Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 2175–2183
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.22306.2348

On the comparative growth analysis of solutions of complex linear differential equations with entire and meromorphic coefficients of $[p, q] - \varphi$ order

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(Communicated by Mugur Alexandru Acu)

Abstract

Let φ be a non-decreasing unbounded function and p, q be any two positive integers with $p \geq q \geq 1$. The relations between the growth of entire or meromorphic coefficients and the growth of entire or meromorphic solutions of general complex linear differential equation with entire or meromorphic coefficients of finite $[p, q]$-φ order are investigated in this paper. Improving and extending some earlier results of J. Liu, J. Tu, L.Z. Shi, L.M. Li, T.B. Cao and others, we obtain some more results here.

Keywords: Entire function, Meromorphic function, $[p, q] - \varphi$ order, $[p, q] - \varphi$ exponent of convergence, Linear differential equations
2020 MSC: 34M10, 30D35

1 Introduction and preliminaries

We do not explain in details the value distribution theory of entire and meromorphic functions and the theory of complex differential equations as those are available in [15], [9] and [11]. In addition let us recall some notations such as $m(r, f)$ and $N(r, f)$. Let $n(r, f)$ be the number of poles of a function f (counting multiplicities) in $|z| \leq r$. Then we define the integrated counting function $N(r, f)$ by

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r,$$

and we define the proximity function $m(r, f)$ by

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(re^{i\phi}\right) \right| d\phi,$$

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Received: January 2021 *Accepted:* July 2021

where $\log^+ x = \max\{0, \log x\}$. We should think of $m(r,f)$ as a measure that how close f is to infinity on $|z| = r$. Nevertheless, within that context, we recall that $T(r,f)$ stands for the Nevanlinna characteristic function of the meromorphic function f defined on each positive real value r by

$$T(r,f) = m(r,f) + N(r,f).$$

And $M(r,f)$ stands for the so called maximum modulus function defined for each non-negative real value r by

$$M(r,f) = \max_{|z|=r} |f(z)|.$$

In order to describe the growth of order of entire or meromorphic functions more precisely, we use some notations for $r \in [0,\infty)$, $\exp_1 r = e^r$ and $\exp_{p+1} r = \exp\left(\exp_p r\right)$, $p \in \mathbb{N}$. For all sufficiently large r, we define $\log_1 r = \log r$ and $\log_{p+1} r = \log\left(\log_p r\right)$, $p \in \mathbb{N}$. Also $\exp_0 r = r = \log_0 r$ and $\exp_{-1} r = \log_1 r$ and $\log_{-1} r = \exp_1 r$. Moreover, we denote the linear measure for a set $E \subset [0,\infty)$, by $mE = \int_E dt$ and logarithmic measure for a set $E \subset (1,\infty)$, by $m_l E = \int_E \frac{dt}{t}$.

In last few decades, the properties of meromorphic solutions of complex differential equations have become a subject of great interest from the viewpoint of Nevanlinna's theory and its difference analogues. Recently, X. Shen, J. Tu and H. Y. Xu [14] introduced the new concept of $[p,q]-\varphi$ order of meromorphic functions in the complex plane to study the growth and zeros of second order linear differential equations, where p,q are positive integers satisfying $p \geq q \geq 1$, since then, many authors investigated the growth properties of solutions of linear differential equations using the concept of $\varphi-$ order ([2, 5, 8]). In this connection, we recall the following definitions:

Definition 1.1. ([14])Let $\varphi : [0,\infty) \to (0,\infty)$ be a non-decreasing unbounded function. Then $[p,q]-\varphi$ order and $[p,q]-\varphi$ lower order of a meromorphic function f are defined by

$$\sigma_{[p,q]}(f,\varphi) = \overline{\lim_{r\to\infty}} \frac{\log_p T(r,f)}{\log_q \varphi(r)},$$
$$\mu_{[p,q]}(f,\varphi) = \underline{\lim_{r\to\infty}} \frac{\log_p T(r,f)}{\log_q \varphi(r)}.$$

Definition 1.2. ([3]) Let f be a meromorphic function satisfying $0 < \sigma_{[p,q]}(f,\varphi) = \sigma < \infty$. Then the $[p,q]-\varphi$ type of f is defined by

$$\tau_{[p,q]}(f,\varphi) = \overline{\lim_{r\to\infty}} \frac{\log_{p-1} T(r,f)}{\left[\log_{q-1} \varphi(r)\right]^\sigma}.$$

Definition 1.3. ([14]) Let f be a meromorphic function. Then the $[p,q]-\varphi$ exponent of convergence of zero-sequence (distinct zero-sequence) of f is defined by

$$\lambda_{[p,q]}(f,\varphi) = \overline{\lim_{r\to\infty}} \frac{\log_p n\left(r,\frac{1}{f}\right)}{\log_q \varphi(r)},$$
$$\overline{\lambda}_{[p,q]}(f,\varphi) = \overline{\lim_{r\to\infty}} \frac{\log_p \overline{n}\left(r,\frac{1}{f}\right)}{\log_q \varphi(r)}.$$

Remark 1.4. If $\varphi(r) = r$ in the definitions 1.1-1.3, then we obtain the standard definitions of the $[p,q]-$order, $[p,q]-$type and $[p,q]-$exponent of convergence.

Remark 1.5. ([14]) Throughout this paper, we assume that $\varphi : [0,\infty) \to (0,\infty)$ is a non-decreasing unbounded function and always satisfies the following two conditions:

(i) $\lim_{r\to+\infty} \frac{\log_{p+1} r}{\log_q \varphi(r)} = 0.$

(ii) $\lim_{r\to+\infty} \frac{\log_q \varphi(\alpha r)}{\log_q \varphi(r)} = 1$ for some $\alpha > 1$.

The theory of complex linear differential equations (see [11]) has been developed since 1960's. Many authors has been investigated the complex linear differential equations for $k \geq 2$

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + ... + A_1(z) f' + A_0(z) f = 0 \tag{1.1}$$

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + ... + A_1(z) f' + A_0(z) f = F(z) \tag{1.2}$$

and obtained many results when the coefficients $A_0(z)$, $A_1(z)$, ... , $A_{k-1}(z)$ and $F(z) (\not\equiv 0)$ are entire or meromorphic functions.(see [1, 4, 7, 12, 16].). When the coefficients in (1.1) or(1.2) are entire functions, Several interesting and important results are recalled in the following theorems.

Theorem A. [13] Let $A_j(z)$ $(j = 0, 1, \ ... \ , \ k-1)$ be entire functions satisfying

$$\max\left\{\sigma_{[p,q]}(A_j) \mid j \neq 0\right\} < \sigma_{[p,q]}(A_0) < \infty,$$

then every nontrivial solution $f(z)$ of (1.1) satisfies

$$\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0).$$

Theorem B. [13] Let $A_j(z)$ $(j = 0, 1, \ ... \ , \ k-1)$ be entire functions satisfying

$$\max\left\{\sigma_{[p,q]}(A_j) \mid j \neq 0\right\} \leq \sigma_{[p,q]}(A_0) < \infty$$

and

$$\max\left\{\tau_{[p,q]}(A_j) \mid \sigma_{[p,q]}(A_j) = \sigma_{[p,q]}(A_0) > 0\right\} < \tau_{[p,q]}(A_0),$$

then every nontrivial solution $f(z)$ of (1.1) satisfies

$$\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0).$$

When the coefficients in (1.1) or (1.2) are meromorphic functions, we have:

Theorem C. [12] Let $A_j(z)$ $(j = 0, 1, \ ... \ , \ k-1)$, $F(z) (\not\equiv 0)$ be meromorphic functions, let $f(z)$ be a meromorphic solution of equation (1.2) satisfying

$$\max\left\{\sigma_{[p+1,q]}(A_j), \sigma_{[p+1,q]}(F) \mid j = 0, 1, \ ... \ , \ k-1\right\} < \sigma_{[p+1,q]}(f),$$

then we have

$$\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \sigma_{[p+1,q]}(f).$$

Here the following natural question is occurred: When the coefficients of the equations (1.1) and (1.2) are entire or meromorphic functions of $[p,q]-\varphi$ order, what would the growth properties of solutions of the linear differential equations (1.1) and (1.2) be like? In this paper, for an answer to this question we obtain the following results which are improvement and extension of the previous results.

2 Main Results

Here we present our main results:

Theorem 2.1. Let p, q be positive integers such that $p \geq q \geq 1$, and let $A_0(z)$, $A_1(z)$, ... , $A_{k-1}(z)$ be entire functions satisfying $\sigma_{[p,q]}(A_0, \varphi) = \sigma_1$ and $\overline{\lim_{r\to\infty}} \sum_{j=1}^{k-1} \frac{m(r,A_j)}{m(r,A_0)} < 1$, and where φ satisfies the conditions $\lim_{r\to+\infty} \frac{\log_{p+1} r}{\log_q \varphi(r)} = 0$ and $\lim_{r\to+\infty} \frac{\log_q \varphi(\alpha r)}{\log_q \varphi(r)} = 1$ for some $\alpha > 1$, then every nontrivial solution $f(z)$ of (1.1) satisfies

$$\sigma_{[p+1,q]}(f, \varphi) = \sigma_{[p,q]}(A_0, \varphi) = \sigma_1.$$

Theorem 2.2. Let p, q be positive integers such that $p \geq q \geq 1$, and let $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ be entire functions and $A_0(z)$ be transcendental function satisfying

$$\max\left\{\sigma_{[p,q]}(A_j,\varphi) \mid j \neq 0\right\} \leq \mu_{[p,q]}(A_0,\varphi) = \sigma_{[p,q]}(A_0,\varphi),$$

and $\underline{\lim}_{r\to\infty} \sum_{j=1}^{k-1} \frac{m(r,A_j)}{m(r,A_0)} < 1$ $(r \notin E_1)$, where E_1 is a set of r of finite linear measure and where φ satisfies the conditions $\lim_{r\to+\infty} \frac{\log_{p+1} r}{\log_q \varphi(r)} = 0$ and $\lim_{r\to+\infty} \frac{\log_q \varphi(\alpha r)}{\log_q \varphi(r)} = 1$ for some $\alpha > 1$, then every nontrivial solution f of (1.1) satisfies

$$\sigma_{[p+1,q]}(f,\varphi) = \mu_{[p,q]}(A_0,\varphi) = \sigma_{[p,q]}(A_0,\varphi).$$

Remark 2.3. In the Theorem A and Theorem B and our Theorem 2.1, the authors investigated the growth of the solutions of (1.1) under the same case that the coefficient $A_0(z)$ in (1.1) grows faster than other coefficients $A_j(z)$ $(j = 1, 2, \ldots, k-1)$ and obtain the same conclusion. We have to note that the condition $\max\left\{\sigma_{[p,q]}(A_j) \mid j \neq 0\right\} < \sigma_{[p,q]}(A_0)$ in the Theorem A and the conditions

$$\max\left\{\sigma_{[p,q]}(A_j) \mid j \neq 0\right\} \leq \sigma_{[p,q]}(A_0) < \infty$$

and

$$\max\left\{\tau_{[p,q]}(A_j) \mid \sigma_{[p,q]}(A_j) = \sigma_{[p,q]}(A_0) > 0\right\} < \tau_{[p,q]}(A_0)$$

in The Theorem B are stronger than our condition $\overline{\lim}_{r\to\infty} \sum_{j=1}^{k-1} \frac{m(r,A_j)}{m(r,A_0)} < 1$ in Theorem 2.1. Thus our Theorem 2.1 is an improvement of Theorem A and the Theorem B. In Theorem 2.2 we replace the condition $\overline{\lim}_{r\to\infty} \sum_{j=1}^{k-1} \frac{m(r,A_j)}{m(r,A_0)} < 1$ with $\underline{\lim}_{r\to\infty} \sum_{j=1}^{k-1} \frac{m(r,A_j)}{m(r,A_0)} < 1$ and $\mu_{[p,q]}(A_0,\varphi) = \sigma_{[p,q]}(A_0,\varphi)$ and we get the same conclusion as Theorem 2.1, therefore Theorem 2.2 is supplement of Theorem 2.1.

Theorem 2.4. Let p, q be positive integers such that $p \geq q \geq 1$, and let $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ and $F(\not\equiv 0)$ be meromorphic functions. If $f(z)$ is a meromorphic solution of (1.2) satisfying

$$\overline{\lim_{r\to\infty}} \frac{\sum_{j=0}^{k-1} T(r,A_j) + T(r,F)}{T(r,f)} < 1,$$

then

$$\overline{\lambda}_{[p+1,q]}(f,\varphi) = \lambda_{[p+1,q]}(f,\varphi) = \sigma_{[p+1,q]}(f,\varphi).$$

where $\varphi(z)$ satisfies the conditions (i) and (ii) of Remark 1.5.

3 Preliminary Lemmas

Some lemmas which are needed for the theorems:

Lemma 3.1. [11] Let $g : (0,\infty) \to \mathbb{R}$ and $h : (0,\infty) \to \mathbb{R}$ be monotone increasing functions such that

1) $g(r) \leq h(r)$ outside of an exceptional set E_2 of finite linear measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.

2) $g(r) \leq h(r)$ outside of an exceptional set E_2 of finite logarithmic measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(r^\alpha)$ for all $r > r_0$.

Lemma 3.2. [11] Let $f(z)$ be a transcendental entire function, and let z be a point with $|z| = r$ at which $|f(z)| = M(r,f)$. Then for all $|z|$ outside a set E_3 of r of finite logarithmic measure, we have

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^n (1 + o(1)) \qquad (n \in \mathbb{N},\ r \notin E_3)$$

where $\nu_f(r)$ is the central index of f.

Lemma 3.3. ([3]) Let p, q be positive integers such that $p \geq q \geq 1$, and let $f(z)$ be a meromorphic function satisfying $\sigma_{[p,q]}(f,\varphi) = \sigma < \infty$, where $\varphi(r)$ only satisfies $\lim\limits_{r\to+\infty} \frac{\log_{q-1}\varphi(\alpha r)}{\log_q \varphi(r)} = 1$ for some $\alpha > 1$, then there exists a set $E_4 \subset (1,\infty)$ having infinite logarithmic measure such that for all $r \in E_4$, we have

$$\lim_{r\to\infty} \frac{\log_p T(r,f)}{\log_q \varphi(r)} = \sigma_{[p,q]}(f,\varphi) = \sigma.$$

Lemma 3.4. [10] Let $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ be entire coefficients in (1.1), and at least one of them is transcendental. If $A_s(z)$ $(s \in \{0,1,\ldots,k-1\})$ is the first one (according to the sequence of $A_0(z), A_1(z), \ldots, A_{k-1}(z)$) satisfying $\varliminf\limits_{r\to\infty} \sum\limits_{j=s+1}^{k-1} \frac{m(r,A_j)}{m(r,A_s)} < 1$ $(r \notin E_5)$, where $E_5 \subset (1,\infty)$ is a set having finite linear measure, then (2.1) possesses at most s linearly independent entire solutions satisfying $\varlimsup\limits_{r\to\infty} \frac{\log T(r,f)}{m(r,A_s)} = 0$ $(r \notin E_5)$.

Lemma 3.5. ([14]) Let $f(z)$ be an entire function of $[p,q]-\varphi$ order, and let $\nu_f(r)$ be the central index of $f(z)$, then

$$\varlimsup_{r\to\infty} \frac{\log_p \nu_f(r)}{\log_q \varphi(r)} = \sigma_{[p,q]}(f,\varphi).$$

4 Proof of Main Results

Proof .[Proof of Theorem 2.1] From the Equation (1.1), we get

$$-A_0(z) = \frac{f^{(k)}(z)}{f(z)} + \sum_{j=1}^{k-1} A_j(z) \frac{f^{(j)}(z)}{f(z)}. \tag{4.1}$$

By the Lemma of the logarithmic derivative and (4.1), we have

$$m(r,A_0) \leq \sum_{j=1}^{k-1} m(r,A_j) + O\{\log rT(r,f)\} \qquad (r \notin E), \tag{4.2}$$

where E is a set of r of finite linear measure.

Assume that

$$\varlimsup_{r\to\infty} \sum_{j=1}^{k-1} \frac{m(r,A_j)}{m(r,A_0)} = \alpha < \beta < 1,$$

then for sufficiently large r, we get

$$\sum_{j=1}^{k-1} m(r,A_j) < \beta m(r,A_0), \tag{4.3}$$

where $\beta \in (0,1)$.

By (4.2) and (4.3), we get

$$(1-\beta) m(r,A_0) \leq O\{\log rT(r,f)\} \qquad (r \notin E). \tag{4.4}$$

By Lemma 3.3 and $\sigma_{[p,q]}(A_0,\varphi) = \sigma_1$, there exists a set $E_4 \subset (1,\infty)$ of r of infinite logarithmic measure such that for all z satisfying $|z| = r \in E_4 \backslash E$ and for any $\varepsilon > 0$, we have

$$(1-\beta)\exp_{p-1}\left\{\left(\log_{q-1}\varphi(r)\right)^{\sigma_1-\varepsilon}\right\} \leq (1-\beta) m(r,A_0) \leq O\{\log rT(r,f)\} \qquad (r \notin E). \tag{4.5}$$

Therefore from (4.5) and Lemma 3.1, we obtain

$$\sigma_{[p+1,q]}(f,\varphi) \geq \sigma_{[p,q]}(A_0,\varphi) = \sigma_1. \tag{4.6}$$

On the other hand from the equation (1.1), we have

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq \sum_{j=1}^{k-1} |A_j(z)| \left|\frac{f^{(j)}(z)}{f(z)}\right| + |A_0(z)|. \tag{4.7}$$

By the Lemma 3.2, there exists a set E_3 of r of finite logarithmic measure such that for all z satisfying $|z| = r \notin E_3$ and $|f(z)| = M(r, f)$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^j (1 + o(1)) \qquad (j = 1, 2, \ \dots\ , \ k). \tag{4.8}$$

By (4.3) and the fact $\sigma_{[p,q]}(A_0, \varphi) = \sigma_1$, we have

$$\sigma_{[p,q]}(A_j, \varphi) < \sigma_{[p,q]}(A_0, \varphi) = \sigma_1 \quad (j = 0, 1, \ \dots\ , \ k-1),$$

$$|A_j(z)| \leq \exp_p\left\{\left(\log_{q-1}\varphi(r)\right)^{\sigma_{[p,q]}(A_0,\varphi)+\varepsilon}\right\} \qquad (j = 0, 1, \ \dots\ , \ k-1), \tag{4.9}$$

Again by definitions of $[p, q] - \varphi$ order, we have

$$|A_0(z)| \leq \exp_p\left\{\left(\log_{q-1}\varphi(r)\right)^{\sigma_{[p,q]}(A_0,\varphi)+\varepsilon}\right\} \tag{4.10}$$

Substitute (4.8), (4.9) and (4.10) into (4.7), we obtain

$$\begin{aligned}\left(\frac{|\nu_f(r)|}{r}\right)^k |1 + o(1)| &\leq k \exp_p\left\{\left(\log_{q-1}\varphi(r)\right)^{\sigma_{[p,q]}(A_0,\varphi)+\varepsilon}\right\} \left(\frac{|\nu_f(r)|}{r}\right)^{k-1} |1 + o(1)| \\ \nu_f(r) &\leq k \exp_p\left\{\left(\log_{q-1}\varphi(r)\right)^{\sigma_{[p,q]}(A_0,\varphi)+\varepsilon}\right\} \end{aligned} \tag{4.1}$$

By Lemma 3.1 and (4.1), we get

$$\overline{\lim_{r\to\infty}} \frac{\log_{p+1}\nu_f(r)}{\log_q \varphi(r)} \leq \sigma_{[p,q]}(A_0, \varphi) + \varepsilon \tag{4.12}$$

since $\varepsilon > 0$ is arbitrary, by Lemma 3.5 and (4.12), we have

$$\sigma_{[p+1,q]}(f, \varphi) \leq \sigma_{[p,q]}(A_0, \varphi) = \sigma_1. \tag{4.13}$$

From (4.6) and (4.13), we obtain

$$\sigma_{[p+1,q]}(f, \varphi) = \sigma_{[p,q]}(A_0, \varphi) = \sigma_1.$$

This completes the proof of the theorem. □

Proof .[Proof of Theorem 2.2] By Lemma 3.4, we obtain that every linearly independent solution $f(z)$ of the equation (1.1) satisfying $\overline{\lim_{r\to\infty}} \frac{\log T(r,f)}{m(r,A_0)} > 0$ $(r \notin E_5)$. This implies that every solution $f(z)$ $(\not\equiv 0)$ of the equation (1.1) satisfying $\overline{\lim_{r\to\infty}} \frac{\log T(r,f)}{m(r,A_0)} > 0$ $(r \notin E_5)$, then there exists a $\delta > 0$ and a sequence $\{r_n\}_{n=1}^{\infty}$ tending to infinity such that for sufficiently large $r_n \notin E_5$ and for every solution $f(z)$ $(\not\equiv 0)$ of the equation (1.1) we have

$$\log T(r_n, f) > \delta m(r_n, A_0). \tag{4.14}$$

Again by Lemma 3.3, there exists a set $E_4 \subset (1, \infty)$ of r of infinite logarithmic measure such that for all z satisfying $|z| = r \in E_4 \backslash E_5$ and for any $\varepsilon > 0$, we have

$$\delta \exp_{p-1}\left\{\left(\log_{q-1}\varphi(r_n)\right)^{\sigma_{[p,q]}(A_0,\varphi)-\varepsilon}\right\} \leq \delta m(r_n, A_0) \tag{4.15}$$

From (4.14) and (4.15), we get

$$\delta \exp_{p-1}\left\{\left(\log_{q-1}\varphi(r_n)\right)^{\sigma_{[p,q]}(A_0,\varphi)-\varepsilon}\right\} \leq \delta m(r_n, A_0) < \log T(r_n, f) \tag{4.16}$$

By Lemma 3.1 and (4.16), we have

$$\sigma_{[p+1,q]}(f, \varphi) \geq \sigma_{[p,q]}(A_0, \varphi)$$

Since $\mu_{[p,q]}(A_0,\varphi)=\sigma_{[p,q]}(A_0,\varphi)$, we have

$$\sigma_{[p+1,q]}(f,\varphi)\geq\sigma_{[p,q]}(A_0,\varphi)=\mu_{[p,q]}(A_0,\varphi) \tag{4.17}$$

On the other hand from the equation (1.1), we have

$$\left|\frac{f^{(k)}(z)}{f(z)}\right|\leq\sum_{j=1}^{k-1}|A_j(z)|\left|\frac{f^{(j)}(z)}{f(z)}\right|+|A_0(z)|. \tag{4.18}$$

Since $\max\left\{\sigma_{[p,q]}(A_j,\varphi)\mid j\neq 0\right\}\leq\mu_{[p,q]}(A_0,\varphi)=\sigma_{[p,q]}(A_0,\varphi)$, for sufficiently large r and for any given $\varepsilon>0$, we have

$$|A_j(z)|\leq\exp_p\left\{\left(\log_{q-1}\varphi(r)\right)^{\sigma_{[p,q]}(A_0,\varphi)+\varepsilon}\right\}\qquad(j=1,2,\ \dots\ ,\ k-1) \tag{4.19}$$

Again by definitions of $[p,q]-\varphi$ order, we have

$$|A_0(z)|\leq\exp_p\left\{\left(\log_{q-1}\varphi(r)\right)^{\sigma_{[p,q]}(A_0,\varphi)+\varepsilon}\right\} \tag{4.20}$$

By Lemma 3.2, there exists a set E_3 of r of finite logarithmic measure such that for all z satisfying $|z|=r\notin E_3$ and $|f(z)|=M(r,f)$, we have

$$\left(\frac{f^{(j)}(z)}{f(z)}\right)=\left(\frac{\nu_f(r)}{z}\right)^j(1+o(1))\qquad(j=1,2,\ \dots\ ,\ k). \tag{4.21}$$

Substitute (4.19), (4.20) and (4.21) into (4.18), we get

$$\begin{aligned}\left(\frac{|\nu_f(r)|}{r}\right)^k|1+o(1)| &\leq k\exp_p\left\{\left(\log_{q-1}\varphi(r)\right)^{\sigma_{[p,q]}(A_0,\varphi)+\varepsilon}\right\}\left(\frac{|\nu_f(r)|}{r}\right)^{k-1}|1+o(1)|\\ \nu_f(r) &\leq k\exp_p\left\{\left(\log_{q-1}\varphi(r)\right)^{\sigma_{[p,q]}(A_0,\varphi)+\varepsilon}\right\}.\end{aligned} \tag{4.2}$$

By Lemma 3.1 and (4.2), we get

$$\overline{\lim_{r\to\infty}}\frac{\log_{p+1}\nu_f(r)}{\log_q\varphi(r)}\leq\sigma_{[p,q]}(A_0,\varphi)+\varepsilon. \tag{4.23}$$

since $\varepsilon>0$ is arbitrary, by Lemma 3.5 and (4.23), we have

$$\sigma_{[p+1,q]}(f,\varphi)\leq\sigma_{[p,q]}(A_0,\varphi).$$

Since $\mu_{[p,q]}(A_0,\varphi)=\sigma_{[p,q]}(A_0,\varphi)$, we have

$$\sigma_{[p+1,q]}(f,\varphi)\leq\sigma_{[p,q]}(A_0,\varphi)=\mu_{[p,q]}(A_0,\varphi). \tag{4.24}$$

From (4.17) and (4.24), we obtain

$$\sigma_{[p+1,q]}(f,\varphi)=\sigma_{[p,q]}(A_0,\varphi)=\mu_{[p,q]}(A_0,\varphi).$$

This completes the proof of the theorem. □

Proof .[Proof of Theorem 2.3] From the equation (1.2), we get

$$\frac{1}{f}=\frac{1}{F}\left(\frac{f^{(k)}(z)}{f(z)}+\sum_{j=1}^{k-1}A_j(z)\frac{f^{(j)}(z)}{f(z)}+A_0(z)\right). \tag{4.25}$$

It is easy to see that if f has a zero at z_0 of order $\beta\ (\beta>k)$, and if $A_0(z),A_1(z),\ \dots\ ,A_k(z)$ are all analytic at z_0, then F must have a zero at z_0 of order $\beta-k$. Hence

$$N\left(r,\frac{1}{f}\right)\leq k\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{F}\right)+\sum_{j=0}^{k-1}N(r,A_j). \tag{4.26}$$

By theorem on logarithmic derivative and (4.25), we have

$$m\left(r,\frac{1}{f}\right)\leq m\left(r,\frac{1}{F}\right)+\sum_{j=0}^{k-1}m\left(r,A_j\right)+O\left(\log rT\left(r,f\right)\right), \tag{4.27}$$

holds for all z satisfying $|z|=r\notin E$, where E is a set of r of finite linear measure.

From (4.26) and (4.27), we get

$$\begin{aligned} T\left(r,f\right) &= T\left(r,\frac{1}{f}\right)+O\left(1\right)=N\left(r,\frac{1}{f}\right)+m\left(r,\frac{1}{f}\right)+O\left(1\right) \\ &\leq k\overline{N}\left(r,\frac{1}{f}\right)+T\left(r,F\right)+\sum_{j=0}^{k-1}T\left(r,A_j\right)+O\left(\log rT\left(r,f\right)\right). \end{aligned} \tag{4.3}$$

holds for all z satisfying $|z|=r\notin E$, where E is a set of r of finite linear measure.

Suppose that

$$\varlimsup_{r\to\infty}\frac{\sum\limits_{j=0}^{k-1}T\left(r,A_j\right)+T\left(r,F\right)}{T\left(r,f\right)}=\delta<c<1$$

Thus for sufficiently large r and for any given $\varepsilon\left(0<\varepsilon<c-\delta\right)$, we have

$$\sum_{j=0}^{k-1}T\left(r,A_j\right)+T\left(r,F\right)\leq\left(\delta+\varepsilon\right)T\left(r,f\right)<cT\left(r,f\right) \tag{4.29}$$

Substitute (4.29) into (4.3), we obtain

$$T\left(r,f\right)\leq\frac{k}{1-c-\varepsilon}\overline{N}\left(r,\frac{1}{f}\right)\leq\frac{2k}{1-c}\overline{N}\left(r,\frac{1}{f}\right)\qquad\left(r\notin E\right) \tag{4.30}$$

By Lemma 3.1 and (4.30), we obtain

$$\overline{\lambda}_{[p+1,q]}\left(f,\varphi\right)\geq\sigma_{[p+1,q]}\left(f,\varphi\right)$$

Again by definitions we get

$$\overline{\lambda}_{[p+1,q]}\left(f,\varphi\right)\leq\lambda_{[p+1,q]}\left(f,\varphi\right)\leq\sigma_{[p+1,q]}\left(f,\varphi\right)$$

Therefore we get

$$\overline{\lambda}_{[p+1,q]}\left(f,\varphi\right)=\lambda_{[p+1,q]}\left(f,\varphi\right)=\sigma_{[p+1,q]}\left(f,\varphi\right).$$

This completes the proof of the theorem. □

5 Acknowledgement

The authors are grateful to the referee for his valuable suggestions which has considerably improved the presentation of the paper

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