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Dynamics of a system of higher order difference equations with a period-two coefficient

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Abstract

The aim of this paper is to study the dynamics of the system of two rational difference equations:

$$x_{n+1} = \alpha_n + \frac{y_{n-k}}{y_n}, \quad y_{n+1} = \alpha_n + \frac{x_{n-k}}{x_n}, \quad n = 0, 1, \dots$$

where $\{\alpha_n\}_{n\geq 0}$ is a two periodic sequence of nonnegative real numbers and the initial conditions x_i, y_i are arbitrary positive numbers for $i = -k, -k+1, -k+2, \dots, 0$ and $k \in \mathbb{N}$. We investigate the boundedness character of positive solutions. In addition, we establish some sufficient conditions under which the local asymptotic stability and the global asymptotic stability are assured. Furthermore, we determine the rate of the convergence of the solutions. Some numerical are considered in order to confirm our theoretical results.

Keywords: System of difference equations, Periodic solutions, Global asymptotic stability, Boundedness
2020 MSC: 39A10, 39A23, 39A22

1 Introduction

Difference equations and systems of difference equations emerge in mathematical models that describe problems in biology, economics and engineering [6]. In the last few decades, they have captivated the interest of the researchers. Particularly, there has been great interest in the study of the dynamics of rational difference equations and systems, (for example, see [1, 5, 8, 9, 10, 11, 13]).

In [14], Zhang *et al.* considered the symmetrical system of the rational difference equation

$$x_{n+1} = A + \frac{y_{n-k}}{y_n}, \quad y_{n+1} = A + \frac{x_{n-k}}{x_n}, \qquad n = 0, 1, \dots \tag{1.1}$$

with parameter $A > 0$, and the initial conditions x_i, y_i are arbitrary positive real numbers for $i = -k, -k+1, \dots, 0$ and $k \in \mathbb{Z}^+$. They investigated the asymptotic behavior of positive solutions of the system in the cases $0 < A < 1$, $A = 1$ and $A > 1$. When $0 < A < 1$, they established the existence of unbounded solutions of the system (1.1), and when $A = 1$ they proved that the system (1.1) has two periodic solutions. They found that any positive solution is

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bounded and persists. They also proved that the unique positive equilibrium point $(\bar{x}, \bar{y}) = (A+1, A+1)$ attracts all the positive solutions when $A > 1$.

Later, Gümüs [7] investigated the semi-cycles of the positive solutions for the system (1.1). The author also proved that if $A > 1$ then the unique positive equilibrium point $(\bar{x}, \bar{y}) = (A+1, A+1)$ is globally asymptotically stable. He suggested three open problems. In [2], Abualrub and Aloqeili considered the first open problem. They studied the oscillatory behavior, the boundedness, the persistence of the positive solutions and the global asymptotic stability of the unique positive equilibrium point of the system of two rational difference equations:

$$x_{n+1} = A + \frac{y_{n-k}}{y_n}, \quad y_{n+1} = B + \frac{x_{n-k}}{x_n}, \qquad n = 0, 1, \dots \tag{1.2}$$

with the parameters $A > 0, B > 0$ and the initial conditions x_i, y_i are arbitrary positive real numbers for $i = -k, -k+1, \dots, 0$ and $k \in \mathbb{Z}^+$.

In this paper, we give an answer to the second open problem in [7]:

Open Problem 2: *Investigate the dynamical behaviors of the system of difference equations*

$$x_{n+1} = \alpha_n + \frac{y_{n-k}}{y_n}, \quad y_{n+1} = \alpha_n + \frac{x_{n-k}}{x_n}, \qquad n = 0, 1, \dots \tag{1.3}$$

where $\{\alpha_n\}$ is a periodic sequence of nonnegative real numbers and the initial conditions x_i, y_i are arbitrary positive numbers for $i = -k, -k+1, -k+2, \dots, 0$ and $k \in \mathbb{Z}^+$.

In this work, we consider the system (1.3) when the period of $\{\alpha_n\}$ is two; namely, $\alpha_{2n} = \alpha$ and $\alpha_{2n+1} = \beta$. Then, we obtain

$$x_{2n+1} = \alpha + \frac{y_{2n-k}}{y_{2n}}, \tag{1.4}$$

$$x_{2n+2} = \beta + \frac{y_{2n+1-k}}{y_{2n+1}}, \tag{1.5}$$

$$y_{2n+1} = \alpha + \frac{x_{2n-k}}{x_{2n}}, \tag{1.6}$$

$$y_{2n+2} = \beta + \frac{x_{2n+1-k}}{x_{2n+1}}. \tag{1.7}$$

If $\alpha_n = \alpha = \beta = A$, then the system (1.3) turns into the symmetrical system (1.1)

$$x_{n+1} = A + \frac{y_{n-k}}{y_n}, \quad y_{n+1} = A + \frac{x_{n-k}}{x_n}, \qquad n = 0, 1, \dots$$

with the parameter $A > 0$, and the initial conditions x_i, y_i are arbitrary positive real numbers for $i = -k, -k+1, ..., 0$ and $k \in \mathbb{Z}^+$, which was studied in ([7],[14]).

Throughout this paper, we assume that $\alpha \neq \beta$. We study the boundedness character of the system (1.3) in the cases: $0 < \alpha$, $\beta < 1$ and α, $\beta > 1$. We use the linearization method to give a necessary and sufficient conditions for the local stability. In addition, we investigate the global behavior of the system (1.3). Furthermore, we determine the rate of the convergence of the solutions and we give some numerical examples that support our theoretical results.

2 Boundedness character

In this section, we investigate the boundedness character of (1.3). We show that if $k \in \mathbb{Z}^+$, α, $\beta > 1$, then every positive solution of the system (1.3) is bounded. When $0 < \alpha$, $\beta < 1$ and k is odd, then there exist unbounded solutions of the system (1.3).

Theorem 2.1. Suppose that

$$\alpha > 1 \quad \textit{and} \quad \beta > 1. \tag{2.1}$$

Then every positive solution of the system (1.3) is bounded.

Proof . It is clear from equations (1.4), (1.5), (1.6) and (1.7) that

$$x_{2n} > \beta, \quad y_{2n} > \beta, \quad x_{2n-1} > \alpha, \quad y_{2n-1} > \alpha, \quad \text{for every} \quad n \geq k. \tag{2.2}$$

We assume that k is odd. Then, from the equations (1.4), (1.5), (1.6), (1.7) and (2.1), we obtain

$$x_{2n+1} = \alpha + \frac{y_{2n-k}}{y_{2n}} < \alpha + \frac{y_{2n-k}}{\beta}, \tag{2.3}$$

$$x_{2n} = \beta + \frac{y_{2n-k-1}}{y_{2n-1}} < \beta + \frac{y_{2n-k-1}}{\alpha}, \tag{2.4}$$

$$y_{2n+1} = \alpha + \frac{x_{2n-k}}{x_{2n}} < \alpha + \frac{x_{2n-k}}{\beta}, \tag{2.5}$$

$$y_{2n} = \beta + \frac{x_{2n-k-1}}{x_{2n-1}} < \beta + \frac{x_{2n-k-1}}{\alpha}. \tag{2.6}$$

From (2.3), (2.5) and using induction we get

$$\begin{aligned} x_{2n+1} &< \alpha\left(1 + \frac{1}{\beta} + \frac{1}{\beta^2} + \frac{1}{\beta^3} + \dots\right) + \mu_1 \\ &= \frac{\alpha\beta}{\beta - 1} + \mu_1, \end{aligned}$$

$$\begin{aligned} y_{2n+1} &< \alpha\left(1 + \frac{1}{\beta} + \frac{1}{\beta^2} + \frac{1}{\beta^3} + \dots\right) + \mu_1 \\ &= \frac{\alpha\beta}{\beta - 1} + \mu_1, \end{aligned}$$

where $\mu_1 = \max\{x_{-k}, y_{-k}, x_{-k+2}, y_{-k+2}, x_{-k+4}, y_{-k+4}, \dots, x_k, y_k\}$.
Similarly, we get

$$\begin{aligned} x_{2n+2} &< \beta\left(1 + \frac{1}{\alpha} + \frac{1}{\alpha^2} + \frac{1}{\alpha^3} + \dots\right) + \mu_2 \\ &= \frac{\alpha\beta}{\alpha - 1} + \mu_2, \end{aligned}$$

$$\begin{aligned} y_{2n+2} &< \beta\left(1 + \frac{1}{\alpha} + \frac{1}{\alpha^2} + \frac{1}{\alpha^3} + \dots\right) + \mu_2 \\ &= \frac{\alpha\beta}{\alpha - 1} + \mu_2, \end{aligned}$$

where $\mu_2 = \max\{x_{-k+1}, y_{-k+1}, x_{-k+3}, y_{-k+3}, x_{-k+5}, y_{-k+5}, \dots, x_{k+1}, y_{k+1}\}$.

Now, we suppose that k is even and $\alpha, \beta > 1$. Then, from the equations (2.3), (2.4), (2.5), (2.6) and using induction we obtain

$$\begin{aligned} x_{2n+1} &< \alpha + 1 + \frac{1}{\alpha\beta}\left[1 + \frac{1}{\alpha\beta} + \frac{1}{(\alpha\beta)^2} + \frac{1}{(\alpha\beta)^3} + \dots\right] \\ &+ \frac{1}{\beta}\left[1 + \frac{1}{\alpha\beta} + \frac{1}{(\alpha\beta)^2} + \frac{1}{(\alpha\beta)^3} + \dots\right] + \mu_3 \\ &= \alpha + 1 + \frac{1}{\alpha\beta}\left(\frac{\alpha\beta}{\alpha\beta - 1}\right) + \frac{1}{\beta}\left(\frac{\alpha\beta}{\alpha\beta - 1}\right) + \mu_3 \\ &= \frac{\alpha\beta(\alpha + 1)}{\alpha\beta - 1} + \mu_3, \end{aligned}$$

$$\begin{aligned}
y_{2n+1} &< \alpha+1+\frac{1}{\alpha\beta}\left[1+\frac{1}{\alpha\beta}+\frac{1}{(\alpha\beta)^2}+\frac{1}{(\alpha\beta)^3}+\dots\right]\\
&+\frac{1}{\beta}\left[1+\frac{1}{\alpha\beta}+\frac{1}{(\alpha\beta)^2}+\frac{1}{(\alpha\beta)^3}+\dots\right]+\mu_4\\
&=\alpha+1+\frac{1}{\alpha\beta}\left(\frac{\alpha\beta}{\alpha\beta-1}\right)+\frac{1}{\beta}\left(\frac{\alpha\beta}{\alpha\beta-1}\right)+\mu_4\\
&=\frac{\alpha\beta(\alpha+1)}{\alpha\beta-1}+\mu_4,
\end{aligned}$$

$$\begin{aligned}
x_{2n+2} &< \beta+1+\frac{1}{\alpha\beta}\left[1+\frac{1}{\alpha\beta}+\frac{1}{(\alpha\beta)^2}+\frac{1}{(\alpha\beta)^3}+\dots\right]\\
&+\frac{1}{\alpha}\left[1+\frac{1}{\alpha\beta}+\frac{1}{(\alpha\beta)^2}+\frac{1}{(\alpha\beta)^3}+\dots\right]+\mu_4\\
&=\beta+1+\frac{1}{\alpha\beta}\left(\frac{\alpha\beta}{\alpha\beta-1}\right)+\frac{1}{\alpha}\left(\frac{\alpha\beta}{\alpha\beta-1}\right)+\mu_4\\
&=\frac{\alpha\beta(\beta+1)}{\alpha\beta-1}+\mu_4,
\end{aligned}$$

$$\begin{aligned}
y_{2n+2} &< \beta+1+\frac{1}{\alpha\beta}\left[1+\frac{1}{\alpha\beta}+\frac{1}{(\alpha\beta)^2}+\frac{1}{(\alpha\beta)^3}+\dots\right]\\
&+\frac{1}{\alpha}\left[1+\frac{1}{\alpha\beta}+\frac{1}{(\alpha\beta)^2}+\frac{1}{(\alpha\beta)^3}+\dots\right]+\mu_3\\
&=\beta+1+\frac{1}{\alpha\beta}\left(\frac{\alpha\beta}{\alpha\beta-1}\right)+\frac{1}{\alpha}\left(\frac{\alpha\beta}{\alpha\beta-1}\right)+\mu_3\\
&=\frac{\alpha\beta(\beta+1)}{\alpha\beta-1}+\mu_3,
\end{aligned}$$

where

$$\mu_3=\max\{x_{-k+1},y_{-k},x_{-k+3},y_{-k+2},x_{-k+5},y_{-k+4},\dots,x_{k+1},y_k\}$$

and

$$\mu_4=\max\{x_{-k},y_{-k+1},x_{-k+4},y_{-k+3},x_{-k+6},y_{-k+5},\dots,x_k,y_{k+1}\}.$$

The proof now is completed. □

Next, we study the existence of unbounded positive solutions of system (1.3) when $0<\alpha<1$ and $0<\beta<1$.

Theorem 2.2. Suppose that $0<\alpha<1$ and $0<\beta<1$. Let $\gamma=\max\{\alpha,\beta\}$ and $\{x_n,y_n\}_{n=-k}^{\infty}$ be a positive solution of (1.3). Then the following statements are true:

(a) If k is odd, $0<x_{-k},x_{-k+2},\dots,x_{-1},y_{-k},y_{-k+2},\dots,y_{-1}<1$ and $x_{-k+1},x_{-k+3},\dots,x_0,y_{-k+1},y_{-k+3},\dots,y_0>\frac{1}{1-\gamma}$, then

$$\lim_{n\to\infty}x_{2n}=\infty,\quad \lim_{n\to\infty}y_{2n}=\infty,\quad \lim_{n\to\infty}x_{2n+1}=\alpha,\quad \lim_{n\to\infty}y_{2n+1}=\alpha.$$

(b) If k is odd, $0<x_{-k+1},x_{-k+3},\dots,x_0,y_{-k+1},y_{-k+3},\dots,y_0<1$ and $x_{-k},x_{-k+2},\dots,x_{-1},y_{-k},y_{-k+2},\dots,y_{-1}>\frac{1}{1-\gamma}$, then

$$\lim_{n\to\infty}x_{2n+1}=\infty,\quad \lim_{n\to\infty}y_{2n+1}=\infty,\quad \lim_{n\to\infty}x_{2n}=\beta,\quad \lim_{n\to\infty}y_{2n}=\beta.$$

Proof . (a) Since $\gamma\ge\alpha$ and $\gamma\ge\beta$ then

$$0<x_1=\alpha+\frac{y_{-k}}{y_0}<\alpha+\frac{1}{y_0}<\alpha+1-\gamma<1,$$

$$0 < y_1 = \alpha + \frac{x_{-k}}{x_0} < \alpha + \frac{1}{x_0} < \alpha + 1 - \gamma < 1,$$

$$x_2 = \beta + \frac{y_{-k+1}}{y_1} > \beta + y_{-k+1} > y_{-k+1} > \frac{1}{1-\gamma},$$

$$y_2 = \beta + \frac{x_{-k+1}}{x_1} > \beta + x_{-k+1} > x_{-k+1} > \frac{1}{1-\gamma}.$$

By induction, we get

$$0 < x_{2n-1}, y_{2n-1} < 1 \text{ and } x_{2n}, y_{2n} > \frac{1}{1-\gamma} \quad \text{for} \quad n = 1, 2, \ldots$$

So for $l > \frac{k+3}{2}$

$$\begin{aligned} x_{2l} &= \alpha + \frac{y_{2l-(k+1)}}{y_{2l-1}} > \alpha + y_{2l-(k+1)} = \alpha + \alpha + \frac{x_{2l-(2k+2)}}{x_{2l-(k+2)}} \\ &> 2\alpha + x_{2l-(2k+2)}, \end{aligned}$$

$$\begin{aligned} x_{4l} &= \alpha + \frac{y_{4l-(k+1)}}{y_{4l-1}} > \alpha + y_{4l-(k+1)} = \alpha + \alpha + \frac{x_{4l-(2k+2)}}{x_{4l-(k+2)}} \\ &> 2\alpha + x_{4l-(2k+2)} = 2\alpha + \alpha + \frac{y_{4l-(3k+3)}}{y_{4l-2k-3}} \\ &> 3\alpha + y_{4l-(3k+3)} = 3\alpha + \alpha + \frac{x_{4l-(4k+4)}}{x_{4l-(3k+4)}} \\ &> 4\alpha + x_{4l-(4k+4)}. \end{aligned}$$

Similarly, we obtain $x_{6l} > 6\alpha + x_{6l-(6k+6)}$. So for all $r = 1, 2, \ldots$

$$x_{2rl} > 2r\alpha + x_{2rl-2r(k+1)}$$

Hence, if $n = rl$, then since $r \to \infty$ and so $\lim_{n\to\infty} x_{2n} = \infty$. In the same, we get $\lim_{n\to\infty} y_{2n} = \infty$. We consider the system (1.3) and we take the limits on both sides of each equation in the system

$$x_{2n+1} = \alpha + \frac{y_{2n-k}}{y_{2n}}, \quad y_{2n+1} = \alpha + \frac{x_{2n-k}}{x_{2n}}$$

we obtain $\lim_{n\to\infty} x_{2n+1} = \alpha$ and $\lim_{n\to\infty} y_{2n+1} = \alpha$. This completes the proof of statement (a). Now, we prove the statement (b). Since $\gamma \geq \alpha$ and $\gamma \geq \beta$ then, we have

$$0 < x_2 = \beta + \frac{y_{-k+1}}{y_1} < \beta + \frac{1}{y_1} < \beta + 1 - \gamma < 1,$$

$$0 < y_2 = \beta + \frac{x_{-k+1}}{x_1} < \beta + \frac{1}{x_1} < \beta + 1 - \gamma < 1,$$

$$x_1 = \alpha + \frac{y_{-k}}{y_0} > \alpha + y_{-k} > y_{-k} > \frac{1}{1-\gamma},$$

$$y_1 = \alpha + \frac{x_{-k}}{x_0} > \alpha + x_{-k} > x_{-k} > \frac{1}{1-\gamma}.$$

By induction, we get

$$0 < x_{2n}, y_{2n} < 1 \text{ and } x_{2n-1}, y_{2n-1} > \frac{1}{1-\gamma} \quad \text{for} \quad n = 1, 2, \ldots$$

So for $l > \frac{k+3}{2}$

$$\begin{aligned} x_{2l+1} &= \beta + \frac{y_{2l-k}}{y_{2l}} > \beta + y_{2l-k} = \beta + \beta + \frac{x_{2l-(2k+1)}}{x_{2l-(k+1)}} \\ &> 2\beta + x_{2l-(2k+1)}, \end{aligned}$$

$$\begin{aligned} x_{4l+1} &= \beta + \frac{y_{4l-k}}{y_{4l}} > \beta + y_{4l-k} = \beta + \beta + \frac{x_{4l-(2k+1)}}{x_{4l-(k+1)}} \\ &> 2\beta + x_{4l-(2k+1)} = 2\beta + \beta + \frac{y_{4l-(3k+2)}}{y_{4l-2k-2}} \\ &> 3\beta + y_{4l-(3k+2)} = 3\beta + \beta + \frac{x_{4l-(4k+3)}}{x_{4l-(3k+3)}} \\ &> 4\beta + x_{4l-(4k+3)}. \end{aligned}$$

Similarly, we get $x_{6l+1} > 6\beta + x_{6l-(6k+5)}$. So for all $r = 1, 2, \ldots$

$$x_{2rl+1} > 2r\beta + x_{2rl-(2r(k+1)-1)}.$$

Consequently, if $n = rl$, then since $r \to \infty$, $\lim_{n\to\infty} x_{2n+1} = \infty$. Similarly, we get $\lim_{n\to\infty} y_{2n+1} = \infty$. Now, we consider the system (1.3) and we take the limits on both sides of each equation in the system,

$$x_{2n+2} = \beta + \frac{y_{2n+1-k}}{y_{2n+1}}, \quad y_{2n+2} = \beta + \frac{x_{2n+1-k}}{x_{2n+1}},$$

we obtain

$$\lim_{n\to\infty} x_{2n+2} = \beta \quad \text{and} \quad \lim_{n\to\infty} y_{2n+2} = \beta.$$

This completes the proof of statement (b). □

3 Local asymptotic stability

The system (1.3) can be converted into a four-dimensional discrete system with constant coefficients. To this end, let

$$u_n = x_{2n-1}, \quad v_n = x_{2n}, \quad t_n = y_{2n-1}, \quad w_n = y_{2n}, \quad n = 0, 1, 2, \ldots$$

We consider the case $k = 2m$. Hence, for $n \geq 0$ we have

$$\begin{aligned} u_{n+1} &= \alpha + \frac{w_{n-m}}{w_n}, \quad v_{n+1} = \beta + \frac{t_{n-m+1}}{t_{n+1}}, \\ t_{n+1} &= \alpha + \frac{v_{n-m}}{v_n}, \quad w_{n+1} = \beta + \frac{u_{n-m+1}}{u_{n+1}}. \end{aligned}$$

So, for $n = 0, 1, \ldots$, the system (1.3) is equivalent to the system

$$\begin{cases} u_{n+1} = \alpha + \frac{w_{n-m}}{w_n} \\ v_{n+1} = \beta + \frac{t_{n-m+1}v_n}{\alpha v_n + v_{n-m}} \\ t_{n+1} = \alpha + \frac{v_{n-m}}{v_n} \\ w_{n+1} = \beta + \frac{u_{n-m+1}w_n}{\alpha w_n + w_{n-m}} \end{cases} \tag{3.1}$$

where the initial conditions are $w_0 = y_0$, $w_{-1} = y_{-2}, \ldots, w_{-m} = y_{-2m}$, $v_0 = x_0$, $v_{-1} = x_{-2}, \ldots, v_{-m} = x_{-2m}$, $t_0 = y_{-1}$, $t_{-1} = y_{-3}, \ldots, t_{-m+1} = y_{-2m+1}$, $u_0 = x_{-1}$, $u_{-1} = x_{-3}, \ldots, u_{-m+1} = x_{-2m+1}$. One can easily see that the system (3.1) has a unique equilibrium point $E = (\alpha+1, \beta+1, \alpha+1, \beta+1)$. In this section, we use the linearization method to give necessary and sufficient conditions for the local asymptotic stability.

Theorem 3.1. If $\alpha > 1$, $\beta > 1$ and k is even, then the unique positive equilibrium point $E = (\alpha+1, \beta+1, \alpha+1, \beta+1)$ of the system (3.1) is locally asymptotically stable.

Proof . The system (3.1) can be formulated as a system of first order recurrence equations as follows:

$$\begin{aligned} u_n^{(1)} &= u_n, u_n^{(2)} = u_{n-1}, \dots, u_n^{(m)} = u_{n-m+1}, \\ v_n^{(1)} &= v_n, v_n^{(2)} = v_{n-1}, \dots, v_n^{(m+1)} = v_{n-m}, \\ t_n^{(1)} &= t_n, t_n^{(2)} = t_{n-1}, \dots, t_n^{(m)} = t_{n-m+1}, \\ w_n^{(1)} &= w_n, w_n^{(2)} = w_{n-1}, \dots, w_n^{(m+1)} = w_{n-m}. \end{aligned}$$

The linearization of the system (3.1) about the equilibrium point E is given by $Z_{n+1} = AZ_n$, where

$$Z_n = (u_n^{(1)}, u_n^{(2)}, \dots, u_n^{(m)}, v_n^{(1)}, v_n^{(2)}, \dots, v_n^{(m+1)}, t_n^{(1)}, t_n^{(2)}, \dots, t_n^{(m)}, w_n^{(1)}, w_n^{(2)}, \dots, w_n^{(m+1)})^T$$

and

$$A = \begin{pmatrix}
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \frac{-1}{\beta+1} & 0 & \dots & 0 & \frac{1}{\beta+1} \\
1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & \frac{1}{(\beta+1)(\alpha+1)} & 0 & \dots & 0 & \frac{-1}{(\beta+1)(\alpha+1)} & 0 & \dots & 0 & \frac{1}{\alpha+1} & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & \frac{-1}{\beta+1} & 0 & \dots & 0 & \frac{1}{\beta+1} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & \frac{1}{\alpha+1} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \frac{1}{(\beta+1)(\alpha+1)} & 0 & \dots & 0 & \frac{-1}{(\beta+1)(\alpha+1)} \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0
\end{pmatrix}_{(4m+2)\times(4m+2)}$$

Let $\lambda_1, \lambda_2, \dots, \lambda_{4m+2}$ be the eigenvalues of the matrix A. Define $D = diag(d_1, d_2, \dots, d_{4m+2})$ be a diagonal matrix such that $d_1 = d_{m+1} = d_{2m+2} = d_{3m+2} = 1$ and

$$d_i = d_{2m+1+i} = 1 - i\varepsilon, \quad \text{for each } i \in \{2, 3, \dots, m, m+2, \dots, 2m+1\}.$$

Since $\alpha, \beta > 1$, we can take a positive number ε such that

$$0 < \varepsilon < \min\left\{ \frac{\beta - 1}{(\beta+1)(2m+1)}, \quad \frac{(\alpha+1)(\beta+1) - 3}{(\alpha+1)(\beta+1)(2m+1)} \right\}. \tag{3.2}$$

Hence, for all i, $1 - i\varepsilon > 0$, and so D is invertible. Now, simple calculations lead to

$$DAD^{-1} = \begin{pmatrix}
0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \delta_1^{(3m+2)} & \dots & 0 & \delta_1^{(4m+2)} \\
\delta_2^{(1)} & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \dots & \delta_m^{(m-1)} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & \delta_{m+1}^{(m+1)} & \dots & 0 & \delta_{m+1}^{(2m+1)} & 0 & \dots & 0 & \delta_{m+1}^{(3m+1)} & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & \delta_{m+2}^{(m+1)} & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & \dots & \delta_{2m+1}^{(2m)} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & \delta_{2m+2}^{(m+1)} & \dots & 0 & \delta_{2m+2}^{(2m+1)} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \delta_{2m+3}^{(2m+2)} & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \delta_{3m+1}^{(3m)} & 0 & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & \delta_{3m+2}^{(m)} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \delta_{3m+2}^{(3m+2)} & \dots & 0 & \delta_{3m+2}^{(4m+2)} \\
0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \delta_{3m+3}^{(3m+2)} & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \delta_{4m+2}^{(4m+1)} & 0
\end{pmatrix}$$

where

$$\delta_1^{(3m+2)} = \frac{-d_1}{(\beta+1)d_{3m+1}}, \quad \delta_1^{(4m+2)} = \frac{d_1}{(\beta+1)d_{4m+2}}, \quad \delta_2^{(1)} = \frac{d_2}{d_1},$$
$$\delta_m^{(m-1)} = \frac{d_m}{d_{m-1}}, \quad \delta_{m+1}^{(m+1)} = \frac{d_{m+1}}{(\beta+1)(\alpha+1)d_{m+1}}, \quad \delta_{m+1}^{(2m+1)} = \frac{-d_{m+1}}{(\beta+1)(\alpha+1)d_{2m+1}},$$
$$\delta_{m+1}^{(3m+1)} = \frac{d_{m+1}}{(\alpha+1)d_{3m+1}}, \quad \delta_{m+2}^{(m+1)} = \frac{d_{m+2}}{d_{m+1}}, \quad \delta_{2m+1}^{(2m)} = \frac{d_{2m+1}}{d_{2m}},$$
$$\delta_{2m+2}^{(m+1)} = \frac{-d_{2m+2}}{(\beta+1)d_{m+1}}, \quad \delta_{2m+2}^{(2m+1)} = \frac{d_{2m+2}}{(\beta+1)d_{2m+1}}, \quad \delta_{2m+3}^{(2m+2)} = \frac{d_{2m+3}}{d_{2m+2}},$$
$$\delta_{3m+1}^{(3m)} = \frac{d_{3m+1}}{d_{3m}}, \quad \delta_{3m+2}^{(m)} = \frac{d_{3m+2}}{(\alpha+1)d_m}, \quad \delta_{3m+2}^{(3m+2)} = \frac{d_{3m+2}}{(\beta+1)(\alpha+1)d_{3m+2}}$$
$$\delta_{3m+2}^{(4m+2)} = \frac{-d_{3m+2}}{(\beta+1)(\alpha+1)d_{4m+2}}, \quad \delta_{3m+3}^{(3m+2)} = \frac{d_{3m+3}}{d_{3m+2}}, \quad \delta_{4m+2}^{(4m+1)} = \frac{d_{4m+2}}{d_{4m+1}}.$$

From the following four inequalities

$$1 = d_1 > d_2 > \cdots > d_m > 0,$$

$$1 = d_{m+1} > d_{m+2} > \cdots > d_{2m} > d_{2m+1} > 0,$$

$$1 = d_{2m+2} > d_{2m+3} > \cdots > d_{3m} > d_{3m+1} > 0,$$

$$1 = d_{3m+2} > d_{3m+3} > \cdots > d_{4m+1} > d_{4m+2} > 0,$$

we get

$$\frac{d_2}{d_1} < 1, \quad \frac{d_3}{d_2} < 1, \ldots, \frac{d_m}{d_{m-1}} < 1, \quad \frac{d_{m+2}}{d_{m+1}} < 1, \ldots, \frac{d_{2m+1}}{d_{2m}} < 1,$$
$$\frac{d_{2m+3}}{d_{2m+2}} < 1, \ldots, \frac{d_{3m+1}}{d_{3m}} < 1, \quad \frac{d_{3m+3}}{d_{3m+2}} < 1, \ldots, \frac{d_{4m+2}}{d_{4m+1}} < 1.$$

Furthermore, since $\alpha, \beta > 1$ and by using (3.2) we have

$$\begin{aligned}\frac{1}{\beta+1} + \frac{1}{(\beta+1)(1-(2m+1)\varepsilon)} &< \frac{1}{(\beta+1)(1-(2m+1)\varepsilon)} + \frac{1}{(\beta+1)(1-(2m+1)\varepsilon)} \\ &< \frac{2}{(1-(2m+1)\varepsilon)(\beta+1)} \\ &< 1\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{(\alpha+1)(\beta+1)} + \frac{1}{(\beta+1)(\alpha+1)(1-(2m+1)\varepsilon)} + \frac{1}{(\alpha+1)(1-m\varepsilon)} \\ &< \frac{3}{(\beta+1)(\alpha+1)(1-(2m+1)\varepsilon)} \\ &< 1.\end{aligned}$$

Since A and DAD^{-1} have the same eigenvalues, we have

$$\begin{aligned}\max\{|\lambda_j|\} &\leq \|DAD^{-1}\|_\infty \\ &= \max\left\{\frac{1}{(\beta+1)} + \frac{1}{(\beta+1)(1-(2m+1)\varepsilon)}, \quad \frac{d_2}{d_1}, \frac{d_3}{d_2}, \ldots, \frac{d_m}{d_{m-1}},\right. \\ &\quad \frac{d_{m+2}}{d_{m+1}}, \ldots, \frac{d_{2m+1}}{d_{2m}}, \quad \frac{d_{2m+3}}{d_{2m+2}}, \ldots, \frac{d_{3m+1}}{d_{3m}}, \quad \frac{d_{3m+3}}{d_{3m+2}}, \ldots, \frac{d_{4m+2}}{d_{4m+1}}, \\ &\quad \left.\frac{1}{(\beta+1)(\alpha+1)} + \frac{1}{(\beta+1)(\alpha+1)(1-(2m+1)\varepsilon)} + \frac{1}{(\alpha+1)(1-m\varepsilon)}\right\} \\ &< 1.\end{aligned}$$

So, the modulus of every eigenvalue of A is less than one. Hence, the unique equilibrium point $E = (\alpha+1, \beta+1, \alpha+1, \beta+1)$ of the system (3.1) is locally asymptotically stable. Thus, the proof is completed. □

4 Global asymptotic stability

In this section, we show that all positive solutions of (1.3) are attracted by a period-two solution.

Theorem 4.1. If $\alpha > 1$, $\beta > 1$, then every positive solution of the system (1.3) converges to the period-two solution $(\alpha+1, \alpha+1)$, $(\beta+1, \beta+1)$,... as $n \to \infty$.

Proof . Let $\{x_n, y_n\}$ be an arbitrary positive solution of the system (1.3) and let

$$u_1 = \limsup_{n\to\infty} x_{2n+1}, \quad l_1 = \liminf_{n\to\infty} x_{2n+1} \quad u_2 = \limsup_{n\to\infty} x_{2n}, \quad l_2 = \liminf_{n\to\infty} x_{2n}$$

$$u_3 = \limsup_{n\to\infty} y_{2n+1}, \quad l_3 = \liminf_{n\to\infty} y_{2n+1}, \quad u_4 = \limsup_{n\to\infty} y_{2n}, \quad l_4 = \liminf_{n\to\infty} y_{2n}$$

Using Theorem (3.1), we get

$$l_1 \le u_1 < +\infty, \quad l_2 \le u_2 < +\infty, \quad l_3 \le u_3 < +\infty, \quad l_4 \le u_4 < +\infty.$$

Now, we assume that k is even. Then the system (1.3) implies that

$$u_1 \le \alpha + \frac{u_4}{l_4}, \quad u_2 \le \beta + \frac{u_3}{l_3}, \quad u_3 \le \alpha + \frac{u_2}{l_2}, \quad u_4 \le \beta + \frac{u_1}{l_1},$$
$$l_1 \ge \alpha + \frac{l_4}{u_4}, \quad l_2 \ge \beta + \frac{l_3}{u_3}, \quad l_3 \ge \alpha + \frac{l_2}{u_2}, \quad l_4 \ge \beta + \frac{l_1}{u_1}.$$

which implies that

$$\beta u_1 + l_1 \le l_4 u_1 \le \alpha l_4 + u_4, \quad \alpha u_4 + l_4 \le l_1 u_4 \le \beta l_1 + u_1$$

and

$$\alpha u_2 + l_2 \le l_3 u_2 \le \beta l_3 + u_3, \quad \beta u_3 + l_3 \le l_2 u_3 \le \alpha l_2 + u_2.$$

Therefore, we obtain

$$(\beta-1)(u_1 - l_1) + (\alpha-1)(u_4 - l_4) \le 0$$

and

$$(\beta-1)(u_3 - l_3) + (\alpha-1)(u_2 - l_2) \le 0.$$

Since $\alpha > 1$, $\beta > 1$ and $u_1 - l_1$, $u_2 - l_2$, $u_3 - l_3$, $u_4 - l_4 \ge 0$, we get

$$u_1 - l_1 = 0, \quad u_2 - l_2 = 0, \quad u_3 - l_3 = 0 \text{ and } u_4 - l_4 = 0.$$

Now, we assume that k is odd. Then the system (1.3) implies that

$$u_1 \le \alpha + \frac{u_3}{l_4}, \quad u_2 \le \beta + \frac{u_4}{l_3}, \quad u_3 \le \alpha + \frac{u_1}{l_2}, \quad u_4 \le \beta + \frac{u_2}{l_1},$$
$$l_1 \ge \alpha + \frac{l_3}{u_4}, \quad l_2 \ge \beta + \frac{l_4}{u_3}, \quad l_3 \ge \alpha + \frac{l_1}{u_2}, \quad l_4 \ge \beta + \frac{l_2}{u_1}$$

which implies that

$$\beta u_1 + l_2 \le l_4 u_1 \le \alpha l_4 + u_3, \quad \alpha u_4 + l_3 \le l_1 u_4 \le \beta l_1 + u_2$$

and

$$\alpha u_2 + l_1 \le l_3 u_2 \le \beta l_3 + u_4, \quad \beta u_3 + l_4 \le l_2 u_3 \le \alpha l_2 + u_1.$$

Consequently, we obtain

$$(\beta-1)u_1+(1-\alpha)l_2 \leq (\alpha-1)l_4+(1-\beta)u_3$$

and

$$(1-\beta)l_1+(\alpha-1)u_2 \leq (\beta-1)l_3+(1-\alpha)u_4.$$

By addition, we get

$$(\beta-1)(u_1-l_1)+(\alpha-1)(u_2-l_2)+(\alpha-1)(u_4-l_4)+(\beta-1)(u_3-l_3) \leq 0.$$

But $\alpha-1, \beta-1>0$ and $u_1-l_1, u_2-l_2, u_3-l_3, u_4-l_4 \geq 0$. Thus

$$u_1-l_1=0, \quad u_2-l_2=0, \quad u_3-l_3=0 \text{ and } u_4-l_4=0.$$

So, we use (3.1) to get

$$l_1=u_1=\alpha+1, \quad l_2=u_2=\beta+1, \quad l_3=u_3=\alpha+1, \quad l_4=u_4=\beta+1$$

Moreover, it is obvious that since $\alpha \neq \beta$, then from equations (1.4), (1.5), (1.6), and (1.7)

$$\lim_{n\to\infty} x_{2n+1} \neq \lim_{n\to\infty} x_{2n}, \quad \lim_{n\to\infty} y_{2n+1} \neq \lim_{n\to\infty} y_{2n}.$$

Finally, since $l_1=u_1$, $l_2=u_2$, $l_3=u_3$, $l_4=u_4$, it is clear that $\{x_n, y_n\}$ converges to the period-two solution $(\alpha+1, \alpha+1)$, $(\beta+1, \beta+1)$,... as $n \to \infty$. $\square$

From Theorems (3.1) and (4.1) we obtain the following result.

Theorem 4.2. If $\alpha, \beta > 1$ and k is even then the period-two solution $\{(\alpha+1, \alpha+1), (\beta+1, \beta+1), ...\}$ of the system (1.3) is globally asymptotically stable.

5 Rate of convergence

In this section, we investigate the rate of convergence of a solution that converges to the equilibrium point $E = (\alpha+1, \beta+1, \alpha+1, \beta+1)$ of the system (3.1) when $\alpha, \beta > 1$ and k is even.

The following result gives the rate of convergence of the solutions of a system of difference equations

$$U_{n+1} = (A+B(n))U_n \tag{5.1}$$

where U_n is a (2k+2) dimensional vector, $A \in C^{(2k+2)\times(2k+2)}$ is a constant matrix and $B : \mathbb{Z}^+ \longrightarrow C^{(2k+2)\times(2k+2)}$ is a matrix function satisfying

$$||B(n)|| \to 0, \ \textit{when } n \longrightarrow \infty, \tag{5.2}$$

where $||.||$ denotes any matrix norm which is associated with the vector norm.

Theorem 5.1 (Perron's Theorem). Consider system (5.1) and suppose condition (5.2) holds. If U_n is a solution of (5.1), then either $U_n = 0$ for all large n or

$$\theta = \lim_{n\to\infty} \sqrt[n]{||U_n||}$$

or

$$\theta = \lim_{n\to\infty} \frac{||U_{n+1}||}{||U_n||}$$

exist and θ is equal to the modulus of one the eigenvalues of the matrix A.

Here is our main result in this part.

Theorem 5.2. Assume that a solution $\{(u_n, v_n, t_n, w_n)\}$ of the system (3.1) converges to the equilibrium point $(\bar{u}, \bar{v}, \bar{t}, \bar{w})$ which is globally asymptotically stable. Then, the error vector

$$\begin{aligned} e_n &= \left(e_n^{(1)}, ..., e_{n-m+1}^{(1)}, e_n^{(2)}, ..., e_{n-m}^{(2)}, e_n^{(3)}, ..., e_{n-m+1}^{(3)}, e_n^{(4)}, ..., e_{n-m}^{(4)}\right)^T \\ &= (u_n - \bar{u}, ..., u_{n-m+1} - \bar{u}, v_n - \bar{v}, ..., v_{n-m} - \bar{v}, t_n - \bar{t}, ..., t_{n-m+1} - \bar{t}, w_n - \bar{w}, ..., w_{n-m} - \bar{w})^T \end{aligned}$$

of every solution of the system (3.1) satisfies both of the following asymptotic relations:

$$\lim_{n\to\infty} \sqrt[n]{||e_n||} = |\lambda_i J_F(\bar{u}, \bar{v}, \bar{t}, \bar{w})|, \text{ for some } i = 1, 2, \ldots, k$$

or

$$\lim_{n\to\infty} \frac{||e_{n+1}||}{||e_n||} = |\lambda_i J_F(\bar{u}, \bar{v}, \bar{t}, \bar{w})|, \text{ for some } i = 1, 2, \ldots, k$$

where $|\lambda_i J_F(\bar{u}, \bar{v}, \bar{t}, \bar{w})|$ is equal to the modulus of one the eigenvalues of the Jacobian matrix evaluated at the equilibrium point $(\bar{u}, \bar{v}, \bar{t}, \bar{w})$.

Proof . We will obtain a system of limiting equations for the system (3.1). The error terms are given as

$$\begin{aligned} u_{n+1} - \bar{u} &= \sum_{i=0}^{k} A_i(u_{n-i} - \bar{u}) + \sum_{i=0}^{k} B_i(w_{n-i} - \bar{w}), \\ v_{n+1} - \bar{v} &= \sum_{i=0}^{k} E_i(v_{n-i} - \bar{v}) + \sum_{i=0}^{k} F_i(t_{n-i} - \bar{t}), \\ t_{n+1} - \bar{t} &= \sum_{i=0}^{k} C_i(t_{n-i} - \bar{t}) + \sum_{i=0}^{k} D_i(v_{n-i} - \bar{v}), \\ w_{n+1} - \bar{w} &= \sum_{i=0}^{k} G_i(w_{n-i} - \bar{w}) + \sum_{i=0}^{k} H_i(u_{n-i} - \bar{u}). \end{aligned}$$

Set

$$e_n^{(1)} = u_n - \bar{u},\ e_n^{(2)} = v_n - \bar{v},\ e_n^{(3)} = t_n - \bar{t},\ e_n^{(4)} = w_n - \bar{w}.$$

Hence we obtain

$$\begin{aligned} e_{n+1}^{(1)} &= \sum_{i=0}^{k} A_i e_{n-i}^{(1)} + \sum_{i=0}^{k} B_i e_{n-i}^{(4)}, \quad e_{n+1}^{(2)} = \sum_{i=0}^{k} E_i e_{n-i}^{(2)} + \sum_{i=0}^{k} F_i e_{n-i}^{(3)}, \\ e_{n+1}^{(3)} &= \sum_{i=0}^{k} C_i e_{n-i}^{(3)} + \sum_{i=0}^{k} D_i e_{n-i}^{(2)}, \quad e_{n+1}^{(4)} = \sum_{i=0}^{k} G_i e_{n-i}^{(4)} + \sum_{i=0}^{k} H_i e_{n-i}^{(1)}, \end{aligned}$$

where

$$\begin{aligned} &A_i = 0, i \in \{0, 1, \ldots, m-1\}, \quad B_0 = \frac{-w_{n-m}}{w_n^2}, \quad B_i = 0, i \in \{1, 2, \ldots, m-1\}, \\ &B_m = \frac{1}{w_n}, \quad E_0 = \frac{t_{n-m+1}v_{n-m}}{(\alpha v_n + v_{n-m})^2}, \quad E_i = 0, i \in \{1, 2, \ldots, m-1\}, \quad E_m = \frac{-t_{n-m+1}v_n}{(\alpha v_n + v_{n-m})^2}, \\ &F_i = 0, i \in \{0, 1, \ldots, m-2, m\}, \quad F_{m-1} = \frac{v_n}{\alpha v_n + v_{n-m}}, \quad C_i = 0, i \in \{0, 1, \ldots, m-1\}, \\ &D_0 = \frac{-v_{n-m}}{v_n^2}, \quad D_i = 0, i \in \{1, 2, \ldots, m-1\}, \quad D_m = \frac{1}{v_n}, \quad G_0 = \frac{u_{n-m+1}w_{n-m}}{(\alpha w_n + w_{n-m})^2}, \\ &G_i = 0, i \in \{1, 2, \ldots, m-1\}, \quad G_m = \frac{-u_{n-m+1}w_n}{(\alpha w_n + w_{n-m})^2}, \quad H_i = 0, i \in \{0, 1, \ldots, m-2, m\} \\ &H_{m-1} = \frac{w_n}{\alpha w_n + w_{n-m}}. \end{aligned}$$

Taking the limits, we have

$$\begin{aligned}
&\lim_{n\to\infty} A_i = 0 \text{ for } i\in\{0,1,\dots,m-1\}, \quad \lim_{n\to\infty} B_0 = \frac{-1}{\bar{w}},\\
&\lim_{n\to\infty} B_i = 0 \text{ for } i\in\{1,\dots,m-1\}, \quad \lim_{n\to\infty} B_m = \frac{1}{\bar{w}}, \quad \lim_{n\to\infty} E_0 = \frac{\bar{t}}{(\alpha+1)^2\bar{v}},\\
&\lim_{n\to\infty} E_i = 0 \text{ for } i\in\{1,\dots,m-1\}, \quad \lim_{n\to\infty} E_m = \frac{-\bar{t}}{(\alpha+1)^2\bar{v}},\\
&\lim_{n\to\infty} F_i = 0 \text{ for } i\in\{0,1,\dots,m-2,m\}, \quad \lim_{n\to\infty} F_{m-1} = \frac{\bar{v}}{(\alpha+1)\bar{v}},\\
&\lim_{n\to\infty} C_i = 0 \text{ for } i\in\{0,1,\dots,m-1\}, \quad \lim_{n\to\infty} D_0 = \frac{-1}{\bar{v}}, \quad \lim_{n\to\infty} D_i = 0 \text{ for } i\in\{1,\dots,m-1\},\\
&\lim_{n\to\infty} D_m = \frac{1}{\bar{v}}, \quad \lim_{n\to\infty} G_0 = \frac{\bar{u}}{(\alpha+1)^2\bar{w}}, \quad \lim_{n\to\infty} G_i = 0 \text{ for } i\in\{1,\dots,m-1\},\\
&\lim_{n\to\infty} G_m = \frac{-\bar{u}}{(\alpha+1)^2\bar{w}}, \quad \lim_{n\to\infty} H_i = 0 \text{ for } i\in\{0,1,\dots,m-2,m\}, \quad \lim_{n\to\infty} H_{m-1} = \frac{\bar{w}}{(\alpha+1)\bar{w}}.
\end{aligned}$$

Hence

$$\begin{aligned}
&B_0 = \frac{-1}{\bar{w}} + a_n, \quad B_m = \frac{1}{\bar{w}} + b_n, \quad E_0 = \frac{\bar{t}}{(\alpha+1)\bar{v}} + c_n, \quad E_m = \frac{-\bar{t}}{(\alpha+1)\bar{v}} + d_n,\\
&D_0 = \frac{-1}{\bar{v}} + f_n, \quad D_m = \frac{1}{\bar{v}} + g_n, \quad G_0 = \frac{\bar{u}}{(\alpha+1)\bar{w}} + h_n, \quad G_m = \frac{-\bar{u}}{(\alpha+1)\bar{w}} + k_n,\\
&F_{m-1} = \frac{\bar{v}}{(\alpha+1)\bar{v}} + p_n, \quad H_{m-1} = \frac{\bar{w}}{(\alpha+1)\bar{w}} + q_n,
\end{aligned}$$

where $a_n, b_n, c_n, d_n, f_n, g_n, h_n, k_n, p_n, q_n \longrightarrow 0$ for $n \longrightarrow \infty$. Consequently, we obtain a system of the form of the equation (5.1)

$$e_{n+1} = (A + B(n))e_n,$$

where

$$B(n) = \begin{pmatrix}
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & a_n & 0 & \dots & 0 & b_n\\
1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0\\
0 & 0 & \dots & 0 & 0 & c_n & 0 & \dots & 0 & d_n & 0 & \dots & 0 & p_n & 0 & 0 & \dots & 0 & 0\\
0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0\\
0 & 0 & \dots & 0 & 0 & f_n & 0 & \dots & 0 & g_n & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0\\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0\\
0 & 0 & \dots & 0 & q_n & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & h_n & 0 & \dots & 0 & k_n\\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0
\end{pmatrix}$$

$$A = \begin{pmatrix}
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \frac{-1}{\beta+1} & 0 & \dots & 0 & \frac{1}{\beta+1}\\
1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0\\
0 & 0 & \dots & 0 & 0 & \frac{1}{(\beta+1)(\alpha+1)} & 0 & \dots & 0 & \frac{-1}{(\beta+1)(\alpha+1)} & 0 & \dots & 0 & \frac{1}{\alpha+1} & 0 & 0 & \dots & 0 & 0\\
0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0\\
0 & 0 & \dots & 0 & 0 & \frac{-1}{\beta+1} & 0 & \dots & 0 & \frac{1}{\beta+1} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0\\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0\\
0 & 0 & \dots & 0 & \frac{1}{\alpha+1} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \frac{1}{(\beta+1)(\alpha+1)} & 0 & \dots & 0 & \frac{-1}{(\beta+1)(\alpha+1)}\\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0
\end{pmatrix}$$

with $||B(n)|| \longrightarrow 0$. Therefore, we can write the limiting system of error terms about the equilibrium point $(\bar{u}, \bar{v}, \bar{t}, \bar{w})$ as follows:

$$e_{n+1} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \frac{-1}{\beta+1} & 0 & \dots & 0 & \frac{1}{\beta+1} \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & \frac{1}{(\beta+1)(\alpha+1)} & 0 & \dots & 0 & \frac{-1}{(\beta+1)(\alpha+1)} & 0 & \dots & 0 & \frac{1}{\alpha+1} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & \frac{-1}{\beta+1} & 0 & \dots & 0 & \frac{1}{\beta+1} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \frac{1}{\alpha+1} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \frac{1}{(\beta+1)(\alpha+1)} & 0 & \dots & 0 & \frac{-1}{(\beta+1)(\alpha+1)} \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ \vdots \\ e_{n-m+1}^1 \\ e_n^2 \\ e_{n-1}^2 \\ \vdots \\ e_{n-m}^2 \\ e_n^3 \\ e_{n-1}^3 \\ \vdots \\ e_{n-m+1}^3 \\ e_n^4 \\ e_{n-1}^4 \\ \vdots \\ e_{n-m}^4 \end{pmatrix}$$

which is the same as the linearized system of system (3.1) about equilibrium point $(\bar{u}, \bar{v}, \bar{t}, \bar{w})$. Finally, we apply Perron's theorem to obtain the desired result. □

6 Numerical examples

In this section, in order to confirm our theoretical results, we consider some numerical examples.

Example 6.1. Consider the system (1.3) with $k = 9$ and the initial conditions $x_{-9} = 3$, $x_{-8} = 4$, $x_{-7} = 0.6$, $x_{-6} = 1.3$,$x_{-5} = 0.1$, $x_{-4} = 2.7$ $x_{-3} = 9$, $x_{-2} = 5$, $x_{-1} = 2.8$, $x_0 = 5.7$, $y_{-9} = 2$, $y_{-8} = 0.4$, $y_{-7} = 3$, $y_{-6} = 1.01$, $y_{-5} = 7$, $y_{-4} = 4.2$, $y_{-3} = 1.9$, $y_{-2} = 7$, $y_{-1} = 6.7$, $y_0 = 3$. Moreover, we take the parameters $\alpha = \frac{5}{6}$, $\beta = \frac{1}{6}$, i.e.,

$$\alpha_n = \begin{cases} \frac{5}{6} & \text{if } n \text{ even }, \\ \frac{1}{6} & \text{if } n \text{ odd }. \end{cases}$$

In this case $0 < \alpha$, $\beta < 1$ and k is odd. Then, by virtue of Theorem 2.2, the solution of the system (1.3) is unbounded (see Figure 1).

Example 6.2. Consider the system (1.3) with $k = 4$ and the initial conditions $x_{-4} = 4$, $x_{-3} = 3$, $x_{-2} = 1.06$, $x_{-1} = 2$, $x_0 = 0.8$, $y_{-4} = 2$, $y_{-3} = 1.4$, $y_{-2} = 4$, $y_{-1} = 1$, $y_0 = 4$. Moreover, we take the parameters $\alpha = \frac{7}{3}$, $\beta = \frac{5}{3}$. In this case α, $\beta > 1$ and k is even. Then, by virtue of Theorem 4.1 the solution of the system (1.3) converges to the period two solution $\{(\frac{10}{3}, \frac{10}{3}), (\frac{8}{3}, \frac{8}{3}), ...\}$ (see Figure 2).

Example 6.3. Consider the system (1.3) with $k = 3$ and the initial conditions $x_{-3} = 1.4$, $x_{-2} = 2.6$, $x_{-1} = 1.4$, $x_0 = 1.1$, $y_{-3} = 2.1$, $y_{-2} = 1.4$, $y_{-1} = 3.1$, $y_0 = 0.8$. In addition, we take the parameters $\alpha = 5$, $\beta = 3$. In this case α, $\beta > 1$ and k is odd. Then, by virtue of Theorem 4.1 the solution of the system (1.3) converges to the period two solution $\{(6, 6), (4, 4), ...\}$ (see Figure 3).

Example 6.4. Consider the system (1.3) with $k = 5$ and the initial conditions $x_{-5} = 1.4$, $x_{-4} = 2.6$, $x_{-3} = 1.4$, $x_{-2} = 1.1$, $x_{-1} = 2.4$, $x_0 = 1.8$, $y_{-5} = 2.01$, $y_{-4} = 1.4$, $y_{-3} = 3.1$, $y_{-2} = 0.8$, $y_{-1} = 2.3$, $y_0 = 5.9$. Moreover, we take the sequence $\{\alpha_n\}$ as follows

$$\alpha_n = \begin{cases} 10 & \text{if } n \text{ even} \\ 8 & \text{if } n \text{ odd} \end{cases}$$

In this case $\alpha, \beta > 1$ and k is odd. Then, by virtue of Theorem 4.1 the solution of the system (1.3) converges to the period two solution $\{(11, 11), (9, 9), ...\}$ (see Figure 4).

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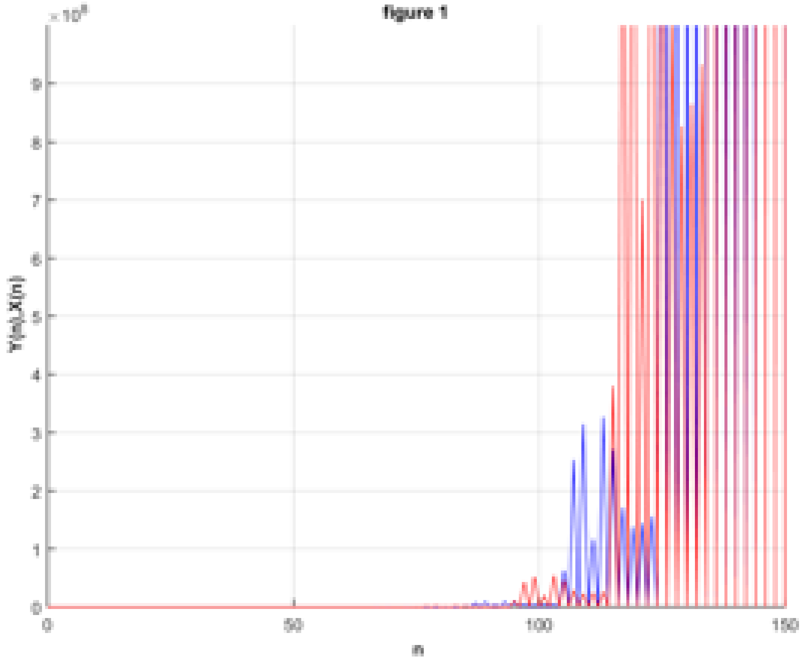

Figure 1: Plot of the solution $\{(x_n, y_n)\}_{n\geq 0}$ of the system (1.3) with $k = 9$ and the initial values $x_{-9} = 3$, $x_{-8} = 4$, $x_{-7} = 0.6$, $x_{-6} = 1.3$,$x_{-5} = 0.1$, $x_{-4} = 2.7$ $x_{-3} = 9$, $x_{-2} = 5$, $x_{-1} = 2.8$, $x_0 = 5.7$, $y_{-9} = 2$, $y_{-8} = 0.4$, $y_{-7} = 3$, $y_{-6} = 1.01$, $y_{-5} = 7$, $y_{-4} = 4.2$, $y_{-3} = 1.9$, $y_{-2} = 7$, $y_{-1} = 6.7$, $y_0 = 3$.

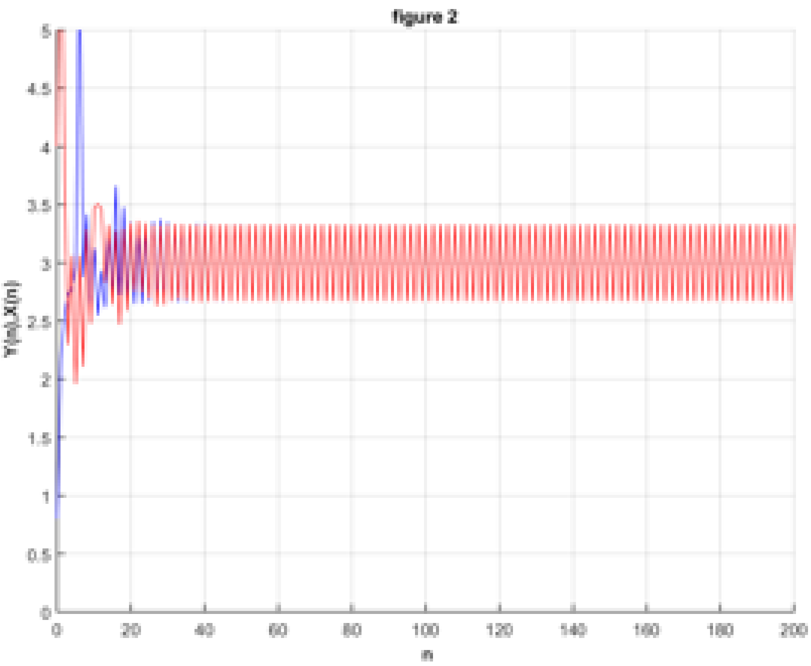

Figure 2: Plot of the solution $\{(x_n, y_n)\}_{n\geq 0}$ of the system (1.3) with $k = 4$ and the initial values $x_{-4} = 4$, $x_{-3} = 3$, $x_{-2} = 1.06$, $x_{-1} = 2$, $x_0 = 0.8$, $y_{-4} = 2$, $y_{-3} = 1.4$, $y_{-2} = 4$, $y_{-1} = 1$, $y_0 = 4$.

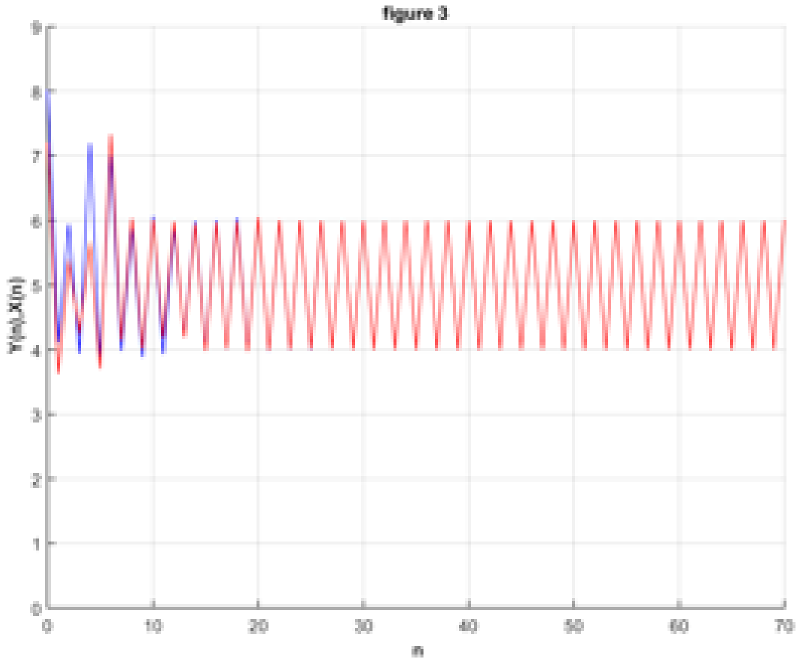

Figure 3: Plot of the solution $\{(x_n, y_n)\}_{n\geq 0}$ of the system (1.3) with $k = 3$ and the initial values $x_{-3} = 1.4$, $x_{-2} = 2.6$, $x_{-1} = 1.4$, $x_0 = 1.1$, $y_{-3} = 2.1$, $y_{-2} = 1.4$, $y_{-1} = 3.1$, $y_0 = 0.8$.

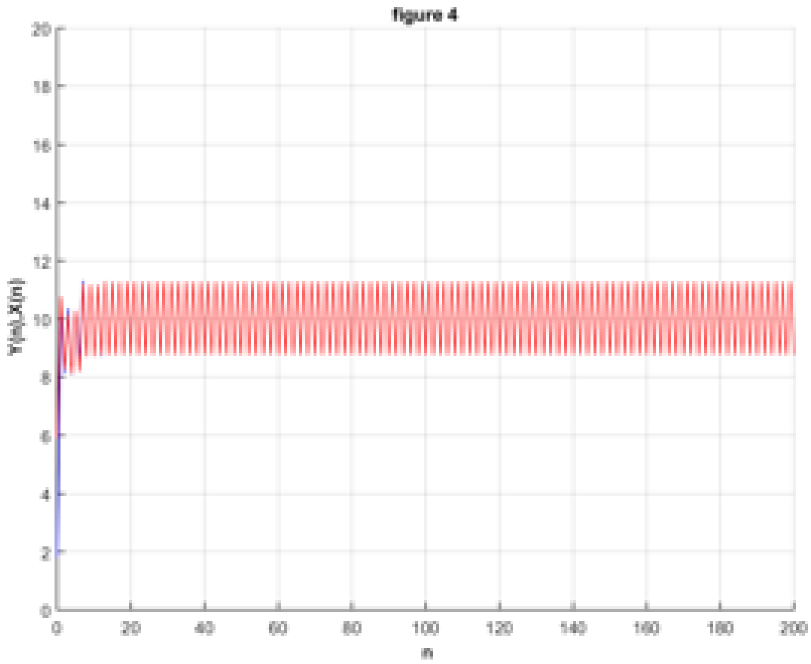

Figure 4: Plot of the solution $\{(x_n, y_n)\}_{n\geq 0}$ of the system (1.3) with $k = 5$ and the initial values $x_{-5} = 1.4$,$x_{-4} = 2.6$, $x_{-3} = 1.4$ $x_{-2} = 1.1$, $x_{-1} = 2.4$, $x_0 = 1.8$, $y_{-5} = 2.01$, $y_{-4} = 1.4$,$y_{-3} = 3.1$, $y_{-2} = 0.8$, $y_{-1} = 2.3$, $y_0 = 5.9$.

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