Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 2763–2770
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.27521.3637

On the co-intersection graph of subsemimodules of a semimodule

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(Communicated by Madjid Eshaghi Gordji)

Abstract

Let S be a semiring with identity and U be a unitary left S-semimodule. The co-intersection graph of an S-semimodule U, denoted by $\Gamma(U)$, is defined to be the undirected simple graph whose vertices are in one-to-one correspondence with all non-trivial subsemimodules of U, and there is an edge between two distinct vertices N and L if and only if $N + L \neq U$. We study these graphs to relate the combinatorial properties of $\Gamma(U)$ to the algebraic properties of the S-semimodule U. We study the connectedness of $\Gamma(U)$. We investigate some properties of $\Gamma(U)$ for instance, we find the domination number and clique number of $\Gamma(U)$. Also, we study cycles in $\Gamma(U)$.

Keywords: Semimodule, Co-intersection graph, Connectivity, Domination number, Clique number
2020 MSC: 16Y60, 05C75, 05C69

1 Introduction

In 1964, Bosak [9] defined the graph of subsemigroups of a semigroup. In 2012, the intersection graph of submodules of a module was considered by Akbari et. al. in [1]. Recently many structures of graphs related to the module and semiring structure are found in [5]-[7]. The co-intersection graph of the proper submodules of a module is studied in [12]. Encouraged by preceding studies on the intersection graph of algebraic constructions, we describe the co-intersection graph of subsemimodules of a semimodule in [13]. Here we revise more aspects of the co-intersection graph of subsemimodules of a semimodule and obtain more results.

A semiring S is algebraic system $(S,+,\cdot)$ where $(S,+)$ and $(S,\cdot)$ are commutative semigroups, connected by $z(x+y) = zx + zy$ for all $x, y, z \in S$ and there exist $0 \neq 1 \in S$ such that $s + 0 = s, s0 = 0s = 0$ and $s1 = 1s = s$ for all $s \in S$ [2].

Let $(U,+)$ be an additive abelian monoid with additive identity 0_U. Then U is called an S-semimodule (a semimodule over a semiring S) if there exists a scalar multiplication $S \times U \to U$ denoted by $(s,u) \mapsto su$, such that $(ss')\,u = s\,(s'u)$; $s\,(u+u') = su + su'$; $(s+s')\,u = su + s'u; 1u = u$ and $s0_U = 0_U = 0u$ for all $s, s' \in S$ and all $u, u' \in U$. A nonempty subset N of a left S-semimodule U is a subsemimodule of S if and only if N is closed under addition and scalar multiplication [4]. All semiring in this paper are commutative with non-zero identity and U be a left S-semimodule.

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Received: April 2022 *Accepted:* June 2022

We mean from a non-trivial subsemimodule of U is a nonzero proper subsemimodule of U. A semimodule U is simple if it has no non-trivial subsemimodule. A semimodule U is called indecomposable, if it is not a direct sum of two non-zero subsemimodules. A subsemimodule of U is minimal if and only if it does not have any subsemimodule of U other than 0 and itself. We mean by $\min(U)$ the set of minimal subsemimodules of U. The length of U is the length of the composition series of U, represented by $l_S(U)$. A subsemimodule N of U is called small in U (we write $N \ll U$), if for every subsemimodule $X \subseteq U$, with $N + X = U$ implies that $X = U$, i.e., N is called small in U, if $N + X \neq U$ for every proper subsemimodule X of U. The radical of an S-semimodule U, denoted by $\mathrm{Rad}(U)$, is the sum of all small subsemimodules of U [11]. A semimodule U is called hollow, if every proper subsemimodule of U is small in U [6].

A subsemimodule M of U is called maximal in U if and only if it is not properly contained in any other subsemimodule of U. If U has a unique maximal subsemimodule then U is named local. $\max(U)$ is the set of all maximal subsemimodules of U. A subsemimodule N of U is called subtractive if $x, x + y \in N$, implies $y \in N$ for all $x, y \in U$. If each subsemimodule of U is subtractive, we say that U is subtractive see for example [3]. If each proper subtractive subsemimodule of U is contained in a maximal subtractive subsemimodule, we say that U is coatomic [8].

For the definitions of semirings and semimodules we refer [10]. For graph theory, the reference is [14].

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a graph with the set of vertices $V(\Gamma)$ and the edge set $E(\Gamma)$ where an edge is an unordered pair of distinct vertices of Γ. Graph Γ is finite, if $\mathrm{card}(V(\Gamma)) < \infty$, otherwise Γ is infinite. For two distinct vertices Q and P represented by $Q - P$ means that Q and P are adjacent. The graph whose vertices can be separated into two sets W and V such that every edge joins a vertex in W to one in V, and W and V are each independent is called bipartite. The vertices u and v of a graph Γ are named joined in Γ if a path exists between them. If a path exists between any two distinct vertices of Γ, we say that a graph Γ is joined (or connected). Or else, Γ is called disconnected. Let Γ be a joined graph. An E, D-path is a path with starting vertex E and ending vertex D. The distance between two distinct vertices E and D, indicated by $d(E, D)$, is the length of the shortest $E, D-$ path joining them if a path exists.

A null graph is a graph with no edges. If there is a path between every pair of vertices of Γ we say that a graph Γ is named connected.

A complete graph is a graph in which every pair of distinct vertices are adjacent. The complete graph with n distinct vertices, denoted by K_n.

A complete subgraph in Γ is called a clique in Γ. The number of vertices in the biggest clique of Γ, denoted by $\omega(\Gamma)$, is called the clique number of Γ. The smallest number of colours which can be assigned to the vertices of Γ such that every two adjacent vertices have dissimilar colours is called the chromatic number of Γ, denoted by $\chi(\Gamma)$. A graph Γ in which $\omega(\Gamma) = \chi(\Gamma)$ is called weakly perfect. A graph Γ is planar if it can be drawn in the plane consequently that its edges intersect only on their ends.

2 Connectivity of $\Gamma(U)$

Let U be an S-semimodule. In this section, we characterize all semimodules for which the co-intersection graph of subsemimodules is not connected. Finally, we study some semimodules whose intersection graphs are complete.

Theorem 2.1. [13, Theorem 2.3] Let U be a subtractive S-semimodule. Then the graph $\Gamma(U)$ is not connected if and only if $U = T_1 \oplus T_2$, where T_1, T_2 are two simple S-semimodules.

Now, we give some corollaries which are direct consequences of Theorem 2.1.

Corollary 2.2. Assume a subtractive S-semimodule U is not simple. Then $\Gamma(U)$ is connected if and only if either U is not semisimple or $U = \bigoplus_{i=1}^{n} U_i$, wherever $n \geq 3$ and U_i is a simple semimodule for all $1 \leq i \leq n$.

Corollary 2.3. Let U be a subtractive S-semimodule and $|\Gamma(U)| > 2$. If $\Gamma(U)$ has at least one edge, then $\Gamma(U)$ is a connected graph.

Corollary 2.4. Let U be a subtractive S-semimodule and $|\Gamma(U)| > 2$. Then $\Gamma(U)$ is a null graph if and only if $l_S(U) = 2$.

Remark 2.5. [13, Remark 2.8] If U is a coatomic subtractive S-semimodule, then every non-maximal subsemimodule is adjacent to at least one maximal subsemimodule in $\Gamma(U)$.

Proof . Let $T \in V(\Gamma(U)) \backslash \max(U)$. So $T \subset M$, for some $M \in \max(U)$. Then clearly, $T + M = M$. So T is adjacent to M. $\square$

For any semimodule U, We mean of $|\max(U)|$ and $|\min(U)|$ are the number of maximal and minimal subsemimodules of U, respectively.

Theorem 2.6. Let U be a subtractive S-semimodule with $\Gamma(U)$ and let N be a minimal subsemimodule of U, such that $\deg(N) < \infty$. If $\Gamma(U)$ is connected, then $|\min(U)| < \infty$.

Proof . Let $\Omega = \{K \leq U \mid K$ be a minimal subsemimodule of $U\}$. Clearly, $\Omega \neq \emptyset$. Since $\Gamma(U)$ is connected, according to [13, Corollary 2.7(2)], for all $K \in \Omega, K + N \neq U$, for N and every $K \in \Omega$ are minimal subsemimodules of U and adjacent vertices of $\Gamma(U)$ with $\deg(N) < \infty$. Thus, $|\Omega| < \infty$, this ends the proof. $\square$

Theorem 2.7. Let U be a Noetherian subtractive S-semimodule. Then $\Gamma(U)$ is complete if and only if U contains a unique maximal subsemimodule.

Proof . Assume that U is a subtractive Noetherian S-semimodule, then $\max(U) \neq \emptyset$. In addition, each nonzero subsemimodule T of U there is $P \in \max(U)$ such that $T \subseteq P$. Hence, if U possesses a unique maximal subsemimodule, say M, then M contains every nonzero subsemimodule of U. Suppose that N and L are two different vertices of $\Gamma(U)$. So $N \subseteq M$ and $L \subseteq M$, hence $N + L \subseteq M \neq U$. Thus, $\Gamma(U)$ is complete.

Conversely, assume that $\Gamma(U)$ is complete. Let $X, Y \in \max(U)$. Then $X + Y \neq U$, since $X \subseteq X + Y$ and $Y \subseteq X + Y$, by maximality of X and Y, we have $X + Y = X = Y$, a contradiction. So, U contains a unique maximal subsemimodule. $\square$

Corollary 2.8. Let U be an S-semimodule. Then $\Gamma(U)$ is complete, if one of the following holds:

(i) if U is an indecomposable S-semimodule, such that $N \cap L = (0)$ for any nontrivial subsemimodules N, L of U.

(ii) if U is a local S-semimodule.

Proof . (i) Clear.
(ii) Since local S-semimodules are hollow, by [13, Theorem 2.12], $\Gamma(U)$ is complete. $\square$

Example 2.9. For all $n \in \mathbb{Z}^+$with $n \geq 2$ and each prime number p, the $\mathbb{Z}$-semimodule $\mathbb{Z}_{p^n}$ is local. By Corollary 2.8, $\Gamma(\mathbb{Z}_{p^n})$ is complete. Also, $\Gamma(\mathbb{Z}_{p^n})$ has $n-1$ vertices (See Figure 1 for $p = 2$ and $n = 5$).

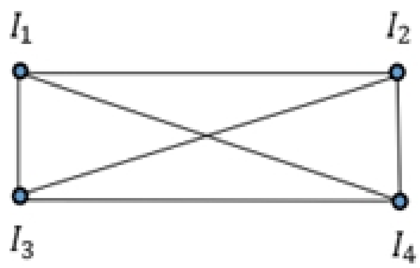

Figure 1. $\Gamma(\mathbb{Z}_{32})$

Example 2.10. Assume $\mathbb{Z}_0 = \mathbb{Z}^+ U\{0\}$ is the semiring of non-negative integers, then the $\mathbb{Z}_0$- semimodule $\mathbb{Z}_0$ is local with maximal subsemimodule $N = \mathbb{Z}_0 \backslash \{1\}$. By Corollary 2.8, $\Gamma(\mathbb{Z}_0)$ is a complete graph.

Example 2.11. Set $S = B(p^h, 0) = \{0, 1, \cdots, p^h - 1\}$, where p is a prime integer and $h \in \mathbb{Z}^+$ and define an operation $\oplus$ on S as follows: If $a, b \in S$ then $a \oplus b = a + b$ if $a + b \leq p^h - 1$ and, otherwise, $a \oplus b$ is the unique element c of S satisfying $c \equiv a + b \,(\bmod p^h)$. Define the an operation $\odot$ on S similarly. Then, $(S, \oplus, \odot)$ is a local semiring [10, Example 6.1, p. 65]. So, by Corollary 2.8, the graph of the S-semimodule S is complete.

3 Cycles in Co-intersection Graphs of semimodules

In this section, the existence of cycles in $\Gamma(U)$ are studied.

Definition 3.1. In any graph, a cycle is a path of length at least 3 through distinct vertices which begins and ends at the same vertex.

Remark 3.2. A cycle of n vertices is denoted by C_n and is called an n-cycle. By (x, y, z) we mean a 3-cycle.

Proposition 3.3. Assume that U be a subtractive semimodule and $\Gamma(U)$ be a connected graph. If U has at least three minimal subsemimodules, then $\Gamma(U)$ contains a cycle.

Proof . Suppose that M_1, M_2 and M_3 are three minimal subsemimodules of U and $\Gamma(U)$ is a connected graph. Then by [13, Corollary 2.7(2)], (M_1, M_2, M_3) is a 3-cycle of $\Gamma(U)$. □

Proposition 3.4. Let U be a subtractive S-semimodule and $\Gamma(U)$ be a connected graph. If U has at least three minimal subsemimodules, then $\Gamma(U)$ is not bipartite graph.

Proof . Suppose M_1, M_2 and M_3 are three minimal subsemimodules of U and $\Gamma(U)$ is a connected graph. Then by [13, Corollary 2.7(2)], (M_1, M_2, M_3) is a 3-cycle of $\Gamma(U)$. It is clear that a simple graph is bipartite if and only if it has no odd cycle. Hence, $\Gamma(U)$ is not bipartite graph. □

Proposition 3.5. Suppose H_1 and H_2 are two adjacent non-maximal subsemimodules of a semimodule U such that $H_1 \nsubseteq H_2 \wedge H_2 \nsubseteq H_1$, then the set $\{H_1, H_2, H_1 + H_2\}$ forms a triangle in $\Gamma(U)$

Proof . It is clear. □

Example 3.6. (1) A graph $\Gamma(\mathbb{Z}_n)$ has a cycle if and only if $n = km$, where k is a positive integer and m is one of the forms: p^4, p^2q or pqr, where p, q and r are different primes.

(2) A graph $\Gamma(\mathbb{Z}_n)$ contains no a cycle if and only if $n = pq, p^2$ or p^3 such that p and q are two different primes. It contains a 3 -cycle, in all other cases.

4 Clique Number and Domination Number of $\Gamma(U)$

Let U be an S-semimodule. In this section, we get some results on the clique number of $\Gamma(U)$. Lastly, it is showed that $\chi(\Gamma(U)) < \infty$, provided $\omega(\Gamma(U)) < \infty$. We determine the domination number in the co-intersection graphs of $\Gamma(U)$. Also, we note here that $\Gamma(U)$ has a minimal dominating set for any S-semimodule U.

Lemma 4.1. Let U be a subtractive S-semimodule such that $1 < \omega(\Gamma(U)) < \infty$. Then $|\min(U)| < \infty$.

Proof . As $\omega(\Gamma) > 1$, by [13, Proposition 3.1(2)] , U is not a direct sum of two simple S semimodules. Then, $\Gamma(U)$ is not joined according to Theorem 2.1. Thus, by [12, Corollary 2.7(2)], $N + L \neq U$ for any $N, L \in \min(U)$. Assume that $\Gamma^\star(U)$ is a subgraph of $\Gamma(U)$ has the vertex set $V^\star = \{L \leq U \mid L$ is minimal subsemimodule of $U\}$. Now $\Gamma^\star(U)$ is a clique in U, and $|V^\star| = \omega(\Gamma^\star(U)) \leq \omega(\Gamma(U)) < \infty$. So, $|\min(U)| < \infty$. □

Theorem 4.2. If $|\Gamma(U)| = \infty$ and $\omega(\Gamma(U)) < \infty$. Then the following hold.

(1) $|\max(U)| = \infty$;

(2) $\chi(\Gamma(U)) < \infty$

Proof . (1) By the contrary way, assume $|\max(U)| < \infty$. Since $|\Gamma(U)| = \infty$, so $\Gamma(U)$ has an infinite clique, a contradiction because $\omega(\Gamma(U)) < \infty$.
(2) If $\omega(\Gamma(U)) = 1$, there is nothing to show. Assume $\omega(\Gamma(U)) > 1$. As, $M + N = U$ for any $M, N \in \max(U)$, so M, N are not two adjacent vertices of $\Gamma(U)$. Now, by Part (1), $|\max(U)| = \infty$. Henceforth, we can color all $N \in \max(U)$ by a color, and other vertices, which are finite number, by a new color, to get a proper vertex coloring of $\Gamma(U)$. So, $\chi(\Gamma(U)) < \infty$. □

Now, we prove that if $\omega(\Gamma(U))$ is infinite, then there is an infinite clique in $\Gamma(U)$.

Proposition 4.3. If $|\Gamma(U)| = \infty$. Then there is an infinite clique of $\Gamma(U)$, if one of the following holds.

(1) The subsemimodules of U form a chain.

(2) The semimodule U is hollow or local.

Proof .

(1) Use Remark 2.11 in [13].

(2) Use [13, Theorem 2.12] and part 2 of Corollary 2.8.

□

Example 4.4. For every prime number p, we consider the graph $\Gamma\left(\mathbb{Z}_{p^\infty}\right)$. Since the $\mathbb{Z}$-semimodule $\mathbb{Z}_{p^\infty}$ is hollow, by Proposition 4.3 (2), $\Gamma\left(\mathbb{Z}_{p^\infty}\right)$ contains an infinite clique.

Example 4.5. Any induced subgraph on the set of any finitely generated subsemimodules of the $\mathbb{Z}$-semimodule $\mathbb{Q}$ is clique in $\Gamma(\mathbb{Q})$. Toward realize this, suppose that $N \subseteq \mathbb{Q}$ and N is finitely generated. Let $N =< x_1, x_2, \cdots, x_n >$, where $x_i \in \mathbb{Q}$, for $1 \leq i \leq n$. Hence $N = x_1\mathbb{Z} + x_2\mathbb{Z} + \cdots + x_n\mathbb{Z}$. Note, for $L \subseteq \mathbb{Q}$, if $N + L = \mathbb{Q}$, then as N has a spanning set $\{x_1, x_2, \cdots, x_n\}$, thus $\{x_1, x_2, \cdots, x_n\} \cup L$ is a generated set of $\mathbb{Q}$ and it can be possible if L is a generated set of $\mathbb{Q}$. Hence $L = \mathbb{Q}$. So, all finitely generated subsemimodule of $\mathbb{Q}$ as a $\mathbb{Z}$ semimodule, is a small subsemimodule. Thus it obtains from [13, Theorem 2.12].

Definition 4.6. A graph Γ is called weakly perfect graph if $\omega(\Gamma) = \chi(\Gamma)$.

At present we provide particular examples of co-intersection graphs of non-semisimple semimodules.

Example 4.7. Let p and q be two different prime numbers. Clearly, $\mathbb{Z}_{pq^3}$ as a $\mathbb{Z}$-semimodule is not semisimple and $\Gamma\left(\mathbb{Z}_{pq^3}\right)$ is a weakly perfect (See Figure 2).

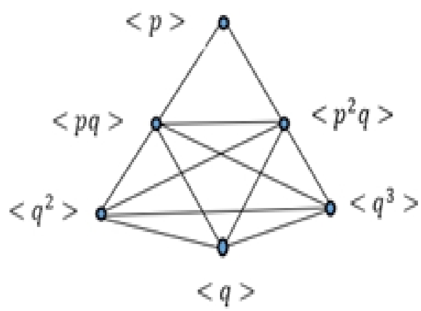

Figure 2. $\Gamma(\mathbb{Z}_{pq^3})$

Example 4.8. We consider $\mathbb{Z}_8$ as $\mathbb{Z}$-semimodule. The non-trivial subsemimodules of $\mathbb{Z}_8$ are $L = \{\overline{0}, \overline{4}\}$ and $N = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}\}$. Since $L + N = N \neq \mathbb{Z}_8$, L and N are adjacent vertices in $\Gamma\left(\mathbb{Z}_8\right)$. Hence, $\Gamma\left(\mathbb{Z}_8\right) \cong K_2$, then $\omega(\Gamma(U)) = \chi(\Gamma(U)) = 2$.

Definition 4.9. Let Γ be a graph. By a DS (dominating set) for Γ we mean a subset D of $V(\Gamma(U))$ such that each vertex not in D is joined to at least one vertex in D by some edge. A dominating set DS is named a minimal dominating set if D' is not a dominating set for every subset D' of D with $D' \neq D$. The smallest of the cardinalities of the minimal dominating sets for Γ, indicated by $\gamma(\Gamma)$ is named the domination number of Γ. An infinite graph may not have a minimal dominating set, in which case, the domination number is not defined. For references on Domination Theory, see [14].

Remark 4.10. In our case, a set D of non-trivial S-subsemimodules of U is a dominating set for $\Gamma(U)$ if and only if for any non-trivial S-subsemimodule X of U there is a Y in D such that $X + Y \neq U$.

Definition 4.11. If a vertex u is adjacent to any vertex in $\Gamma(U)$ then u is called universal.

Lemma 4.12. Let U be an S-semimodule and N a non-trivial subsemimodule of U. Then N is a universal vertex in $\Gamma(U)$ if and only if $N \ll U$.

Proof . Clear. □

We begin by the next obvious lemma.

Lemma 4.13. For a semimodule U with $|\Gamma(U)| \geq 2$, then the next hold:

(1) If a subset D of $V(\Gamma(U))$ which contains a universal vertex, then D is a DS in $\Gamma(U)$.

(2) If $\Gamma(U)$ contains a universal vertex, then for each universal vertex M of $\Gamma(U), \{M\}$ is a minimal dominating set and $\gamma(\Gamma(U)) = 1$.

Proof . This is clear. □

Example 4.14. Consider $\mathbb{Q}$ as $\mathbb{Z}$-semimodule and $\mathbb{Z}_{pq^2}$ as $\mathbb{Z}_{pq^2}$-semimodule such that p and q are two distinct prime numbers. Since $\mathbb{Z} \ll \mathbb{Q}$ and $< pq > \ll \mathbb{Z}_{pq^2}$, it is not difficult to see that $\mathbb{Z}$ is a universal vertex of $\Gamma(\mathbb{Q})$ and $< pq >$ is a universal vertex of $\Gamma\left(\mathbb{Z}_{pq^2}\right)$. Then by Lemma 4.12, we conclude that:

(1) $\{< pq >\}$ is a minimal dominating set and $\gamma\left(\Gamma\left(\mathbb{Z}_{pq^2}\right)\right) = 1$.

(2) $\{\mathbb{Z}\}$ is a minimal dominating set and $\gamma(\Gamma(\mathbb{Q})) = 1$.

Corollary 4.15. Let U be a subtractive semimodule over a semiring S and $\mathrm{Rad}(U) \neq (0)$. Then the next situations hold:

(1) $\{\mathrm{Rad}(U)\}$ is a minimal dominating set if U is finitely generated.

(2) The graph $\Gamma(U)$ has a minimal dominating set if every non-trivial subsemimodule of U is contained in a maximal subsemimodule.

Proof .

(1) Let U be a finitely generated subtractive semimodule. So, $\mathrm{Rad}(U) \ll U$. Since $\mathrm{Rad}(U) \neq (0)$, $\mathrm{Rad}(U)$ is a vertex of $\Gamma(U)$. Thus $\mathrm{Rad}(U)$ is adjacent to every other vertex of $\Gamma(U)$. So, $\{\mathrm{Rad}(U)\}$ is a dominating set.

(2) From the assumption, we have $\mathrm{Rad}(U) \ll U$. From part (1), $\{\mathrm{Rad}(U)\}$ is a dominating set.

□

Theorem 4.16. Let U be a subtractive S-semimodule with $\mathrm{Rad}(U) \neq (0)$ and $|\Gamma(U)| \geq 2$. If U is hollow, then the next hold:

(1) Every subset of $V(\Gamma(U))$ is a dominating set in $\Gamma(U)$.

(2) $\gamma(\Gamma(U)) = 1$

(3) If $\Gamma(U)$ is a finite graph, then the number of the dominating set is equal to $2^{|\Gamma(U)|} - 2$.

(4) If $\Gamma(U)$ is an infinite graph, then the number of the dominating set is infinite.

Proof . (1) Assume the semimodule U is hollow. By [13, Theorem 2.12], $\Gamma(U)$ is complete. Thus, every subset of $V(\Gamma(U))$ is a DS in $\Gamma(U)$.

(2) Since the semimodule U is hollow, $L \ll U$ for every non-trivial subsemimodule L of U. Henceforth, by Lemmas 4.12 and 4.13, $\gamma(\Gamma(U)) = 1$.

(3) Let $|\Gamma(U)| = n$, where $2 \leq n < \infty$. As $\Gamma(U)$ is a complete graph with n vertices, then the number of non-empty proper subsets of the vertex set $V(\Gamma(U))$, which are DS, is equal to $\sum_{t=1}^{k-1} C(k,t) = 2^k - 2$, where $C(k,t)$ is an t-combination of $V(\Gamma(U))$ with k elements, and for a positive integer $t \leq k$.

(4) It follows from (3). □

Example 4.17. For each prime number p and $n \in \mathbb{Z}, n \geq 2$, we get:

(1) $\gamma\left(\Gamma\left(\mathbb{Z}_{p^n}\right)\right) = \gamma\left(\Gamma\left(\mathbb{Z}_{p^\infty}\right)\right) = 1$

(2) The number of the dominating set of $\Gamma\left(\mathbb{Z}_{p^n}\right)$ is $2^{n-1} - 2$.

(3) The number of the dominating set of $\Gamma\left(\mathbb{Z}_{p^\infty}\right)$ is infinite.

Corollary 4.18. Let U be a subtractive S-semimodule. Then each $D \subsetneq V(\Gamma(U))$ is a dominating set of $\Gamma(U)$ and $\gamma(\Gamma(U)) = 1$, if one of the next statements holds:

(a) The semimodule U contains a unique maximal subsemimodule and it is Noetherian.

(b) The semimodule U is an indecomposable semimodule such that $N \cap L = (0)$ for any non-trivial subsemimodules N and L of U.

(c) U is local semimodule.

Proof . (a) It follows from Theorem 2.7 and Lemmas 4.12 and 4.13.

(b) By Corollary 2.8.

(c) It follows from Corollary 2.8 and Theorem 4.16, since every local semimodule is hollow as in [15]. □

Acknowledgments

The authors are deeply grateful to the referees for careful reading of the manuscript and helpful suggestions.

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