcal{S}(1,\phi) = \mathcal{R}(\phi) := \left\{ f \in \mathcal{A} : \left|(f'(z))^2 - 1\right| < 1,\ z \in \Delta \right\}$$

was studied by Sahoo and Patel [21].

In our work we have used the techniques of Libera and Zlotkiewicz [9] and Koepf [7], we obtained the initial coefficient estimates for a_2, a_3, Fekete-Szegő inequality results for $f \in \mathcal{M}_q(\vartheta,\Phi)$ and $f^{-1} \in \mathcal{M}_q(\vartheta,\Phi)$. As a special case of our result, we obtain Fekete-Szegő inequality for a class of functions defined through Poisson distribution and further with the help of MAPLE™ software we find Hankel determinant for$f \in \mathcal{M}_q(\vartheta,\Phi)$.

2 Preliminaries

To establish our main results, we shall need the followings lemmas. The first lemma is the well-known *Carathéodory's lemma* (see also [16, Corollary 2.3.]):

Lemma 2.1. [1] If $\phi \in \mathcal{P}$ and given by (1.2), then $|p_k| \leq 2$ for all $k \geq 1$, and the result is best possible for the function $\phi_1(z) = \dfrac{1+\eta z}{1-\eta z}$, $|\eta| = 1$.

The next lemma gives us a majorant for the coefficients of the functions of the class $\mathcal{P}$, and more details may be found in [11, Lemma 1]:

Lemma 2.2. [10] Let the function p given by (1.2) be a member of the class $\mathcal{P}$. Then,

$$\left|p_2 - \nu\, p_1^2\right| \leq 2\max\left\{1; |2\nu - 1|\right\}, \text{ where } \nu \in \mathbb{C}. \tag{2.1}$$

The result is sharp for the functions given by

$$\phi_2(z) = \frac{1+\eta^2 z^2}{1-\eta^2 z^2} \quad \text{and} \quad \phi_2(z) = \frac{1+\eta z}{1-\eta z},\ |\eta| = 1.$$

Lemma 2.3. [11] If $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is a function with positive real part in Δ, then

$$\left|c_2 - v c_1^2\right| \leq \begin{cases} -4v + 2, & \text{if} \quad v \leq 0, \\ 2, & \text{if} \quad 0 \leq v \leq 1, \\ 4v - 2, & \text{if} \quad v \geq 1. \end{cases} \tag{2.2}$$

When $v < 0$ or $v > 1$, the equality holds if and only if $p_1(z)$ is $\dfrac{1+z}{1-z}$ or one of its rotations. If $0 < v < 1$, then equality holds if and only if $p_1(z)$ is $\dfrac{1+z^2}{1-z^2}$ or one of its rotations. If $v = 0$, the equality holds if and only if

$$p_1(z) = \left(\frac{1}{2} + \frac{\eta}{2}\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{\eta}{2}\right)\frac{1-z}{1+z} \quad (0 \leq \eta \leq 1)$$

or one of its rotations. If $v = 1$, the equality holds if and only if p_1 is the reciprocal of one of the functions such that the equality holds in the case of $v = 0$. Although the above upper bound is sharp, when $0 < v < 1$, it can be improved as follows:

$$\left|c_2 - v c_1^2\right| + v|c_1|^2 \leq 2, \quad \text{if} \quad 0 < v \leq \frac{1}{2}, \tag{2.3}$$

$$\left|c_2 - v c_1^2\right| + (1-v)|c_1|^2 \leq 2, \quad \text{if} \quad \frac{1}{2} < v \leq 1. \tag{2.4}$$

Lemma 2.4. [10] Let $\phi \in \mathcal{P}$ given by (1.2). Then,

$$p_2 = \frac{1}{2}\left[p_1^2 + \left(4 - p_1^2\right)x\right], \tag{2.5}$$

and

$$p_3 = \frac{1}{4}\left[p_1^3 + 2\left(4 - p_1^2\right)p_1 x - \left(4 - p_1^2\right)p_1 x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right)z\right] \tag{2.6}$$

for some complex numbers x, z satisfying $|x| \leq 1$ and $|z| \leq 1$.

3 Fekete-Szegő inequality for $f \in \mathcal{M}_q(\vartheta, \Phi)$

In our first result we will determine an upper bound for $\left|a_3 - \mu\, a_2^2\right|$, and this tends to solve the Fekete-Szegő problem for the subclass $\mathcal{M}_q(\vartheta, \Phi)$.

Theorem 3.1. For $f \in \mathcal{M}_q(\vartheta, \Phi)$ and is in the form given by (1.1) then,

$$|a_2| \le \frac{1+q}{4(q+\vartheta)},$$
$$|a_3| \le \frac{1+q}{4(q^2+q+\vartheta)} \max\left\{1; \left|\frac{3q^2+(\vartheta-3)q+(2-7\vartheta)}{8(q+\vartheta)}\right|\right\},$$

and for any $\mu \in \mathbb{C}$ we have

$$\left|a_3 - \mu\, a_2^2\right| \le \frac{1+q}{4\,(q^2+q+\vartheta)} \times \max\left\{1; \left|\frac{2\mu(1+q)(q^2+q+\vartheta) - \left[3q^2+(\vartheta-3)q+(2-7\vartheta)\right](q+\vartheta)}{8(q+\vartheta)^2}\right|\right\}. \tag{3.1}$$

Proof . If $f \in \mathcal{M}_q(\vartheta, \Phi)$, from (1.3) it follows that there exists a function $\psi \in \mathcal{H}(\Delta)$ satisfying the conditions $\psi(0) = 0$ and $|\psi(z)| < 1$, $z \in \Delta$, such that

$$\frac{z\mathfrak{D}_q f(z)}{(1-\vartheta)f(z)+\vartheta z} \prec \sqrt{\frac{2(1+z)}{2+(1-q)z}}. \tag{3.2}$$

Setting

$$p(z) := \frac{1+\psi(z)}{1-\psi(z)} = 1 + p_1 z + p_2 z^2 + \dots,\ z \in \Delta,$$

then $p \in \mathcal{P}$. From the above relation, we get

$$\psi(z) = \frac{p(z)-1}{p(z)+1},\ z \in \Delta,$$

and from (3.2) it follows that

$$\frac{z\mathfrak{D}_q f(z)}{(1-\vartheta)f(z)+\vartheta z} = \left(\frac{4p(z)}{(1+q)+(3-q)p(z)}\right)^{\frac{1}{2}},\ z \in \Delta.$$

It is easy to show that

$$\left(\frac{4p(z)}{(1+q)+(3-q)p(z)}\right)^{\frac{1}{2}} = 1 + \frac{1+q}{8}p_1 z + \frac{1+q}{128}\left[16p_2 + (3q-13)p_1^2\right] z^2$$
$$+\frac{1+q}{1024}\left[128p_3 + (48q-208)p_1p_2 + (5q^2-38q+85)p_1^3\right] z^3 + \dots,\ z \in \Delta, \tag{3.3}$$

and

$$\frac{z\mathfrak{D}_q f(z)}{(1-\vartheta)f(z)+\vartheta z} = 1 + \left[[2]_q - (1-\vartheta)\right] a_2 z + \left[([3]_q-(1-\vartheta))a_3 - [(1-\vartheta)[2]_q - (1-\vartheta)^2]a_2^2\right] z^2$$
$$+\left\{([4]_q-(1-\vartheta))a_4 - (1-\vartheta)\left[[3]_q + [2]_q - 2(1-\vartheta)\right] a_2a_3 + (1-\vartheta)^2[(1+q)-(1-\vartheta)]a_2^3\right\} z^3 + \dots.$$
$$= 1 + (q+\vartheta)a_2 z + \left[\left(q+q^2+\vartheta\right) a_3 - (1-\vartheta)(q+\vartheta)a_2^2\right] z^2$$
$$+\left[(q+q^2+q^3+\vartheta)a_4 - (1-\vartheta)(q^2+2q+2\vartheta)a_2a_3 + (1-\vartheta)^2(q+\vartheta)a_2^3\right] z^3 + \dots. \tag{3.4}$$

Equating the corresponding coefficients of (3.3) and (3.4), we deduce that

$$a_2 = \frac{(1+q)p_1}{8(q+\vartheta)}, \tag{3.5}$$
$$a_3 = \frac{1+q}{8(q^2+q+\vartheta)}\left[p_2 + \frac{3q^2+(\vartheta-11)q+(2-15\vartheta)}{16(q+\vartheta)}p_1^2\right], \tag{3.6}$$

and from Lemma 2.1, we get

$$|a_2| \le \frac{1+q}{4(q+\vartheta)}.$$

Now from (3.6), we have

$$\begin{aligned}|a_3| &\leq \frac{1+q}{8(q^2+q+\vartheta)}\left|p_2+\frac{3q^2+(\vartheta-11)q+(2-15\vartheta)}{16(q+\vartheta)}p_1^2\right| \\ &= \frac{1+q}{8(q^2+q+\vartheta)}\left|p_2-\frac{p_1^2}{2}\left(\frac{-3q^2+(11-\vartheta)q+(15\vartheta-2)}{8(q+\vartheta)}\right)\right|,\end{aligned}$$

and by using the estimate

$$\left|c_2-vc_1^2\right| \leq 2\max(1;|2v-1|)$$

given in Lemma 2.2 we deduce

$$\begin{aligned}|a_3| &\leq \frac{1+q}{4(q^2+q+\vartheta)}\max\left\{1;\left|2\times\frac{1}{2}\left(\frac{-3q^2+(11-\vartheta)q+(15\vartheta-2)}{8(q+\vartheta)}\right)-1\right|\right\} \\ &= \frac{1+q}{4(q^2+q+\vartheta)}\max\left\{1;\left|\frac{-3q^2+(3-\vartheta)q+(7\vartheta-2)}{8(q+\vartheta)}\right|\right\} \\ &= \frac{1+q}{4(q^2+q+\vartheta)}\max\left\{1;\left|-\frac{3q^2+(\vartheta-3)q+(2-7\vartheta)}{8(q+\vartheta)}\right|\right\}.\end{aligned}$$

Thus, from (3.5) and (3.6) we get

$$a_3-\mu\,a_2^2=\frac{1+q}{8(q^2+q+\vartheta)}\left[p_2-p_1^2\left(\frac{2\mu(1+q)(q^2+q+\vartheta)-[3q^2+(\vartheta-11)q+(2-15\vartheta)](q+\vartheta)}{16(q+\vartheta)^2}\right)\right],$$

which with the aid of the inequality (2.1) of Lemma 2.2 yields

$$\left|a_3-\mu\,a_2^2\right| \leq \frac{1+q}{4(q^2+q+\vartheta)}\max\left\{1;\left|\left(\frac{2\mu(1+q)(q^2+q+\vartheta)-[3q^2+(\vartheta-11)q+(2-15\vartheta)](q+\vartheta)}{8(q+\vartheta)^2}\right)-1\right|\right\},$$

that is the required estimate (3.1). □

For $\vartheta=1$ the above theorem reduces to the following special case:

Corollary 3.2. If $f\in\mathcal{R}_q(\Phi)$ and is given by (1.1) then, for any $\mu\in\mathbb{C}$ we have

$$\left|a_3-\mu\,a_2^2\right|=\frac{1+q}{4(q^2+q+1)}\max\left\{1;\left|\frac{2\mu(q^2+q+1)-[3q^2-2q-5]}{8(q+1)}\right|\right\}.$$

If we take $\mu\in\mathbb{R}$ in Theorem 3.1 we get the next special case:

Theorem 3.3. If the function $f\in\mathcal{M}_q(\vartheta,\Phi)$ and is given by (1.1), with $\mu\in\mathbb{R}$, then

$$\left|a_3-\mu\,a_2^2\right| \leq \begin{cases}\dfrac{1+q}{8(q^2+q+\vartheta)}\left(\dfrac{\Lambda(q,\vartheta)}{4\xi}-\dfrac{2\mu(q+1)(q^2+q+\vartheta)}{4\xi^2}\right), & \text{if} \quad \mu<\sigma_1, \\ \dfrac{1+q}{8(q^2+q+\vartheta)}, & \text{if} \quad \sigma_1\leq\mu\leq\sigma_2, \\ \dfrac{1+q}{8(q^2+q+\vartheta)}\left(-\dfrac{\Lambda(q,\vartheta)}{4\xi}+\dfrac{2\mu(q+1)(q^2+q+\vartheta)}{4\xi^2}\right), & \text{if} \quad \mu>\sigma_2,\end{cases}$$

with

$$\begin{aligned}\sigma_1 &:= \frac{[3q^2+(\vartheta-11)q+(2-15\vartheta)](q+\vartheta)}{2(q^2+q+\vartheta)(1+q)}=\frac{\Phi(q,\vartheta)\xi}{2(q^2+q+\vartheta)(1+q)}, \\ \sigma_2 &:= \frac{[3q^2+(\vartheta+5)q+(2+\vartheta)](q+\vartheta)}{2(q^2+q+\vartheta)(1+q)},\end{aligned}$$

where for convenience we let

$$\Lambda(q,\vartheta):=3q^2+(\vartheta-3)q+(2-7\vartheta), \tag{3.7}$$

$$\Psi(q,\vartheta):=3q^2+(\vartheta-11)q+(2-15\vartheta) \tag{3.8}$$

and

$$\xi := q + \vartheta. \tag{3.9}$$

These results are sharp.

Further, if $\sigma_1 \leq \mu \leq \sigma_3$, then

$$|a_3 - \mu a_2^2| + \left(\frac{2\mu(1+q)(q^2+q+\vartheta) - [3q^2 + (\vartheta - 11)q + (2 - 15\vartheta)](q+\vartheta)}{2(q+\vartheta)(q^2+q+\vartheta)}\right)|a_2|^2 \leq \frac{1+q}{4(q^2+q+\vartheta)}. \tag{3.10}$$

If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$|a_3 - \mu a_2^2| + \left(\frac{[3q^2 + (\vartheta + 5)q + (2 + \vartheta)](q+\vartheta) - 2\mu(1+q)(q^2+q+\vartheta)}{2(q+\vartheta)(q^2+q+\vartheta)}\right)|a_2|^2 \leq \frac{1+q}{4(q^2+q+\vartheta)} \tag{3.11}$$

where

$$\sigma_3 := \frac{[3q^2 + (\vartheta - 3)q + (2 - 7\vartheta)](q+\vartheta)}{2(q^2+q+\vartheta)(1+q)} = \frac{\Lambda(q,\vartheta)\xi}{2(q^2+q+\vartheta)(1+q)}.$$

These results are sharp.

Proof . From (3.5) and (3.6) we have

$$a_3 - \mu\, a_2^2 = \frac{1+q}{8(q^2+q+\vartheta)}\left[p_2 + \frac{3q^2 + (\vartheta - 11)q + (2 - 15\vartheta)}{16(q+\vartheta)} p_1^2\right] - \mu \frac{(1+q)^2 p_1^2}{64(q+\vartheta)^2},$$

that is

$$\begin{aligned} a_3 - \mu\, a_2^2 &= \frac{1+q}{8(q^2+q+\vartheta)}\left[p_2 - p_1^2\left(\frac{2\mu(1+q)(q^2+q+\vartheta) - [3q^2 + (\vartheta - 11)q + (2 - 15\vartheta)](q+\vartheta)}{16(q+\vartheta)^2}\right)\right], \\ &= \frac{1+q}{8(q^2+q+\vartheta)}\left(p_2 - \nu p_1^2\right), \end{aligned} \tag{3.12}$$

where

$$\nu = \frac{2\mu(1+q)(q^2+q+\vartheta) - [3q^2 + (\vartheta - 11)q + (2 - 15\vartheta)](q+\vartheta)}{16(q+\vartheta)^2}.$$

From the assumptions, using the second above equality if follows that $\nu \in \mathbb{R}$. We have

$$4\nu - 2 = \frac{1+q}{8(q^2+q+\vartheta)}\left(-\frac{\Lambda(q,\vartheta)}{4\xi} + \frac{2\mu(q+1)(q^2+q+\vartheta)}{4\xi^2}\right),$$

$\nu \geq 1$ is equivalent to $\mu \geq \sigma_2$, and $\nu \leq 0$ is equivalent to $\mu \leq \sigma_1$.

The assertion of Theorem 3.3 now follows by an application of Lemma 2.3.

For the proof of the second part, first we see that $0 < \nu \leq 1/2$ is equivalent to $\sigma_1 < \mu \leq \sigma_3$. Using the relations (3.12) and (3.5), and then applying the inequality (2.3) of Lemma 2.3 we get

$$|a_3 - \mu a_2^2| + (\mu - \sigma_1)|a_2^2| = |a_3 - \mu a_2^2| + |\mu - \sigma_1||a_2^2| = \frac{1+q}{8(q^2+q+\vartheta)}\left[|p_2 - \nu a_1^2| + \nu|p_1^2|\right] \leq \frac{1+q}{4(q^2+q+\vartheta)},$$

that represents the required inequality (3.10).

Further, we easily check that $1/2 \leq \nu < 1$ is equivalent to $\sigma_3 \leq \mu < \sigma_2$. From the relations (3.12) and (3.5), and then applying the inequality (2.4) of Lemma 2.3 we obtain

$$|a_3 - \mu a_2^2| + (\sigma_2 - \mu)|a_2^2| = |a_3 - \mu a_2^2| + |\sigma_2 - \mu||a_2^2| = \frac{1+q}{8(q^2+q+\vartheta)}\left[|p_2 - \nu a_1^2| + (1-\nu)|p_1^2|\right] \leq \frac{1+q}{4(q^2+q+\vartheta)},$$

that is the inequality (3.11).

Clearly, the result is sharp for the function

$$\frac{z\mathfrak{D}_q F_1(z)}{(1-\vartheta)F_1(z) + \vartheta z} = \sqrt{\frac{2(1+z)}{2+(1-q)z}}$$

if $\mu < \sigma_1$ or $\mu > \sigma_2$, or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if and only if

$$\frac{z\mathfrak{D}_q F_1(z)}{(1-\vartheta)F_1(z)+\vartheta z} = \sqrt{\frac{2[2+2(1-q)z+(1+q^2)z^2]}{(2+(1-q)z)^2}}$$

or one of its rotations. □

Remark 3.4. For $\vartheta = 1$ in Theorem 3.3 one can easily deduces the corresponding results for $f \in \mathcal{R}_q(\Phi)$.

4 Coefficient inequalities for some special functions of $\mathcal{M}_q(\vartheta, \Phi)$

Theorem 4.1. If $f \in \mathcal{M}_q(\vartheta, \Phi)$ and $f^{-1}(w) = w + \sum\limits_{n=2}^{\infty} d_n w^n$ is the analytic continuation to Δ of the inverse function of f, then for any complex number μ we have

$$\left|d_3 - \mu d_2^2\right| \leq \frac{1+q}{4(q^2+q+\vartheta)} \max\left\{1; \left|\frac{\Psi(q,\vartheta)}{8\xi} + \frac{(2-\mu)(1+q)(q^2+q+\vartheta)}{4\xi^2} - 1\right|\right\} \tag{4.1}$$

where $\Psi(q,\vartheta)$ and ξ are given by (3.8)and (3.9) respectively.

Proof . As

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n = z + d_2 w^2 + d_3 w^3 + \cdots . \tag{4.2}$$

is the inverse function of f, it can be seen that

$$f^{-1}(f(z)) = f\left(f^{-1}(z)\right) = z. \tag{4.3}$$

From equations (4.2) and (4.3), we have

$$f^{-1}\left(z + \sum_{n=2}^{\infty} a_n z^n\right) = z. \tag{4.4}$$

Thus (4.3) and (4.4) yields

$$z + (a_2 + d_2)z^2 + (a_3 + 2a_2 d_2 + d_3)z^3 + \cdots = z,$$

hence by equating corresponding coefficients of z, z^2, z^3, it can be seen that

$$d_2 = -a_2, \tag{4.5}$$

$$d_3 = 2a_2^2 - a_3. \tag{4.6}$$

From relations (3.5), (3.6), (4.5) and (4.6) it follows that

$$\begin{aligned} d_2 &= -\frac{(1+q)p_1}{8(q+\vartheta)} = -\frac{(1+q)p_1}{8\xi}, \\ d_3 &= \frac{(1+q)^2 p_1^2}{32(q+\vartheta)^2} - \frac{1+q}{8(q^2+q+\vartheta)}\left[p_2 + \frac{3q^2+(\vartheta-11)q+(2-15\vartheta)}{16(q+\vartheta)} p_1^2\right], \\ &= -\frac{1+q}{8(q^2+q+\vartheta)}\left[p_2 - \left(\frac{\Psi(q,\vartheta)}{16\xi} + \frac{(1+q)(q^2+q+\vartheta)}{4\xi^2}\right)p_1^2\right], \end{aligned}$$

where $\Psi(q,\vartheta)$ and ξ are given by (3.8)and (3.9) respectively.

For any complex number μ, consider

$$d_3 - \mu d_2^2 = -\frac{1+q}{8(q^2+q+\vartheta)}\left[p_2 - \left(\frac{\Psi(q,\vartheta)}{16\xi} + \frac{(2-\mu)(1+q)(q^2+q+\vartheta)}{8\xi^2}\right)p_1^2\right], \tag{4.7}$$

where $\Psi(q,\vartheta)$ and ξ are given by (3.8)and (3.9) respectively. Taking modules on the both sides and by applying Lemma 2.2 on the right hand side of (4.7), one can obtain the result as in (4.1). □

Theorem 4.2. If $f \in \mathcal{M}_q(\vartheta, \Phi)$ and $G(z) = \dfrac{z}{f(z)} = 1 + \sum\limits_{n=1}^{\infty} q_n z^n$, $z \in \Delta$, then for any complex number μ we have

$$|q_2 - \mu q_1^2| \leq \frac{1+q}{4(q^2+q+\vartheta)} \left| \frac{\Psi(q,\vartheta)}{8\xi} + \frac{(1+\mu)(1+q)(q^2+q+\vartheta)}{4\xi^2} - 1 \right|, \tag{4.8}$$

where $\Psi(q, \vartheta)$ and ξ are given by (3.8) and (3.9), respectively, and the result is sharp.

Proof . Since $f \in \mathcal{M}_q(\vartheta, \Phi)$ and

$$G(z) = \frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} q_n z^n, \ z \in \Delta, \tag{4.9}$$

by a simple computation one can obtain that

$$\frac{z}{f(z)} = 1 - a_2 z + (a_2^2 - a_3) z^2 + \dots, \ z \in \Delta. \tag{4.10}$$

Upon equating the coefficient of z and z^2, from relations (4.9) and (4.10) it can be seen that

$$q_1 = -a_2, \tag{4.11}$$

$$q_2 = -a_3 + a_2^2. \tag{4.12}$$

From equations (3.5), (3.6), (4.11) and (4.12), we obtain

$$\begin{aligned} q_1 &= -\frac{(1+q)p_1}{8(q+\vartheta)} = -\frac{(1+q)p_1}{8\xi}, \\ q_2 &= \frac{(1+q)^2 p_1^2}{64(q+\vartheta)^2} - \frac{1+q}{8(q^2+q+\vartheta)} \left[p_2 + \frac{3q^2 + (\vartheta - 11)q + (2 - 15\vartheta)}{16(q+\vartheta)} p_1^2 \right], \\ &= -\frac{1+q}{8(q^2+q+\vartheta)} \left[p_2 - \left(\frac{\Psi(q,\vartheta)}{16\xi} + \frac{(1+q)(q^2+q+\vartheta)}{8\xi^2} \right) p_1^2 \right], \\ &= -\frac{1+q}{8(q^2+q+\vartheta)} \left[p_2 - \frac{p_1^2}{2} \left(\frac{\Psi(q,\vartheta)}{8\xi} + \frac{(1+q)(q^2+q+\vartheta)}{4\xi^2} \right) \right]. \end{aligned}$$

For any complex number μ, consider

$$q_2 - \mu q_1^2 = -\frac{1+q}{8(q^2+q+\vartheta)} \left[p_2 - \frac{p_1^2}{2} \left(\frac{\Psi(q,\vartheta)}{8\xi} + \frac{(1+\mu)(1+q)(q^2+q+\vartheta)}{4\xi^2} \right) \right], \tag{4.13}$$

and taking modules on both sides of the above relation, by applying Lemma 2.2 for the right hand side of (4.13), one can obtain the inequality (4.8). The sharpness of the result follows from the sharpness of Lemma 2.2. □

5 Application to Functions Defined by Poisson distribution

A variable $\mathcal{X}$ is said to be Poisson distributed if it takes the values $0, 1, 2, 3, \dots$ with probabilities $e^{-\kappa}$, $\kappa \dfrac{e^{-\kappa}}{1!}$, $\kappa^2 \dfrac{e^{-\kappa}}{2!}$, $\kappa^3 \dfrac{e^{-\kappa}}{3!}$, ... respectively, where κ is called the parameter. Thus,

$$P(\mathcal{X} = \tau) = \frac{\kappa^\tau e^{-\kappa}}{\tau!}, \ \tau = 0, 1, 2, 3, \dots.$$

In [17], Porwal introduced a power series whose coefficients are probabilities of Poisson distribution

$$\mathcal{I}(\kappa, z) = z + \sum_{n=2}^{\infty} \frac{\kappa^{n-1}}{(n-1)!} e^{-\kappa} z^n, \ z \in \Delta,$$

where $\kappa > 0$. We note that by using the familiar ratio test the radius of convergence of the above series is infinity. Lately, Porwal [17] (see also [13, 14, 18]) introduced a new linear operator $\mathcal{I}^\kappa(z) : \mathcal{A} \to \mathcal{A}$ given by

$$\mathcal{I}^\kappa f(z) := \mathcal{I}(\kappa, z) * f(z) = z + \sum_{n=2}^{\infty} \frac{\kappa^{n-1}}{(n-1)!} e^{-\kappa} a_n z^n = z + \sum_{n=2}^{\infty} \varphi_n(\kappa) a_n z^n, \ z \in \Delta,$$

where $\varphi_n = \varphi_n(\kappa) = \dfrac{\kappa^{n-1}}{(n-1)!}e^{-\kappa}$, and "$*$" denotes the convolution (or Hadamard) product of the two power series. In particular,

$$\varphi_2 = \kappa e^{-\kappa} \text{ and } \varphi_3 = \frac{\kappa^2}{2}e^{-\kappa}.$$

We define the class $\mathcal{M}_q^\kappa(\vartheta,\Phi)$ in the following way

$$\mathcal{M}_q^\kappa(\vartheta,\Phi) := \left\{f \in \mathcal{A} : \mathcal{I}^\kappa f \in \mathcal{M}_q(\vartheta,\Phi)\right\},$$

where $\mathcal{M}_q(\vartheta,\Phi)$ is given by Definition 1.1. We could obtain the coefficient estimates for functions of this class $\mathcal{M}_q^\kappa(\vartheta,\Phi)$ from the corresponding estimates for functions of the class $\mathcal{M}_q(\vartheta,\Phi)$. Applying the Theorems 3.1 and 3.3 for the function $\mathcal{I}^\kappa f$ we get the following Theorems 5.1 and 5.2 after an obvious change of the parameter μ:

Theorem 5.1. Let $0 < q < 1$, and $0 \le \vartheta \le 1$. Further, let

$$\mathcal{I}^\kappa f(z) = z + \varphi_2 a_2 z^2 + \varphi_3 a_3 z^3 + \ldots,\ z \in \Delta,$$

where $\varphi_2 = \kappa e^{-\kappa}$ and $\varphi_3 = \dfrac{\kappa^2}{2}e^{-\kappa}$. If $f \in \mathcal{M}_q^\kappa(\vartheta,\Phi)$, then for any complex μ we have

$$\left|a_3 - \mu a_2^2\right| \le \frac{1+q}{4(q^2+q+\vartheta)\varphi_3}\max\left\{1; \left|\left(\frac{2\mu(1+q)(q^2+q+\vartheta)\varphi_3}{4\xi^2\varphi_2^2} - \frac{\Psi(q,\vartheta)}{8\xi}\right) - 1\right|\right\}.$$

In particular, taking $\varphi_2 = \kappa e^{-\kappa}$ and $\varphi_3 = \dfrac{\kappa^2}{2}e^{-\kappa}$, we get

$$\left|a_3 - \mu a_2^2\right| \le \frac{1+q}{2(q^2+q+\vartheta)\kappa^2 e^{-\kappa}}\max\left\{1; \left|\left(\frac{\mu(1+q)(q^2+q+\vartheta)}{4\xi^2 e^{-\kappa}} - \frac{\Psi(q,\vartheta)}{8\xi}\right) - 1\right|\right\},$$

where $\Psi(q,\vartheta)$ and ξ are given by (3.8)and (3.9)respectively.

Theorem 5.2. Let $0 < q < 1$, $0 \le \vartheta \le 1$ and

$$\mathcal{I}^\kappa f(z) = z + \varphi_2 a_2 z^2 + \varphi_3 a_3 z^3 + \ldots,\ z \in \Delta.$$

If $f \in \mathcal{M}_q^\kappa(\vartheta,\Phi)$, and μ a real number, then

$$\left|a_3 - \mu\, a_2^2\right| \le \begin{cases} \dfrac{1+q}{8(q^2+q+\vartheta)\varphi_3}\left(\dfrac{\Lambda(q,\vartheta)}{4\xi} - \dfrac{2\mu(q+1)(q^2+q+\vartheta)\varphi_3}{4\xi^2\varphi_2^2}\right), & \text{if} \quad \mu < \sigma_1, \\ \dfrac{1+q}{8(q^2+q+\vartheta)\varphi_3}, & \text{if} \quad \sigma_1 \le \mu \le \sigma_2, \\ \dfrac{1+q}{8(q^2+q+\vartheta)\varphi_3}\left(-\dfrac{\Lambda(q,\vartheta)}{4\xi} + \dfrac{2\mu(q+1)(q^2+q+\vartheta)\varphi_3}{4\xi^2\varphi_2^2}\right), & \text{if} \quad \mu > \sigma_2, \end{cases}$$

$$\sigma_1 := \frac{\varphi_2^2}{\varphi_3}\frac{\Psi(q,\vartheta)\xi}{2(q^2+q+\vartheta)(1+q)}, \quad \sigma_2 := \frac{\varphi_2^2}{\varphi_3}\frac{[3q^2+(\vartheta+5)q+(2+\vartheta)]\xi}{2(q^2+q+\vartheta)(1+q)},$$

where $\Lambda(q,\vartheta)$, $\Psi(q,\vartheta)$ and ξ are given by (3.7), (3.8) and (3.9)respectively.

In particular, for $\varphi_2 = \kappa e^{-\kappa}$ and $\varphi_3 = \dfrac{\kappa^2}{2}e^{-\kappa}$, we get

$$\left|a_3 - \mu\, a_2^2\right| \le \begin{cases} \dfrac{1+q}{4(q^2+q+\vartheta)\kappa^2 e^{-\kappa}}\left(\dfrac{\Lambda(q,\vartheta)}{4\xi} - \dfrac{\mu(q+1)(q^2+q+\vartheta)}{4\xi^2 e^{-\kappa}}\right), & \text{if} \quad \mu < \sigma_1, \\ \dfrac{1+q}{4(q^2+q+\vartheta)\kappa^2 e^{-\kappa}}, & \text{if} \quad \sigma_1 \le \mu \le \sigma_2, \\ \dfrac{1+q}{4(q^2+q+\vartheta)\kappa^2 e^{-\kappa}}\left(-\dfrac{\Lambda(q,\vartheta)}{4\xi} + \dfrac{\mu(q+1)(q^2+q+\vartheta)}{4\xi^2 e^{-\kappa}}\right), & \text{if} \quad \mu > \sigma_2, \end{cases}$$

$$\sigma_1 := e^{-\kappa}\frac{\Psi(q,\vartheta)\xi}{(q^2+q+\vartheta)(1+q)}, \quad \sigma_2 := e^{-\kappa}\frac{[3q^2+(\vartheta+5)q+(2+\vartheta)]\xi}{(q^2+q+\vartheta)(1+q)},$$

where $\Lambda(q,\vartheta)$,$\Psi(q,\vartheta)$ and ξ are given by (3.7), (3.8) and (3.9)respectively.

Remark 5.3. Suitably fixing the parameters ϑ as stated in Definitions 1.2 and 1.3 in Theorems 3.1, 3.3 and 4.1 one can easily state above result for the function classes defined in Definitions 1.2 to 1.3 subordinated to Bernoulli's lemniscate. We also state analogues results as proved in Theorems 5.1 and 5.2. Further, allowing $q \to 1^-$ one can easily state the results for the function classes mentioned in Remark 1.4.

6 Hankel determinant result for $f \in \mathcal{M}_q(\vartheta, \Phi)$

Theorem 6.1. If $f \in \mathcal{M}_q(\vartheta, \Phi)$ has the form given by (1.1) then,

$$\left|a_2a_4 - a_3^2\right| \le \max\left\{\mathcal{M}(q,\vartheta); \mathcal{N}(q,\vartheta)\right\}, \tag{6.1}$$

with

$$\mathcal{M}(q,\vartheta) = \frac{8|k_3|\sqrt{3}}{9} + 16|A|, \text{ and } \mathcal{N}(q,\vartheta) = \frac{8|k_3|\sqrt{3}}{9} + 4|k_4|, \tag{6.2}$$

where A is given by (6.10), k_3 and k_4 are given by (6.6) and (6.7), respectively, and depends of Λ_1, Λ_2, Ψ, Ω and Υ defined by (6.4), (6.5), (3.8) and (6.8), respectively.

Proof . If $f \in \mathcal{M}_q(\vartheta, \Phi)$, using a similar proof like in the proof of Theorem 3.1, from (3.5) and (3.6) we get

$$\begin{aligned}(q^3 + q^2 + q + \vartheta)a_4 &- (1-\vartheta)(q^2 + 2q + 2\vartheta)a_2a_3 + (1-\vartheta)^2(q+\vartheta)a_2^3 \\ &= \frac{1+q}{1024}\left[128p_3 + (48q - 208)p_1p_2 + (5q^2 - 38q + 85)p_1^3\right],\end{aligned}$$

hence

$$\begin{aligned}a_4 =& \frac{1+q}{1024(q^3+q^2+q+\vartheta)}\left[\left(128p_3 + \frac{16\ \Lambda_2(p,\vartheta)}{(q+v)(q^2+q+\vartheta)}\right)p_1p_2\right. \\ &\left. + \frac{\Lambda_1(p,\vartheta)(q^2+q+\vartheta) + (1-\vartheta)(1+q)(q^2+2q+2\vartheta)\Psi(q,\vartheta)}{(q+v)^2(q^2+q+\vartheta)}p_1^3\right],\end{aligned} \tag{6.3}$$

where

$$\Lambda_1(p,\vartheta) := (5q^2 - 38q + 85)(q+\vartheta)^2 - 2(1-\vartheta)^2(1+q)^2, \tag{6.4}$$

$$\Lambda_2(p,\vartheta) := (3q - 13)(q+\vartheta)(q^2+q+\vartheta) + (1-\vartheta)(1+q)(q^2+2q+2\vartheta), \tag{6.5}$$

and $\Psi(q,\vartheta)$ as assumed in (3.8).

From (3.5), (3.6) and (6.3) we get

$$a_2a_4 - a_3^2 = k_1p_1^4 + k_2p_1^2p_2 + k_3p_1p_3 + k_4p_2^2,$$

where

$$k_1 = \frac{\Lambda_1(q,\vartheta)(q^2+q+\vartheta) + (1-\vartheta)(1+q)(q^2+2q+2\vartheta)\Psi(q,\vartheta)}{8(q+\vartheta)^2(q^2+q+\vartheta)}\Omega - \frac{(\Psi(q,v))^2}{256(q+\vartheta)^2}\Upsilon,$$

$$k_2 = \frac{2\Lambda_2(p,\vartheta)}{(q+\vartheta)(q^2+q+\vartheta)}\Omega - \frac{\Psi(q,v)}{8(q+\vartheta)}\Upsilon,$$

$$k_3 = 16\frac{(1+q)^2}{1024(q^3+q^2+q+\vartheta)} = 16\Omega, \tag{6.6}$$

$$k_4 = -\frac{(1+q)^2}{64(q^2+q+\vartheta)^2} = -\Upsilon, \tag{6.7}$$

and $\Lambda_1(q,\vartheta)$, $\Lambda_2(p,\vartheta)$ and $\Psi(q,\vartheta)$ are assumed as in (6.4), (6.5) and (3.8) respectively, while

$$\Omega = \frac{(1+q)^2}{1024(q^3+q^2+q+\vartheta)} \quad \text{and} \quad \Upsilon = \frac{(1+q)^2}{64(q^2+q+\vartheta)^2}. \tag{6.8}$$

Using the relations (2.5) and (2.6) of Lemma 2.4, we get

$$\left|a_2a_4-a_3^2\right|=\left|Ap_1^4+B\left(4-p_1^2\right)xp_1^2+\left(\frac{k_4}{4}\left(4-p_1^2\right)-\frac{k_3}{4}p_1^2\right)\left(4-p_1^2\right)x^2+\frac{k_3}{2}p_1\left(4-p_1^2\right)\left(1-|x|^2\right)z\right|, \tag{6.9}$$

with $|x|\leq 1$, $|z|\leq 1$, and

$$\begin{aligned} A:=&\frac{1}{4}\left(4k_1+2k_2+k_3+k_4\right)=\frac{\Lambda_1(q,\vartheta)(q^2+q+\vartheta)+(1-\vartheta)(1+q)(q^2+2q+2\vartheta)\Psi(q,\vartheta)}{8(q+\vartheta)^2(q^2+q+\vartheta)}\Omega\\ &-\frac{\Psi(q,\vartheta)^2\Upsilon}{256\left(q+\vartheta\right)^2}+\frac{\Lambda_2(p,\vartheta)}{(q+\vartheta)(q^2+q+\vartheta)}\Omega-\frac{\Psi(q,v)}{16(q+\vartheta)}\Upsilon+4\,\Omega-\frac{\Upsilon}{4},\\ B:=&\frac{1}{2}\left(k_2+k_3+k_4\right)=\frac{\Lambda_2(p,\vartheta)}{(q+\vartheta)(q^2+q+\vartheta)}\Omega-\frac{\Psi(q,v)}{16(q+\vartheta)}\Upsilon+8\,\Omega-\frac{\Upsilon}{2}, \end{aligned} \tag{6.10}$$

where $\Lambda_1(q,\vartheta),\Lambda_2(p,\vartheta)$ and $\Psi(q,\vartheta)$ are assumed as in (6.4), (6.5) and (3.8) respectively Ω and Υ are assumed in (6.8). Since $\phi\in\mathcal{P}$ it follows that $\phi\left(e^{-i\arg p_1}z\right)\in\mathcal{P}$, hence we may assume without loss of generality that $p:=p_1\geq 0$, and according to Lemma 2.1 it follows that $p\in[0,2]$. Now, using the triangle's inequality in (6.9) and substituting $|x|=t$ we get

$$\begin{aligned} \left|a_2a_4-a_3^2\right|\leq&|A|\,p^4+|B|\left(4-p^2\right)p^2t+\frac{|k_4|}{4}\left(4-p^2\right)^2t^2+\frac{|k_3|}{4}p^2\left(4-p^2\right)t^2+\frac{|k_3|}{2}p\left(4-p^2\right)(1-t^2)|z|\\ \leq&|A|\,p^4+|B|\left(4-p^2\right)p^2t+\frac{|k_4|}{4}\left(4-p^2\right)^2t^2+\frac{|k_3|}{4}p^2\left(4-p^2\right)t^2\\ &+\frac{|k_3|}{2}p\left(4-p^2\right)(1-t^2)=:\mathcal{G}(p,t),\quad(0\leq p\leq 2,\ 0\leq t\leq 1). \end{aligned}$$

Denoting

$$\begin{aligned} G_1(p,t):=&|A|\,p^4+|B|\left(4-p^2\right)p^2t,\\ G_2(p,t):=&\frac{|k_4|}{4}\left(4-p^2\right)^2t^2+\frac{|k_3|}{4}p^2\left(4-p^2\right)t^2+\frac{|k_3|}{2}p\left(4-p^2\right)(1-t^2), \end{aligned}$$

then $\mathcal{G}=G_1+G_2$, hence

$$\begin{aligned} \max\left\{\mathcal{G}(p,t):(p,t)\in[0,2]\times[0,1]\right\}\leq&\max\left\{G_1(p,t):(p,t)\in[0,2]\times[0,1]\right\}\\ &+\max\left\{G_2(p,t):(p,t)\in[0,2]\times[0,1]\right\}. \end{aligned} \tag{6.11}$$

Next, we will find maximum of $G_1(p,t)$ and $G_2(p,t)$ on the closed rectangle $[0,2]\times[0,1]$. Using the MAPLE™ software for the following code

```
[> Lambda[1] := (5*q^2 - 38*q + 85)*(q + v)^2 - 2*(1 - v)^2*(1 + q)^2;
[> Lambda[2] := (3*q - 13)*(q + v)*(q^2 + q + v)
+ (1 - v)*(1 + q)*(q^2 + 2*q + 2*v);
[> Phi := 3*q^2 + (v - 11)*q + 2 - 15*v;
[> Omega := (1 + q)^2/(1024*(q^3 + q^2 + q + v));
[> Upsilon := (1 + q)^2/(64*(q^2 + q + v)^2);
[> k[1] := (Lambda[1]*(q^2 + q + v)
+ (1 - v)*(1 + q)*(q^2 + 2*q + 2*v)*Phi)/(8*(q + v)^2*(q^2 + q + v))*Omega
- Phi^2/(256*(q + v)^2)*Upsilon;
[> k[2] := 2*Lambda[2]*Omega/((q + v)*(q^2 + q + v))
- Phi*Upsilon/(8*(q + v));
[> k[3] := 16*Omega;
[> k[4] := -Upsilon;
[> A := (4*k[1] + 2*k[2] + k[3] + k[4])/4;
[> B := (k[2] + k[3] + k[4])/2;
[> G1 := abs(A)*p^4 + abs(B)*(-p^2 + 4)*p^2*t + abs(k4)/4*(-p^2 + 4)^2*t^2
+ abs(k3)/4*p^2*(-p^2 + 4)*t^2;
[> G2 := abs(k3)/2*p*(-p^2 + 4)*(-t^2 + 1);
[> maximize(G1, p = 0 .. 2, t = 0 .. 1, location);
[> maximize(G2, p = 0 .. 2, t = 0 .. 1, location);
```

we get

$$\max\{G_1(p,t):(p,t)\in[0,2]\times[0,1]\}=\max\{G_1(2,0)=16|A|;G_1(0,1)=4|k_4|\},$$
$$\max\{G_2(p,t):(p,t)\in[0,2]\times[0,1]\}=G_2\left(\frac{2\sqrt{3}}{3}\right)=\frac{8|k_3|\sqrt{3}}{9},$$

hence

$$\max\{\mathcal{G}(p,t):(p,t)\in[0,2]\times[0,1]\}\leq\max\{\mathcal{M}(q,\vartheta);\mathcal{N}(q,\vartheta)\},$$

where $\mathcal{M}(q,\vartheta)$ and $\mathcal{N}(q,\vartheta)$ are defined by (6.2). □

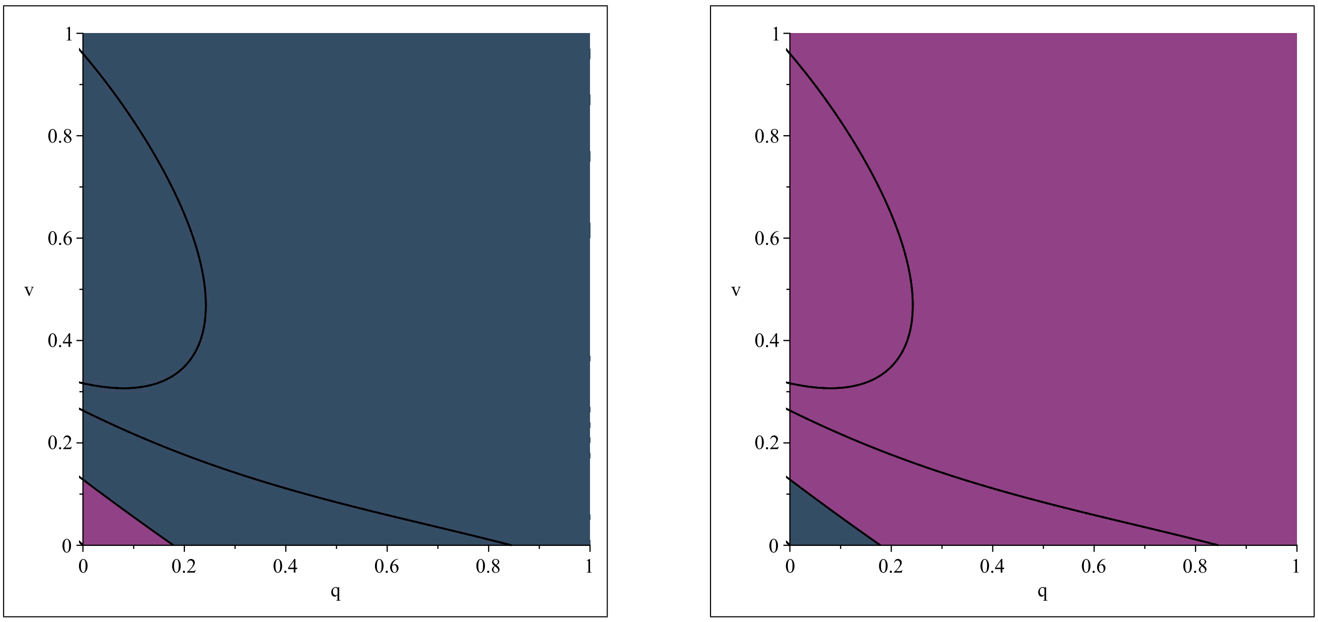
Remark 6.2. 1. Since in the above proof we used the inequality (6.11), and the global maximum for $G_1(p,t)$ and $G_2(p,t)$ are not attained at the same point, it follows that the upper bound of $\mathcal{G}(p,t)$ we found is not the best possible (the lowest one).

The reason we split the function $\mathcal{G}(p,t)$ in the sum of $G_1(p,t)$ and $G_2(p,t)$ is that the maximization of $\mathcal{G}(p,t)$ cannot be obtained by using MAPLE™ since the computation capacity of this software was exceeded.

2. Using the next MAPLE™ software codes

```
[> K := (8*abs(k[3])*sqrt(3))/9;
[> M := K + 16*abs(A);
[> N := K + 4*abs(k[4]);
[> with(plots):
[> inequal(M <= N, q = 0 .. 1, v = 0 .. 1, color = "Nautical 1",
optionsexcluded = [color = "Niagara DarkOrchid"]);
[> inequal(N <= M, q = 0 .. 1, v = 0 .. 1, color = "Nautical 1",
optionsexcluded = [color = "Niagara DarkOrchid"]);
```

the solutions of the inequalities $\mathcal{M}(q,\vartheta)\leq\mathcal{N}(q,\vartheta)$ and $\mathcal{M}(q,\vartheta)\geq\mathcal{N}(q,\vartheta)$ for $(q,\vartheta)\in(0,1)\times[0,1]$ are shown in the below Figure 1(a) and Figure 1(b), respectively, marked with blue colours:



(a) The solution of the inequality $\mathcal{M}(q,\vartheta)\leq\mathcal{N}(q,\vartheta)$

(b) The solution of the inequality $\mathcal{M}(q,\vartheta)\geq\mathcal{N}(q,\vartheta)$

According to the Figure 1(a) if $(q,\vartheta)\in D_{\mathcal{N}}$, where the set $D_{\mathcal{N}}\subset\mathbb{R}^2$ is given by

$$D_{\mathcal{N}}:=\left\{(q,\vartheta)\in\mathbb{R}^2:0<q<1,0\leq\vartheta\leq1,5q+5\vartheta-1\geq0\right\},$$

then, according to (6.1) we have

$$\left|a_2a_4 - a_3^2\right| \leq \mathcal{N}(q,\vartheta) = \frac{(q+1)^2\sqrt{3}}{72(q^3+q^2+q+\vartheta)} + \frac{(q+1)^2}{16(q^2+q+\vartheta)^2},$$

hence we have the following special case:

Corollary 6.3. If $f \in \mathcal{M}_q(\vartheta,\Phi)$ has the form given by (1.1) then,

$$\left|a_2a_4 - a_3^2\right| \leq \frac{(q+1)^2\sqrt{3}}{72(q^3+q^2+q+\vartheta)} + \frac{(q+1)^2}{16(q^2+q+\vartheta)^2},$$

whenever $(q,\vartheta) \in D_{\mathcal{N}}$.

Similarly, we get the next corollary:

Corollary 6.4. If $f \in \mathcal{M}_q(\vartheta,\Phi)$ has the form given by (1.1) then,

$$\left|a_2a_4 - a_3^2\right| \leq \mathcal{M}(q,\vartheta),$$

whenever $(q,\vartheta) \in D_{\mathcal{M}}$, where

$$D_{\mathcal{M}} := \left\{(q,\vartheta) \in \mathbb{R}^2 : 0 < q < 1, 0 \leq \vartheta \leq 1, 10q + 10\vartheta - 1 \leq 0\right\}.$$

References

[1] C. Carathéodory, *Über den Variabilitätsbereich der Fourier'schen Konstanten von positiven harmonischen Funktionen*, Rend. Circ. Mat. Palermo (2) **32** (1911), 193–217.

[2] P.L. Duren, *Univalent functions*, Grundlehren der Mathematischen Wissenschaften, 259, Springer-Verlag, New York, USA, 1983.

[3] M. Fekete and G. Szegő, *Eine Bemerkung über ungerade schlichte Funktionen*, J. Lond. Math. Soc. (2) **8** (1933), 85–89.

[4] M. E. H. Ismail, E. Merkes and D. Styer, *A generalization of starlike functions*, Complex Var. Theory Appl. **14** (1990), no. 1-4, 77–84.

[5] F.H. Jackson, *On q-functions and a certain difference operator*, Trans. Roy. Soc. Edin. **46** (1908), 253–281.

[6] F.R. Keogh and E. P. Merkes, *A coefficient inequality for certain classes of analytic functions*, Proc. Amer. Math. Soc. **20** (1969), 8–12.

[7] W. Koepf, *On the Fekete-Szegő problem for close-to-convex functions*, Proc. Amer. Math. Soc. **101** (1987), no. 1, 89–95.

[8] W. Koepf, *On the Fekete-Szegő problem for close-to-convex functions-II*, Arch. Math. (Basel) **49** (1987), 420–433.

[9] R.J. Libera and E.J. Zlotkiewicz, *Early coefficients of the inverse of a regular convex function*, Proc. Amer. Math. Soc. **85** (1982), no. 2, 225–230.

[10] R.J. Libera and E.J. Zlotkiewicz, *Coefficient bounds for the inverse of a function with derivative in $\mathcal{P}$*, Proc. Amer. Math. Soc. **87** (1983), no. 2, 251–257.

[11] W.C. Ma and D. Minda, *A unified treatment of some special classes of univalent functions*, In: Proceedings of the Conference on Complex Analysis (Tianjin, 1992), Z. Li, F. Ren, L. Yang and S. Zhang (Eds.), Int. Press, Cambridge, MA, (1994), 157–169.

[12] T.H. MacGregor, *Functions whose derivative have a positive real part*, Trans. Amer. Math. Soc. **104** (1962), no. 3, 532–537.

[13] G. Murugusundaramoorthy, *Subclasses of starlike and convex functions involving Poisson distribution series*, Afr. Mat. **28** (2017), 1357–1366.

[14] G. Murugusundaramoorthy, K. Vijaya and S. Porwal, *Some inclusion results of certain subclass of analytic functions associated with Poisson distribution series*, Hacet. J. Math. Stat. **45** (2016), no. 4, 1101–1107.

[15] K. Nazar, M. Shafiq, M. Darus, B.K. Qazai, A. Zahoor, *Upper bound of the third hankel determinant for a subclass of q–starlike functions associated with the lemniscate of Bernoulli*, J. Math. Ineq. **14** (2020), no. 1, 53-–65.

[16] Ch. Pommerenke, *Univalent Functions*, Vanderhoeck & Ruprecht, Göttingen, 1975.

[17] S. Porwal, *An application of a Poisson distribution series on certain analytic functions*, J. Complex Anal. (2014), Art. ID 984135, 1–3.

[18] S. Porwal and M. Kumar, *A unified study on starlike and convex functions associated with Poisson distribution series*, Afr. Mat. **27** (2016), no. 5, 1021–1027.

[19] M. Raza ans S. Malik, *Upper bound of third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli*, J. Inequal. Appl. **2013**:412, (2013), 1–8.

[20] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc. **49** (1975), no. 1, 109–115.

[21] A.K. Sahoo and J. Patel, *Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli*, Int. J. Anal. Appl. **6** (2014), no. 2, 170–177.

[22] J. Sokół, *Coefficient estimates in a class of strongly starlike functions*, Kyungpook Math. J. **49** (2009), no. 2, 349–353.

[23] J. Sokół and J. Stankiewicz, *Radius of convexity of some subclasses of strongly starlike functions*, Folia Scient. Univ. Tech. Resoviensis, Mat. **19** (1996), 101–105.

[24] H.M. Srivastava, *Operators of basic (or q-)calculus and fractional q-calculus and their applications in geometric function theory of complex analysis*, Iran. J. Sci. Technol. Trans. A Sci. **44** (2020), no. 1, 327–344.