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Solution and stability of a fixed point problem for mappings without continuity

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Abstract

In this paper by taking into account three trends, prevalent in metric fixed point theory, namely, use of control functions instead of contraction constants, consideration of relational structure in the metric space and fixed point studies of discontinuous functions, we formulate and solve a new problem in relational metric fixed point theory. Our result extends the well known result of Kannan. The theorems are illustrated with examples. Further the problem is shown to have Hyers-Ulam-Rassias stability property. We make an application of our main result to a problem of a nonlinear integral equation.

Keywords: Kannan type mapping, Geraghty type mapping, Binary relation, Hyers-Ulam-Rassias stability, Integral equation
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1 Introduction and mathematical preliminaries

The present paper is about a problem formulated in metric fixed point theory. It is well known that the aforesaid domain of mathematics is an expanding area of research even after about hundred years of its initiation through the famous result due to Banach in 1922 [8]. Several books like [30, 32, 40] comprehensively describe the subject along with its modern development. Particularly we deal with a discontinuous class of functions which are known as Kannan type mappings. We solve the problem of existence of fixed points for such mappings under certain conditions. Further we study the Hyers-Ulam-Rassias stability of the problem which is answering the question whether an approximate fixed point of the operator has an actual fixed point approximation. This is an instance of a general approach to stability studies, particularly in nonlinear analysis, which has come up prominently in the recent literature [1, 2, 18, 23, 21, 27, 44, 46].

Kannan-type mappings are considered important in the metric fixed point theory. A reason for their important position is that they can be discontinuous. In fact the work of Kannan [25] is considered to have initiated the line of research for investigating fixed points of discontinuous function. A further distinguishing feature of Kannan's contraction is that the existence of fixed points for such mappings on metric spaces implies the completeness of the

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space [50] which is not shared by Banch's contraction. References [15, 20, 26, 31, 50] are some instances of works on Kannan type mappings along with their extensions.

Definition 1.1. [25]A mapping $T : X \to X$, where (X, d) is a metric space, is called a Kannan type mapping if there exists $0 \leq k < \frac{1}{2}$ such that, for all $x, y \in X$,

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)]. \tag{1.1}$$

Theorem 1.2 (Kannan). [25] Let (X, d) be a complete metric space and $T : X \to X$, be a map satisfying (1.1). Then T has a unique fixed point.

If $k \geq \frac{1}{2}$ in the inequality (1.1), then the theorem fails to hold, that is the function T may not have a fixed point. So it is pertinent to search for additional conditions on the behaviour of the function under which the range of k can be extended. One such set of conditions was developed by Górnicki in [15], which, if assumed, allows the value of k to be any value, less than 1.

On the other hand replacing the contractive constants by a suitable function opens up the scope of encompassing more functions to follow the conclusion of that theorem. This is one of the ways by which more generalised results are produced. Few works in this line are [7, 12, 14, 16, 34]. Getting inspired by the work of Rakotch [42], Geraghty in his paper [14] made such attempt and was successful to generalise the results of Rakotch [42] and that of Boyd and Wong [10]. Geraghty used the class $\mathcal{S}$ of all such functions $\beta : [0, \infty) \to [0, 1)$ such that

$$\beta(t_n) \to 1 \text{ implies } t_n \to 0, \tag{1.2}$$

and concluded the following theorem.

Theorem 1.3 (Geraghty). [14] Let $f : X \to X$ be a map on a complete metric space X satisfying $d(f(x), f(y)) \leq \beta(d(x, y))d(x, y)$ where $\beta \in \mathcal{S}$. Then T has a unique fixed point in X.

Here in our paper we replace the contractive constant by a Geraghty type function.

The main results of this paper are derived without using any continuity assumption. So the results are applicable to non continuous functions as well.

Use of binary relation is a new trend of research in fixed point theory. For details we refer the reader to the papers of [3, 4, 5, 11, 28, 29, 47]. The use of binary relations are effective in weakening the contractive map in a way that the contractive condition is not supposed to hold for any pair of points in the space but only for those which are related. In this work we make use of the notion of binary relation to relax the contractive conditions. Furthermore, the assumption in our result lies on the notion of $\mathcal{R}$-complete metric space, which is a more general structure than the complete metric space.

The concept of Hyers-Ulam-Rassias stability has its root in a question which Ulam [53] has raised in 1940. This question was partially addressed affirmatively by Hyers in the context of Banach spaces [17] and was further elaborated by Rassias [43]. Today it is a specialized category of stability which is applicable to diverse domains of mathematics. Several works in this line are [6, 17, 19, 41, 43, 45, 48]. For more information on functional equations we refer, e.g., to [9, 13, 22, 24, 21, 27, 35, 36, 38, 39, 49, 51, 54]. Here we discuss the Hyers-Ulam-Rassias stability of our fixed point problem.

In the last section of this paper we give an application of our main result to a problem of a nonlinear integral equation.

In the following we shall have few definitions which are used in the paper.

Definition 1.4. [37] Any subset $\mathcal{R}$ of $A \times B$ is called a relation from A to B.
The domain and range of the relation $\mathcal{R}$ are defined respectively as, Dom $\mathcal{R} := \{a \in A \mid \exists\, b \in B$ such that $(a, b) \in \mathcal{R}\}$ and Range $\mathcal{R} = \{b \in B \mid \exists\, a \in A$ such that $(a, b) \in \mathcal{R}\}$.

Definition 1.5. [3] Let X be a nonempty set. A subset $\mathcal{R}$ of $X \times X$ is a binary relation on X. We say that x and y are $\mathcal{R}$-comparative if either $(x,\ y) \in \mathcal{R}$ or $(y,\ x) \in \mathcal{R}$. We denote it by $[x,\ y] \in \mathcal{R}$.

Definition 1.6. [52] Let X be a nonempty set and $\mathcal{R}$ be a binary relation on X. A sequence $\{x_n\} \subset X$ is called $\mathcal{R}$-preserving if $(x_n, x_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}$.

Definition 1.7. [4] Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation on X. We say that (X, d) is $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence in X is convergent.

Definition 1.8. [3] Let $(X,\ d)$ be a metric space. A binary relation $\mathcal{R}$ defined on X is called d-self closed if every $\mathcal{R}$-preserving convergent sequence $\{x_n\}$ (with $x_n \longrightarrow x$, as $n \longrightarrow \infty$), has a subsequence $\{x_{n_k}\}$ such that $[x_{n_k}, x] \in \mathcal{R}$ for all $k \in \mathbb{N}$.

Definition 1.9. [37] Let X be a nonempty set and $\mathcal{R}$ be a binary relation on X.

(1) The inverse, transpose or dual relation of $\mathcal{R}$, denoted by $\mathcal{R}^{-1}$, is defined by

$$\mathcal{R}^{-1} = \{(x, y) \in X \times X \mid (y, x) \in \mathcal{R}\}.$$

(2) The symmetric closure of $\mathcal{R}$, denoted by $\mathcal{R}^s$, is defined by $\mathcal{R}^s := \mathcal{R} \bigcup \mathcal{R}^{-1}$.
Indeed, $\mathcal{R}^s$ is the smallest symmetric relation on X containing $\mathcal{R}$.

Definition 1.10. [33] Let X be a nonempty set and $\mathcal{R}$ be a binary relation on X. For $x,\ y \in X$, a path of length k (where k is a natural number) in $\mathcal{R}$ from x to y is a finite sequence $\{z_0, z_1, z_2, ..., z_k\} \subset X$ satisfying the following conditions:

$$(i)\ z_0 = x \text{ and } z_k = y,$$

$$(ii)\ (z_i, z_{i+1}) \in \mathcal{R} \text{ for each } i \text{ where } 0 \le i \le k-1.$$

Note that a path of length k involves $k+1$ elements of X, although they are not necessarily distinct.

Definition 1.11. [5] A relation $\mathcal{R}$ is called transitive if $(x, z) \in \mathcal{R}$, whenever $(x, y), (y, z) \in \mathcal{R}$.

Definition 1.12. [5] Given a mapping $T : X \to X$, a relation $\mathcal{R}$ is called T-transitive if $(Tx, Tz) \in \mathcal{R}$ whenever $(Tx, Ty), (Ty, Tz) \in \mathcal{R}$, for all $x, y, z \in X$.

Definition 1.13. [5] Let (X, d) be a metric space and $\mathcal{R}$ be a relation on it. X is called $\mathcal{R}$-connected if $\Upsilon(x, y, \mathcal{R}) \neq \emptyset$ for all $x, y \in X$, where $\Upsilon(x, y, \mathcal{R})$ is defined to be the class of all paths in $\mathcal{R}$ from x to y.

In the following we use the following notations:
- $F(T) := \{x \mid x = Tx\}$.
- $G(T, \varepsilon) := \{x \mid d(x, Tx) \le \varepsilon\}$.

Definition 1.14. [48] Let X be a nonempty set and T be a self-mapping on X. The fixed point problem -" Solve $x = Tx$" is said to be generalized Hyers-Ulam-Rassias stable if there exists a function $f : [0, \infty) \to [0, \infty)$, which is non-decreasing, continuous at 0 with $f(0) = 0$, such that for each $\varepsilon > 0$ and each $w \in G(T, \varepsilon)$ there exists $x_0 \in F(T)$, with $d(x_0, w) < f(\varepsilon)$.
If $f(t) = ct$ for some $c > 0$ then the problem is called Hyers-Ulam-Rassias stable.

2 Main Results

2.1 Result for the existence of fixed point

Theorem 2.1. Let $(X,\ d)$ be a metric space, equipped with a binary relation $\mathcal{R}$ and a self map T on X. Assume that the following conditions hold -

(i) $(X,\ d)$ is $\mathcal{R}^s$-complete.

(ii) Dom $\mathcal{R} \subseteq$ Range $\mathcal{R}$.

(iii) There exists $a \in [0, 1)$ such that for all $x \in$ Range $\mathcal{R}$, there exists $u \in$ Dom $\mathcal{R}$ such that

$$d(u, Tu) \le a d(x, Tx). \tag{2.1}$$

(iv) $\mathcal{R}^s$ is d-self closed.

(v) There exists a function $\beta : [0,\ \infty) \to [0,\ 1)$ with $\beta(t_n) \to 1 \implies t_n \to 0$, such that for all $x, y \in X,$ with $[x, y] \in \mathcal{R}$ the following condition holds:

$$d(Tx, Ty) \le \beta\left(d\left(x, Tx\right) + d\left(y, Ty\right)\right)\left\{d\left(x, Tx\right) + d\left(y, Ty\right)\right\}. \tag{2.2}$$

Then T has a fixed point.

Proof . Let $x_0 \in$ Range $\mathcal{R}$. Then by (2.1) there exists $x_1 \in$ Dom $\mathcal{R}$ such that $d(x_1, Tx_1) \le ad(x_0, Tx_0)$.
As $x_1 \in$ Dom $\mathcal{R} \subseteq$ Range $\mathcal{R}$, there exists $x_2 \in$ Dom $\mathcal{R}$ such that $d(x_2, Tx_2) \le ad(x_1, Tx_1)$.
Inductively, we can construct a sequence $\{x_n\}$ such that

$$(x_{n+1}, x_n) \in \mathcal{R} \text{ and } d(x_{n+1}, Tx_{n+1}) \le ad(x_n, Tx_n). \tag{2.3}$$

From (2.3) we get, $d(x_{n+1}, Tx_{n+1}) \le ad(x_n, Tx_n) \le ... \le a^{n+1}d(x_0, Tx_0)$.

$$\text{Since } a \in [0, 1), \text{ as } n \to 0,\ d(x_n, Tx_n) \to 0. \tag{2.4}$$

For $m, n \in \mathbb{N}$, with $m < n$ we have

$$\begin{aligned}
d(x_m, x_n) &\le d(x_m, Tx_m) + d(Tx_m, Tx_{m+1}) + ... + d(Tx_{n-1}, Tx_n) + d(Tx_n, x_n)\\
&< d(x_m, Tx_m) + \left\{d\left(x_m, Tx_m\right) + d\left(x_{m+1}, Tx_{m+1}\right)\right\} + ... + \left\{d\left(x_{n-1}, Tx_{n-1}\right) + d\left(x_n, Tx_n\right)\right\} + d(Tx_n, x_n)\\
&= 2\left\{d\left(x_m, Tx_m\right) + d\left(x_{m+1}, Tx_{m+1}\right) + d\left(x_{m+2}, Tx_{m+2}\right) + ... + d\left(x_n, Tx_n\right)\right\}\\
&\le 2\left\{d\left(x_m, Tx_m\right) + ad\left(x_m, Tx_m\right) + a^2 d\left(x_m, Tx_m\right) + ... + a^{n-m} d\left(x_m, Tx_m\right)\right\}\\
&\le 2\left\{1 + a + a^2 + ... + a^{n-m}\right\} d\left(x_m, Tx_m\right)\\
&\le \frac{2}{1-a} d\left(x_m, Tx_m\right) \to 0 \text{ as } m, n \to \infty.
\end{aligned} \tag{2.5}$$

From (2.5) it is seen that $\{x_n\}$ is a Cauchy sequence. Also for every $n \in \mathbb{N},\ (x_{n+1}, x_n) \in \mathcal{R}$. Thus by condition (i), $x_n \to z$ for some $z \in X$ as $n \to \infty$. Now by condition (iv) there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $[x_{n_k},\ z] \in \mathcal{R}$ for all $k \in \mathbb{N}$. Therefore,

$$\begin{aligned}
d(z,\ Tz) &\le d(z, x_{n_k}) + d(x_{n_k}, Tx_{n_k}) + d(Tx_{n_k},\ Tz)\\
&\le d(z, x_{n_k}) + d(x_{n_k}, Tx_{n_k}) + \beta(d(x_{n_k},\ Tx_{n_k}) + d(z,\ Tz))\left\{d(x_{n_k},\ Tx_{n_k}) + d(z,\ Tz)\right\}.
\end{aligned} \tag{2.6}$$

Taking limit $k \to \infty$ in above relation, and using (2.4) we get,

$$\begin{aligned}
d(z, Tz) &\le 0 + 0 + \lim_{k\to\infty} \beta(d(x_{n_k},\ Tx_{n_k}) + d(z,\ Tz))\left\{0 + d(z,\ Tz)\right\}\\
&= \lim_{k\to\infty} \beta(d(x_{n_k},\ Tx_{n_k}) + d(z,\ Tz)) d(z,\ Tz)\\
&\le d(z, Tz).
\end{aligned} \tag{2.7}$$

The relation (2.7) implies either $d(z, Tz) = 0$ or $\lim_{k\to\infty} \beta(d(x_{n_k},\ Tx_{n_k}) + d(z,\ Tz)) = 1$. If $\lim_{k\to\infty} \beta(d(x_{n_k},\ Tx_{n_k}) + d(z,\ Tz)) = 1$, then $\lim_{k\to\infty} \{d(x_{n_k},\ Tx_{n_k}) + d(z,\ Tz)\} = 0$. That is, $d(z,\ Tz) = 0$. So in any case $d(z, Tz) = 0$, which implies $z = Tz$.Hence, z is a fixed point of T. $\square$

Example 2.2. Let us consider the metric space $X = [0, 1)$ with usual metric d on it. Let $T : X \to X$ be defined as

$$Tx = \begin{cases} 0, & \text{if } x \in [0, \frac{1}{3}], \\ \frac{1}{4}, & \text{if } x \in (\frac{1}{3}, 1). \end{cases}$$

We define a relation $\mathcal{R} = [0, \frac{1}{3}] \times ([0, \frac{1}{3}] \cup (\frac{9}{4} - \sqrt{3}, 1))$.

We note that any $\mathcal{R}^s$ preserving Cauchy sequence has at most a finite number of terms in $(\frac{9}{4} - \sqrt{3}, 1)$. For, let $\{x_n\}$ be any $\mathcal{R}^s$ preserving Cauchy sequence. If possible, let $\{x_n\}$ contains infinite number of points in $(\frac{9}{4} - \sqrt{3}, 1)$. This means there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \in (\frac{9}{4} - \sqrt{3}, 1)$ for all $k \in \mathbb{N}$. Also, $\{x_n\}$ is $\mathcal{R}^s$ preserving and for every $k \in \mathbb{N}$, $x_{n_k} \in$ Range $\mathcal{R}$ implies $x_{n_k+1} \in$ Dom $\mathcal{R}$. Now, $\{x_n\}$ being a Cauchy sequence its subsequence

$\{x_{n_k+1}\}$ is also Cauchy in the complete subspace $[0, \frac{1}{3}]$. Thus $\{x_{n_k+1}\}$ converges to some point in $[0, \frac{1}{3}]$. Which means $\{x_n\}$, and thus $\{x_{n_k}\}$ are also convergent to the same point. This is absurd since $[0, \frac{1}{3}]$ has no point in the closure of $(\frac{9}{4}-\sqrt{3}, 1)$ in which $\{x_{n_k}\}$ is fully contained. Thus by the method of contradiction we establish that any $\mathcal{R}^s$-preserving sequence in X has all its elements in $[0, \frac{1}{3}]$, except possibly finitely many. As $[0, \frac{1}{3}]$ is complete it is seen that X is R^s-complete.

Here Dom $\mathcal{R} = [0, \frac{1}{3}]$ and Range $\mathcal{R} = [0, \frac{1}{3}] \cup (\frac{9}{4} - \sqrt{3}, 1)$. Thus, Dom $\mathcal{R} \subseteq$ Range $\mathcal{R}$.

Now let $0 \leq a < 1$. Then for each $x \in [0, \frac{1}{3}]$ we can find $u = ax \in [0, \frac{1}{3}]$ and for each $x \in (\frac{9}{4} - \sqrt{3}, 1)$, we can find $u = \min\{a(x - \frac{1}{4}), \frac{1}{3}\} \in [0, \frac{1}{3}]$ such that (2.1) holds. Thus for each $x \in$ Range $\mathcal{R}$ there exists $u \in$ Dom $\mathcal{R}$ such that (2.1) holds.

Now let $\{x_n\}$ be any $\mathcal{R}^s$ preserving sequence converging to $v \in X$. Clearly $\{x_n\}$ is eventually contained in $[0, \frac{1}{3}]$, which is a closed set. So $v \in [0, \frac{1}{3}]$. Thus $\mathcal{R}^s$ is d-self closed.

We now define $\beta : [0, \infty) \to [0, 1)$ as $\beta(t) = \begin{cases} 1 - \dfrac{t}{4}, & \text{if } 0 \leq t \leq 1, \\ \dfrac{1}{2}, & \text{otherwise.} \end{cases}$.

Let $[x, y] \in \mathcal{R}$. If both $x, y \in [0, \frac{1}{3}]$, then L.H.S of (2.2) becomes 0, and the relation holds trivially. Otherwise, to check the relation (2.2), without the loss of any generality, we consider $x \in [0, \frac{1}{3}]$ and $y \in (\frac{9}{4} - \sqrt{3}, 1)$. Then, $d(Tx, Ty) = \frac{1}{4}$ and $d(x, Tx) + d(y, Ty) = x + y - \frac{1}{4}$.
We consider two cases here.

<u>**Case-I**</u>: When $2 - \sqrt{3} < x + y - \dfrac{1}{4} \leq 1$, $\beta(t) = 1 - \frac{t}{4}$. The inequality (2.2) in this case then becomes $\frac{1}{4} \leq \{1 - \frac{1}{4}(x + y - \frac{1}{4})\}(x + y - \frac{1}{4})$. The value of the right hand side is greater than $\frac{1}{4}$. Thus (2.2) is satisfied.

<u>**Case-II**</u>: When $1 < x + y - \dfrac{1}{4} \leq 1 + \dfrac{1}{3} - \dfrac{1}{4}$, $\beta(t) = \frac{1}{2}$.
Therefore, $\beta\left(d\left(x, Tx\right) + d\left(y, Ty\right)\right)\left\{d\left(x, Tx\right) + d\left(y, Ty\right)\right\} = \frac{1}{2} \times (x + y - \frac{1}{4}) \geq \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = d(Tx, Ty)$.
This shows that the inequality (2.2) is satisfied in this case also.
So for all $[x, y] \in \mathcal{R}$, the inequality (2.2) is satisfied.

Hence all the conditions of Theorem 2.1 are satisfied. It is seen that $'0'$ is a fixed point of T.

Remark 2.3. $X = [0, 1)$ is not complete. So Theorem 2.1 is applicable in non complete spaces also.

Remark 2.4. The map T is not continuous. So Theorem 2.1 is applicable for non continuous maps.

Remark 2.5. It is to be noted that the map T is not of the Kannan Type. For $x = 0$ and $y = \frac{3}{4}$, we have $d(Tx, Ty) = \frac{1}{4}$ and $d(x, Tx) + d(y, Ty) = x + y - \frac{1}{4} = \frac{1}{2}$. In order to satisfy (1.1) one must take $k \geq \frac{1}{2}$.

Remark 2.6. The inequality (2.2) is not satisfied for any pair of points x, y from the set X. For example if we take $x = 0$ and $y = \frac{1}{2}$ then $d(Tx, Ty) = d(x, Tx) + d(y, Ty)$, which is not satisfied for any $\beta \in \mathcal{S}$ as $\beta(t) \neq 1$ for all $t \in [0, \infty)$. However, the inequality (2.2) is satisfied for every pair of points which are related by a relation $\mathcal{R}$. This shows that our result generalizes the result of [15] also.

Remark 2.7. In the Example 2.2 it is seen that there exists no constant $k \in [0, 1)$ such that the inequality (1.1) holds for all $[x, y] \in \mathcal{R}$. Therefore replacement of the contractive constant k by the function β is effective to address a more general fixed point result.

Remark 2.8. Theorem 2.1 requires lesser conditions than Lemma 1.2 of [15]. In that result, they have assumed that there is some $b \geq 0$ such that $d(u, x) \leq bd(x, Tx)$, which we do not require in proving our result.

2.2 Uniqueness result of Fixed points.

In Theorem 2.1 we have only assured the existence of the fixed point. Now we find conditions for such fixed point to be unique. First we give a lemma.

Lemma 2.9. Let $(X,\ d)$ be a metric space equipped with a binary relation $\mathcal{R}$, a self map T on X. Suppose further that there exists a function $\beta : [0,\ \infty) \to [0,\ 1)$ satisfying (1.2), such that for all $x, y \in X,$ with $[x, y] \in \mathcal{R}$, (2.2) holds. If T has two distinct fixed points x_0 and y_0, then $[x_0, y_0] \notin \mathcal{R}$.

Proof . If possible let T has two distinct fixed points x_0, y_0, and $[x_0, y_0] \in \mathcal{R}$.
By equation (2.2) we have,

$$\begin{aligned} d(Tx_0,\ Ty_0) &\leq \beta(d(x_0, Tx_0) + d(y_0\ Ty_0))(d(x_0,\ Tx_0) + d(y_0,\ Ty_0)) \\ &= \beta(d(x_0, Tx_0) + d(y_0\ Ty_0)) \times 0 \qquad\qquad [\text{since } x_0 = Tx_0,\ y_0 = Ty_0] \\ &= 0. \end{aligned}$$

Thus we have $Tx_0 = Ty_0$, that is $x_0 = y_0$, which is a contradiction to our assumption that x_0, y_0 are distinct.
Hence the result is proved. □

Note: If T has fixed points x_0, y_0 with $[x_0, y_0] \in \mathcal{R}$, then $x_0 = y_0$.

Theorem 2.10. In addition to the conditions stated in Theorem 2.1, if the following conditions hold
(vi) $T(X)$ is $\mathcal{R}$-connected;
(vii) $\mathcal{R}$ is T-transitive;
then the fixed point is unique.

Proof . Let, along with the conditions stated in Theorem 2.1, the above conditions (vi) and (vii) also hold and $x,\ y$ be two fixed points of T.
By (vi), we have $\Upsilon(x, y, \mathcal{R}) \neq \emptyset$, that is, there is a path $\{z_0, z_1, ..., z_k\}$ of finite length such that $x = z_0,\ y = z_k,\ z_i \in T(X)$ and $(z_i, z_{i+1}) \in \mathcal{R}$ for $0 \leq i \leq k-1$.
As $\mathcal{R}$ is T-transitive, we have $(x, y) \in \mathcal{R}$. Then, by the Lemma 2.9, we conclude that $x = y$.
Thus the fixed point is unique. □

Theorem 2.1 and Theorem 2.10 do not require completeness assumption. However results for complete metric spaces can also be concluded from these results. We have the following corollaries.

Corollary 2.11. Let (X, d) be a complete metric space. Let the following conditions hold:

(i) There exists $a \in [0, 1)$ such that for each $x \in X$ there exists $u \in X$ such that (2.1) holds.

(ii) There exists a function $\beta : [0,\ \infty) \to [0,\ 1)$ satisfying (1.2), such that for all $x, y \in X$, (2.2) holds.

Then T has a unique fixed point.

Proof . For the metric space X if $\mathcal{R}$ is taken to be $X \times X$, i.e. the universal relation on it, then the notion of $\mathcal{R}^s$-completeness reduces to the completeness. So all the conditions of Theorem 2.1 holds. So there exists a fixed point of T.

Moreover the condition of the Lemma 2.9 is also satisfied. Thus the fixed point would be unique. □

We illustrate the fact with the following example.

Example 2.12. Let us consider the metric space $X = [0, \infty)$ with the usual metric. Let $T : X \to X$ be defined as

$$Tx = \begin{cases} 2, & \text{if } x \neq 3, \\ 1, & \text{if } x = 3. \end{cases}$$

We define a relation $\mathcal{R} = [0, \infty) \times [0, \infty)$. Then X is clearly $\mathcal{R}^s$-complete. Also, Dom $\mathcal{R} \subseteq$ Range $\mathcal{R}$.
Now, let $0 \leq a < 1$. Then for each $x \in [0, \infty)$ we can choose $u = 2 \in [0, \infty)$ such that (2.1) holds. Thus for all $x \in$ Range $\mathcal{R}$, there exists $u \in$ Dom $\mathcal{R}$ such that (2.1) holds.
Clearly $\mathcal{R}^s$ is d-self closed.

We now define $\beta : [0, \infty) \to [0, 1)$ as $\beta(t) = \begin{cases} \frac{2}{3}, & \text{if } \ t \geq 1, \\ e^{-t}, & \text{if } \ 0 < t \leq 1, \\ 0, & t = 0. \end{cases}$ Let $x,\ y \in X$. If $x = y = 3$ or $x \neq 3, y \neq 3$, then L.H.S of (2.2) becomes 0, and thus the inequality holds trivially. Otherwise, to check the inequality (2.2), without the loss of any generality, consider the case when $x = 3$ and $y \neq 3$.
For $x = 3$ and $y \neq 3$, $d(Tx, Ty) = 1$.
$d(x, Tx) + d(y, Ty) = 2 + |y - 2| \geq 2$. In this case $\beta(d(x, Tx) + d(y, Ty)) = \frac{2}{3}$. Thus $d(Tx, Ty) \leq \beta(d(x, Tx) +$

$d(y,Ty))d(x,Tx)+d(y,Ty)$.
This shows that (2.2) is satisfied.
Hence all the conditions of Theorem 2.1 are satisfied. It is seen that '2′ is a fixed point of T.
Also $T(X)=\{2,1\}$. Clearly conditions of Theorem 2.10 are also satisfied. It is seen that ′2′ is the only fixed point of T.

Remark 2.13. The map T in the Example 2.12 is not a Kannan type map. Because for $x=2, y=3$, $d(Tx,Ty)=1$ and $d(x,Tx)+d(y,Ty)=2$, the inequality (1.1) is not satisfied unless $k\geq \frac{1}{2}$.

Corollary 2.14 (Kannan). [25] Let (X,d) be a complete metric space and $T:X\to X$ be a map satisfying inequality (1.1), where $0\leq k<\frac{1}{2}$. Then T has a unique fixed point.

Proof . Let x be any element of X. If we set $u=Tx$ then from (1.1) we have

$$\begin{aligned} d(u,Tu) &= d(Tx,Tu) \\ &\leq k(d(x,Tx)+d(u,Tu)). \end{aligned}$$

Therefore we have,

$$d(u,Tu)\leq \frac{k}{1-k}d(x,Tx). \tag{2.8}$$

As $0\leq k<\frac{1}{2}$, choosing $a=\dfrac{k}{1-k}<1$, we observe that condition (i) of Corollary 2.11 is satisfied.
If we choose $\beta(t)=k$ for all $t\in[0,\infty)$ then such β satisfies condition (ii) of Corollary 2.11.
Thus by Corollary 2.11 we conclude that T has a unique fixed point in X. □

Remark 2.15. Thus we see that Corollary 2.11 is a proper generalization of the result of Kannan[25].

2.3 Hyers-Ulam-Rassias Stability

In this section we show that our problem is Hyers-Ulam-Rassias Stable.

Theorem 2.16. Let X, T and $\mathcal{R}$ be as in Theorem 2.1. Consider the fixed point problem :

$$\text{Solve } x=Tx. \tag{2.9}$$

In addition to the conditions in Theorem 2.1, if for all $\varepsilon>0$, $w\in G(T,\varepsilon)$ implies there exists $z\in F(T)$ such that $[w,z]\in\mathcal{R}$, then the problem (2.9) is Hyers-Ulam-Rassias stable (see definition 1.14).

Proof . Let $\varepsilon>0$ and consider $w\in G(T,\varepsilon)$. Then $d(w,Tw)\leq\varepsilon$. According to the hypothesis of this theorem, there exists $z\in F(T)$ such that $[w,z]\in\mathcal{R}$. So from relation (2.2), we have

$$\begin{aligned} d(Tz,\ Tw) &\leq \beta(d(z,\ Tz)+d(w,\ Tw))(d(z,\ Tz)+d(w,\ Tw)) \\ &\leq \beta(d(z,\ z)+d(w,\ Tw))(d(z,\ z)+d(w,\ Tw)) \\ &\leq \beta(d(w,\ Tw))d(w,\ Tw)). \end{aligned}$$

Then we have,

$$\begin{aligned} d(w,z) \leq& d(w,Tw)+d(Tw,z) \\ =& d(w,Tw)+d(Tw,Tz) \\ \leq& d(w,Tw)+\beta(d(w,\ Tw))d(w,\ Tw)) \\ \leq& \varepsilon+\beta(d(w,\ Tw))\times\varepsilon \\ \leq& \varepsilon+\varepsilon \qquad\qquad [\text{since } \beta(d(w,\ Tw))<1]. \\ =& 2\varepsilon. \end{aligned}$$

Thus taking $f(\varepsilon)=2\varepsilon$, we have for each $\varepsilon>0$ and each $w\in G(T,\varepsilon)$ there exists $x_0\in F(T)$, with $d(x_0,w)<f(\varepsilon)$. Hence the problem is Hyers-Ulam-Rassias stable. □

Remark 2.17. The significance of Theorem 2.16 is that approximate fixed point can be actually approximated by a fixed point.

3 Application

In this section, we present an application of our fixed point results to establish the existence of solution of integral equations.

We consider here a homogeneous nonlinear integral equation as follows

$$x(t) = \int_0^1 h(t,\ s,\ x(s))ds \text{ where } t,\ s \in [0,\ 1] \tag{3.1}$$

where $x(t)$ takes values from $[0, 1]$.

Let $X = C([0,\ 1])$, be the set of all real valued continuous functions defined on $[0,\ 1]$. Consider the Banach space $(X, \|.\|)$ equipped with the norm:

$$\|x\| = \sup_{t\in[0,\ 1]} \mid x(t) \mid .$$

Then the metric space (X, d), where d is the metric induced by the norm $\|.\|$ defined as

$$d(x,\ y) = \max_{t\in[0,\ 1]} \mid x(t) - y(t) \mid \tag{3.2}$$

is complete.

Consider the relation $\precsim$ on $C[0, 1]$, defined by $x \precsim y$ if $x(t) \leq y(t)$ for all $t \in [0, 1]$.

Next, we define a mapping $T : X \to X$ by

$$T(x)(t) = \int_0^1 h(t,\ s,\ x(s))ds, \quad \text{for all} \quad t,\ s \in [0,\ 1] \tag{3.3}$$

with the following assumptions:

A_1 : $h : [0,\ 1] \times [0,\ 1] \times [0,\ 1] \to [0,\ 1]$ is a continuous mapping and $h(t, s, x) \to \theta$ as $x \to \theta$, where θ is the zero function, $\theta(t) = 0$, for all $t \in [0, 1]$;

A_2 : For $x \precsim y$, $\mid h(t, s, x(s)) - h(t, s, y(s)) \mid \leq e^{-\{d(x,Tx)+d(y,Ty)\}}\{d(x, Tx) + d(y, Ty)\}$, for all $x,\ y \in X$ and for all $t,\ s \in [0,\ 1]$].

Theorem 3.1. Let $X,\ \precsim,\ T,\ h(t, s, x)$ satisfy the assumptions A_1 and A_2. Then the nonlinear integral equation (3.1) has a solution $x \in C([0,\ 1])$.

Proof . We first note that the space X being complete, it is $\precsim$-complete also.

Next we note, Dom $\mathcal{R} \subseteq$ Range $\mathcal{R}$.

Now, let $a \in [0, 1)$ and for each $x \in$ Range $\precsim$, take $u = \theta$. Then By assumptions A_1

$$\begin{aligned} d(u, Tu) &= \sup_{t\in[0,1]} \{\mid \theta(t) - \int_0^1 h(t, s, \theta(s))ds \mid\} \\ &= \sup_{t\in[0,1]} \{\mid \theta(t) - \int_0^1 \theta(s)ds \mid\} \\ &= 0 \\ &\leq ad(x, Tx). \end{aligned}$$

Thus the inequality (2.1) is satisfied.

Now, let $\{x_n\}$ be any $\precsim$-preserving sequence in X converging to $x \in X$. Then we have, $x_0(t) \leq x_1(t) \leq x_2(t) \leq .... \leq x_n(t) \leq x_{n+1}(t) \leq ...$, for all $t \in [0, 1]$.

Thus, $x_n(t) \leq x(t)$ for all $t \in [0, 1]$. Therefore, $x_n \precsim x$ for all $n \in \mathbb{N}$. So $\precsim$ is d-self-closed.

Also, for $x, y \in X$ with $x \precsim y$,

$$\begin{aligned}
d(Tx, Ty) &= \sup_{t\in[0,1]} \mid T(x)(t) - T(y)(t) \mid \\
&= \sup_{t\in[0,1]} \mid \int_0^1 [h(t,s,x(s))ds - h(t,s,y(s))]ds \mid \\
&\leq \sup_{t\in[0,1]} \int_0^1 \mid [h(t,s,x(s)) - h(t,s,y(s)))] \mid \ ds \\
&\leq \sup_{t\in[0,1]} \int_0^1 [e^{-\{d(x,Tx)+d(y,Ty)\}}\{d(x,Tx)+d(y,Ty)\}]ds \\
&\leq \sup_{t\in[0,1]} e^{-\{d(x,Tx)+d(y,Ty)\}}\{d(x,Tx)+d(y,Ty)\} \int_0^1 ds \\
&\leq e^{-\{d(x,Tx)+d(y,Ty)\}}\{d(x,Tx)+d(y,Ty)\}.
\end{aligned}$$

So we have,

$$d(Tx, Ty) \leq \beta(d(x,Tx)+d(y,Ty))(d(x,Tx)+d(y,Ty)) \text{ where } \beta(t) = e^{-t} \to 1 \text{ implies } t \to 0.$$

Therefore, all the conditions of Theorem 2.1 are satisfied. Therefore, Theorem 2.1 applies to T, which guarantees the existence of a fixed point x in X. That is, there is a solution of the integral equation (3.1). □

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References

[1] M.R. Abdollahpour, R. Aghayaria, and T.M. Rassias, *Hyers-Ulam stability of associated Laguerre differential equations in a subclass of analytic functions*, J. Math. Anal. Appl. **437** (2016), 605–612.

[2] M.R. Abdollahpour and T.M. Rassias, *Hyers-Ulam stability of hypergeometric differential equations*, Aeq. Math. **93** (2019), no. 4, 691–698.

[3] A. Alam and M. Imdad, *Relation-theoretic contraction principle*, J. Fixed Point Theory Appl. **17** (2015), no. 4, 693–702.

[4] ______, *Relation-theoretic metrical coincidence theorems*, Filomat **31** (2017), 4421—4439.

[5] ______, *Nonlinear contractions in metric spaces under locally T-transitive binary relations*, Fixed Point Theory **19** (2018), 13–24.

[6] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), no. 1-2, 64–66.

[7] Md.I. Ayari, *A best proximity point theorem for α-proximal Geraghty non-self mappings*, Fixed Point Theory Appl. **2019** (2019), no. 10.

[8] S. Banach, *Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrales*, Fund. Math. **3** (1922), 133–181.

[9] D. G. Bourgin, *Classes of transformations and bordering transformations*, Bull. Amer. Math. Soc. **57** (1951), 223–237.

[10] D.W. Boyd and J.S.W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc. **20** (1969), no. 2, 458–464.

[11] P. Chakraborty, B.S. Choudhury, and M. De la Sen, *Relation-theoretic fixed point theorems for generalized weakly contractive mappings*, Symmetry **12** (2020).

[12] B. S. Choudhury and A. Kundu, *A Kannan-like contraction in partially ordered spaces*, Demonst. Math. **XLVI** (2013), no. 2.

[13] S. Czerwik, *Functional equations and inequalities in several variables*, World Scientific, 2002.

[14] M.A. Geraghty, *On contractive mappings*, Proc. Amer. Math. Soc. **40** (1973), 604–608.

[15] J. Górnicki, *Fixed point theorems for Kannan type mappings*, J. Fixed Point Theory Appl. **19** (2017), 2145—2152.

[16] V. Gupta, W. Shatanawi, and N. Mani, *Fixed point theorems for (ψ, β)-Geraghty contraction type maps in ordered metric spaces and some applications to integral and ordinary differential equations*, J. Fixed Point Theory Appl. **19** (2017), 1251—1267.

[17] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA **27** (1941), no. 4, 222–224.

[18] D.H. Hyers, G. Isac, and T.M. Rassias, *Stability of functional equations in several variables*, Birkhauser, Boston, 1998.

[19] D.H. Hyers and T.M. Rassias, *Approximate homomorphisms*, Aeq. Math. **44** (1992), 125–153.

[20] L. Janos, *On mappings contractive in the sense of Kannan*, Proc. Amer. Math. Soc. **61** (1976), no. 1, 171–175.

[21] S.M. Jung, *Hyers-Ulam-Rassias stability of functional equations in nonlinear analysis*, Springer, 2011.

[22] S.M. Jung, C. Mortici, and T.M. Rassias, *On a functional equation of trigonometric type*, Appl. Math. Comput. **252** (2015), 294–303.

[23] S.M. Jung, D. Popa, and T.M. Rassias, *On the stability of the linear functional equation in a single variable on complete metric groups*, J. Glob. Optim. **59** (2014), 165—-171.

[24] S.M. Jung and T.M. Rassias, *A linear functional equation of third order associated to the Fibonacci numbers*, Abstr. Appl. Anal. **2014** (2014), no. 137468.

[25] R. Kannan, *Some results on fixed points*, Bull. Cal. Math. Soc **60** (1968), 71–76.

[26] ______, *Some results on fixed points-II*, Amer. Math. Month. **76** (1969), 405–408.

[27] P. Kannappan, *Functional equations and inequalities with applications*, Springer, 2009.

[28] S. Khalehoghli, H. Rahimi, and M. E. Gordji, *Fixed point theorems in R-metric spaces with applications*, AIMS Mathematics **5** (2020), no. 4, 3125–3137.

[29] ______, *R-topological spaces and sr-topological spaces with their applications*, Math. Sci. **14** (2020), 249–255.

[30] Md.A. Khamsi and W.A. Kirk, *An introduction to metric spaces and fixed point theory*, John Wiley & Sons, 2011.

[31] M. Kikkaw and T. Suzuki, *Some similarity between contractions and Kannan mappings*, Fixed Point Theory Appl. **2008** (2008), no. 649749.

[32] W.A. Kirk and B. Sims, *Handbook of metric fixed point theory*, Springer, Dordrecht, 2001.

[33] B. Kolman, R.C. Busby, and S. Ross, *Discrete mathematical structures*, 3rd ed. ed., PHI Pvt. Ltd., New Delhi, India, 2000.

[34] P. Kuman and C. Mongkolkeha, *Common best proximity points for proximity commuting mapping with Geraghty's functions*, Carpathian J. Math. **31** (2015), no. 3, 359–364.

[35] Y.H. Lee, S.M. Jung, and T.M. Rassias, *On an n-dimensional mixed type additive and quadratic functional equation*, Appl. Math. Comput. **228** (2014), 13—16.

[36] ______, *Uniqueness theorems on functional inequalities concerning cubic-quadratic-additive equation*, J. Math. Inequal. **12** (2018), no. 1, 43–61.

[37] S. Lipschutz, *Schaum's outlines of theory and problems of set theory and related topics*, McGraw-Hill, New York, USA, 1964.

[38] C. Mortici, S.M. Jung, and T.M. Rassias, *On the stability of a functional equation associated with the Fibonacci numbers*, Abstr. Appl. Anal. **2014** (2014), no. 546046.

[39] C. Park and T.M. Rassias, *Additive functional equations and partial multipliers in C*-algebras*, Rev. Real Acad. Cienc. Exactas, Serie A. Mat. **113** (2019), no. 3, 2261–2275.

[40] V. Pata, *Fixed point theorems and applications*, Springer International Publishing, Switzerland, 2019.

[41] S. Phiangsungnoen and P. Kumam, *Generalized Ulam-Hyers stability and well-posedness for fixed point equation via α-admissibility*, J. Inequal. Appl. **2014** (2014), no. 418.

[42] E. Rakotch, *A note on contractive mappings*, Proc. Amer. Math. Soc. **13** (1962), 459–165.

[43] T.M. Rassias, *On the stability of the linear mappings in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[44] ______, *Functional equations and inequalities*, Kluwer Academic Publishers, 2000.

[45] I.A. Rus and M.A. Şerban, *Basic problems of the metric fixed point theory and the relevance of a metric fixed point theorem*, Carpathian J. Math. **29** (2013), no. 2, 239–258.

[46] P.K. Sahoo and P. Kannappan, *Introduction to functional equations*, CRC Press, 2011.

[47] B. Samet and M. Turinici, *Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications*, Commun. Math. Anal. **13** (2012), no. 2, 82–97.

[48] W. Sintunavarat, *Generalized Ulam-Hyers stability, well-posedness and limit shadowing of fixed point problems for $\alpha-\beta$-contraction mapping in metric spaces*, The Scientific World Journal **2014** (2014), no. 569174.

[49] W. Smajdor, *Note on a Jensen type functional equation*, Publ. Math. Debrecen **63** (2003), 703–714.

[50] P. V. Subrahmanyam, *Completeness and fixed points*, Monatsh. Math. **80** (1975), 325–330.

[51] T. Trif, *On the stability of a functional equation deriving from an inequality of popoviciu for convex functions*, J. Math. Anal. Appl. **272** (2002), 604–616.

[52] M. Turinici, *Product fixed points in ordered metric spaces*, arXiv: General Topology (2011).

[53] S.M. Ulam, *Problems in modern mathematics*, Wiley, New York, 1964.

[54] J. Wang, *Some further generalization of the Ulam-Hyers-Rassias stability of functional equations*, J. Math. Anal. Appl. **263** (2001), 406–423.