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n-tuple fixed point theorems via α-series on partially ordered cone metric spaces

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Abstract

In this research, we prove the results of n-tuple fixed point in partially ordered cone metric spaces. We will impose some conditions upon a self-mapping and a sequence of mappings via α-series. This series are wider than the convergent series. Also, at the end of this paper, an example is provided to illustrate the results.

Keywords: α-series, n-tuple fixed point, n-tuple coincidence point, cone metric space, compatible mappings, reciprocally continuous
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1 Introduction

In [12] by replacing an ordered Banach space instead of real numbers, the concept of cone metric space was introduced and some fixed point theorems for contractive mappings in cone metric spaces have been proved. In various methods, many authors later generalized their fixed point theorems (see [10, 11, 13, 14, 19]). Some coincidence point theorems on cone metric spaces have been studied in [1, 3, 9]. In [5] the concept of a coupled coincidence point was introduced and they studied fixed point theorems in partially ordered metric spaces. In [20], Shatanawi proved that coupled coincidence point theorems on cone metric spaces are not necessarily normal.

Throughout this article, $\mathbb{N}$ is the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We establish the results of n-tuple fixed point for a self mapping g and $\{T_m\}_{m\in\mathbb{N}_0}$ that is a sequence of mappings from X^n into X, in partially ordered cone metric spaces via α-series, which introduced in [21]. The α-series are wider than the convergent series. We provide the preliminaries and definitions used throughout the article.

Definition 1.1. ([12]) Let $P \subseteq E$, where E is a real Banach space with $int(P) \neq \emptyset$. If P satisfies

1. P is closed and $P \neq \{\theta\}$, where θ represents zero.
2. $a, b \in \mathbb{R}^+, x, y \in P$ implies $ax + by \in P$.
3. $x \in P \cap -P$ implies $x = \theta$.

Then P is called a cone.

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The cone $P \subseteq E$ is given, we define a partial ordering $\leq$ with respect to P by $x \leq y$ iff $y - x \in P$. We write $x < y$ to show that $x \leq y$ but $x \neq y$. We write $x \ll y$ if $y - x \in IntP$. It is easy to show that $\lambda Int(P) \subseteq Int(P)$ for all positive scalar λ.

Definition 1.2. ([12]) A cone metric space is a pair (X, d), where X is a nonempty set and $d : X^2 \to E$ is a map such that satisfies

1. $\theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ iff $x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

The map d is called a cone metric on X.

Definition 1.3. ([12]) Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X and $x \in X$.

(i) The sequence $\{x_n\}$ is called *converges* to x, if for every $c \in E$ with $\theta \ll c$ there exists a positive integer $N \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n \geq \mathbb{N}$. We denote this by $\lim_{n \to +\infty} x_n = x$.

(ii) The sequence $\{x_n\}$ is called a *Cauchy sequence* in X, if for every $c \in E$ with $\theta \ll c$, there is an $N \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq \mathbb{N}$.

(iii) The space (X, d) is called a *complete cone metric space* if every Cauchy sequence is convergent.

Definition 1.4. ([18]) Let (X, d) be a cone metric space, $f : X \to X$ and $x_0 \in X$. Then f is said to be *continuous at* x_0 if for any sequence $x_n \to x_0$, we have $fx_n \to fx_0$.

Definition 1.5. ([7]) An element $(x, y) \in X^2$ is called a *coupled fixed point* of $F : X^2 \to X$ if

$$F(x, y) = x, \quad F(y, x) = y.$$

Example 1.6. Let $X = [0, \infty)$ and $F : X^2 \to X$ be defined by $F(x, y) = xy$ for all $x, y \in X$. One can easily see that F has a unique coupled fixed point $(1, 1)$.

Definition 1.7. ([17]) An element $(x, y) \in X^2$ is called a *coupled coincidence point* of the mappings $g : X \to X$ and $F : X^2 \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$. In this case, (gx, gy) is called a *coupled coincidence point.*

Definition 1.8. ([2]) An element $(x, y) \in X^2$ is called a *common coupled fixed point* of mappings $g : X \to X$ and $F : X^2 \to X$ if $x = g(x) = F(x, y)$ and $y = g(y) = F(y, x)$.

Definition 1.9. ([16]) Let $X \neq \emptyset$. We say that the mappings $g : X \to X$ and $F : X^2 \to X$ are *commutative* if

$$gF(x, y) = F(gx, gy), \quad gF(y, x) = F(gy, gx)$$

Now, inspired by [22], we generalize the concept of compatible mapping for a self-mapping g and a bivariate mapping F on a cone metric space as follows.

Definition 1.10. The mappings $g : X \to X$ and $F : X^2 \to X$ are called *compatible* if for arbitrary $c \in intP$, there exists $n_0 \in \mathbb{N}$ such that

$$d(F(gx_n, gy_n), g(F(x_n, y_n)) \ll c,$$
$$d(F(gy_n, gx_n), g(F(y_n, x_n)) \ll c$$

whenever $n > n_0$; $\{x_n\}, \{y_n\} \in X$, such that

$$\lim_{n \to +\infty} F(x_n, y_n) = \lim_{n \to +\infty} gx_n = x,$$
$$\lim_{n \to +\infty} F(y_n, x_n) = \lim_{n \to +\infty} gy_n = y,$$

for some $x, y \in X$. It called *weakly compatible* if they commute at coincidence points.

Example 1.11. Let $X = [0,3]$ be endowed with $d(x,y) = |x-y|$.

Define $g : X \to X$ and $F : X^2 \to X$ by

$$g(x) = \begin{cases} x & \textit{if } x \in [0,1) \\ 3 & \textit{if } x \in [1,3] \end{cases}$$

$$F(x,y) = \begin{cases} x+y & \textit{if } x,y \in [0,1) \\ 3 & \textit{elsewhere.} \end{cases}$$

Then for any $x,y \in [1,3]$, $F(gx,gy) = gF(x,y)$ and $F(gy,gx) = gF(y,x)$, show that F *and* g are weakly compatible maps on $[0,3]$.

Example 1.12. Let $X = \mathbb{R}$ be endowed with $d(x,y) = |x-y|$. Define $g : X \to X$ and $F : X^2 \to X$ by

$$g(x) = x^2, \quad F(x,y) = x+y$$

Then g and F are not weakly compatible maps on $\mathbb{R}$.

Definition 1.13. ([22]) The mappings $g : X \to X$ and $F : X^2 \to X$ are called *reciprocally continuous* if

$$\lim_{n\to+\infty} g(F(x_n,y_n)) = g(x), \ \textit{and} \ \lim_{n\to+\infty} F(gx_n,gy_n) = F(x,y)$$
$$\lim_{n\to+\infty} g(F(y_n,x_n)) = g(y), \ \textit{and} \ \lim_{n\to+\infty} F(gy_n,gx_n) = F(y,x)$$

whenever $\{x_n\},\{y_n\} \in X$, such that

$$\lim_{n\to+\infty} F(x_n,y_n) = \lim_{n\to+\infty} g(x_n) = x,$$

$$\lim_{n\to+\infty} F(y_n,x_n) = \lim_{n\to+\infty} g(y_n) = y,$$

for some $x,y \in X$.

Definition 1.14. ([4]) Let $(X,\preceq)$ be a poset (or partially ordered set) and $F : X^2 \to X$. We say that F has the *mixed monotone property* if for any $x,y \in X$

$$x_1,x_2 \in X, \quad x_1 \preceq x_2 \quad \Rightarrow \quad F(x_1,y) \preceq F(x_2,y),$$
$$y_1,y_2 \in X, \quad y_1 \preceq y_2 \quad \Rightarrow \quad F(x,y_1) \succeq F(x,y_2),$$

That is, $F(x,y)$ is monotone non-decreasing in x and is monotone non-increasing in y.

Definition 1.15. ([6]) Let $(X,\preceq)$ be a poset and $g : X \to X$ and $F : X^2 \to X$. We say that F has the *g-mixed monotone property* if for any $x,y \in X$,

$$x_1,x_2 \in X, \quad gx_1 \preceq gx_2 \quad \Rightarrow \quad F(x_1,y) \preceq F(x_2,y),$$
$$y_1,y_2 \in X, \quad gy_1 \preceq gy_2 \quad \Rightarrow \quad F(x,y_1) \succeq F(x,y_2),$$

That is, $F(x,y)$ is g-monotone non-decreasing in x, and it is g-monotone non-increasing in y.

We give an n-dimensional case of this definition as follows:

Definition 1.16. We consider poset $(X, \preceq^\circ)$. We define on X^n the following order:

$$\mathbf{x} = (x^1, \cdots, x^n) \preceq (y^1, \cdots, y^n) = \mathbf{y} \Leftrightarrow x^i \preceq^\circ y^i (\textit{if } i \textit{ is odd}), \textit{ and}$$
$$x^i \,{}^\circ\!\succeq y^i (\textit{if } i \textit{ is even}).$$

Definition 1.17. Let $(X, \preceq^\circ)$ be a poset and $F : X^n \to X$ is given. We say that F is *monotone* if

$$\mathbf{x} \preceq \mathbf{y} \Rightarrow F(\mathbf{x}) \preceq^\circ F(\mathbf{y})$$

where $\mathbf{x} = (x^1, \cdots, x^n), \mathbf{y} = (y^1, \cdots, y^n)$, which $\preceq$ is the same as in definition 1.16.

Definition 1.18. [8]. Let $(X, \preceq^\circ)$ be a poset and $g : X \to X$ and $F : X^n \to X$. We say that F is *g-mixed monotone* if

$$gx^i \preceq^\circ gy^i \text{ (if i is odd)} \Rightarrow F(x^1, \cdots, x^i, \cdots, x^n) \preceq^\circ F(x^1, \cdots, y^i, \cdots, x^n)$$

and

$$gx^i \,{}^\circ\!\succeq gy^i \text{ (if i is even)} \Rightarrow F(x^1, \cdots, x^i, \cdots, x^n) \preceq^\circ F(x^1, \cdots, y^i, \cdots, x^n).$$

Definition 1.19. [8]. Let $X \neq \emptyset$. An element $(x^1, ..., x^n) \in X^n$ is called an *n-tuple fixed point* of the mapping $F : X^n \to X$ if

$$x^i = F(x^i, x^{i+1}, ..., x^n, x^1, ..., x^{i-1}), \textit{where } 1 \leq i \leq n.$$

Definition 1.20. [8]. Let $X \neq \emptyset$. An element $(x^1, ..., x^n) \in X^n$ is called an *n-tuple coincidence point* of the mappings $g : X \to X$ and $F : X^n \to X$ if

$$gx^i = F(x^i, x^{i+1}, ..., x^n, x^1, ..., x^{i-1}), \textit{where } 1 \leq i \leq n.$$

Definition 1.21. [15] Let $X \neq \emptyset$. The mappings $g : X \to X$ and $F : X^n \to X$ are said to be *commutating* if

$$gF(x^1, ..., x^n) = F(gx^1, ..., gx^n),$$

for all $x^1, \ldots, x^n \in X$.

We generalize the definitions of compatibility and weakly reciprocally continuity, for a self-mapping g and n-variate mapping F.

Definition 1.22. Let (X, d) be a cone metric space. The mappings $g : X \to X$ and $F : X^n \to X$ are called *compatible* if for arbitrary $c \in intP$, there exists $m_0 \in \mathbb{N}$ such that

$$d(g(F(x^i_m, x^{i+1}_m, \ldots, x^n_m, x^1_m, \ldots, x^{i-1}_m)),$$
$$F(gx^i_m, gx^{i+1}_m, \ldots, gx^n_m, gx^1_m, \ldots, gx^{i-1}_m)) \ll c,$$

where $1 \leq i \leq n$, whenever $m > m_0$, $\{x^i_m\}$ are sequences in X, such that

$$\lim_{m \to +\infty} F(x^i_m, x^{i+1}_m, \ldots, x^n_m, x^1_m, \ldots, x^{i-1}_m) = \lim_{m \to +\infty} gx^i_m := x^i,$$

for some $x^i \in X$. It is said to be *weakly compatible* if

$$gx^i = F(x^i, x^{i+1}, \ldots, x^n, x^1, \ldots, x^{i-1}),$$

implies

$$g(F(x^i, x^{i+1}, \ldots, x^n, x^1, \ldots, x^{i-1})) = F(gx^i, gx^{i+1}, \ldots, gx^n, gx^1, \ldots, gx^{i-1}),$$

where $1 \leq i \leq n$, for some $(x^1, \cdots x^n) \in X^n$.

Definition 1.23. The mappings $g : X \to X$ and $F : X^n \to X$ are called *reciprocally continuous* if

$$\lim_{m\to+\infty} g(F(x_m^i, x_m^{i+1}, \ldots, x_m^n, x_m^1, \ldots, x_m^{i-1})) = gx^i, \text{ and}$$
$$\lim_{m\to+\infty} F(gx_m^i, gx_m^{i+1}, \ldots, gx_m^n, gx_m^1, \ldots, gx_m^{i-1}) = F(x^i, x^{i+1}, \ldots, x^n, x^1, \ldots, x^{i-1}),$$

whenever $\{x_m^i\}, 1 \le i \le n$, are sequences in X, such that

$$\lim_{m\to+\infty} F(x_m^i, x_m^{i+1}, \ldots, x_m^n, x_m^1, \ldots, x_m^{i-1}) = \lim_{m\to+\infty} gx_m^i := x^i,$$

for some $x^i \in X, 1 \le i \le n$.

The new concept of an α-series was introduced by Sihag et al. [21] as follow.

Definition 1.24. ([21]) Let $\{a_n\}$ be a sequence of positive real numbers. We say that a series $\sum_{n=1}^{+\infty} a_n$ is an α-series, if there exist $0 < \alpha < 1$ and $n_\alpha \in \mathbb{N}$ such that $\sum_{i=1}^{k} a_i \le \alpha k$ for each $k \ge n_\alpha$.

For example, we know that every convergent series is bounded hence, every convergent series of non-negative real terms is an α-series. Moreover, there exists also divergent series that are α-series. For example, $\sum_{n=1}^{+\infty} \frac{1}{n}$ is an α-series.

As a main result of this paper, we shall study the existence and uniqueness of n-tuple common fixed point for self-mapping $g : X \to X$ and the sequence of mappings $T_m : X^n \to X$, where (X, d) is a cone metric space.

2 Main results

According to the definition 1.18 we have the following definition.

Definition 2.1. Let $(X, \preceq)$ be a poset and $g : X \to X$, and $T_m : X^n \to X$, $m \in \mathbb{N}_0$ are given. We say that $\{T_m\}_{m\in\mathbb{N}_0}$ has the *g-mixed monotone property* if for any $x^i, y^i \in X, 1 \le i \le n$,

$$gx^i \preceq gy^i \ (\textit{if } i \textit{ is odd}), \textit{ and } gx^i \succeq gy^i \ (\textit{if } i \textit{ is even}), \textit{ imply}$$
$$T_m(x^i, x^{i+1}, ..., x^n, x^1, ..., x^{i-1}) \preceq T_{m+1}(y^i, y^{i+1}, ..., y^n, y^1, ..., y^{i-1})(\textit{if } i \textit{ is odd}),$$
$$T_{m+1}(y^i, y^{i+1}, ..., y^n, y^1, ..., y^{i-1}) \preceq T_m(x^i, x^{i+1}, ..., x^n, x^1, ..., x^{i-1})(\textit{if } i \textit{ is even}),$$

where $1 \le i \le n$.

Definition 2.2. Let $g : X \longrightarrow X$ and $T_m : X^n \to X$ are given. $\{T_m\}_{m\in\mathbb{N}_0}$ and g enjoy the (K) property if

$$d(T_m(x_1, \ldots, x_n), T_{m'}(y_1, \ldots, y_n)) \le \beta_{m,m'}[d(gx_1, T_m(x_1, \ldots, x_n)) + d(gy_1, T_{m'}(y_1, \ldots, y_n))] + \gamma_{m,m'} d(gy_1, gx_1) \tag{2.1}$$

for all $x_i, y_i \in X$, where $1 \le i \le n$, with $gx_i \preceq gy_i$ (*if i is odd*), *and* $gx_i \succeq gy_i$ (*if i is even*) or $gx_i \succeq gy_i$ (*if i is odd*), *and* $gx_i \preceq gy_i$(*if i is even*), $0 \le \beta_{m,m'}, \gamma_{m,m'} < 1$ for $m, m' \in \mathbb{N}_0$, and $\sum_{m=1}^{+\infty} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1 - \beta_{m,m+1}}\right)$ is an α-series.

Definition 2.3. If T_0 and g have non-decreasing transcendence point in its odd position arguments and non-increasing transcendence point in its even position arguments, then we call T_0 and g have *mixed n-tuple transcendence point*, if there exists $x_0^i \in X^n, 1 \le i \le n$, such that

$$gx_0^i \preceq T_0(x_0^i, x_0^{i+1}, \ldots x_0^n, x_0^1, \ldots x_0^{i-1}), \quad (\textit{if } i \textit{ is odd}),$$
$$gx_0^i \succeq T_0(x_0^i, x_0^{i+1}, \ldots x_0^n, x_0^1, \ldots x_0^{i-1}), \quad (\textit{if } i \textit{ is even}). \tag{2.2}$$

Before presenting the main result, first consider the sequences that are made in the following lemma.

Lemma 2.4. Let $(X, d, \preceq)$ be a partially ordered cone metric space and g and $\{T_m\}_{m\in\mathbb{N}_0}$ are given. Let$\{T_m\}_{m\in\mathbb{N}_0}$ has the *g-mixed monotone property* with $T_m(X^n) \subseteq g(X)$. If T_0 and g have mixed n-tuple transcendence point, then

(a) there are sequences $\{x^i\} \in X, 1 \leq i \leq n$, such that

$$gx^i_m = T_{m-1}(x^i_{m-1}, x^{i+1}_{m-1}, ..., x^n_{m-1}, x^1_{m-1}, ..., x^{i-1}_{m-1}),\ 1 \leq i \leq n,$$

for $m \in \mathbb{N}_0$.

(b) The sequences $\{gx^i_r\}, 1 \leq i \leq n$, are non-decreasing if i is odd and non-increasing if i is even.

(c) if $\{T_m\}_{m\in\mathbb{N}_0}$ and g satisfy the condition (K), then $\{gx^i_r\}, 1 \leq i \leq n$ are Cauchy sequences.

Proof . By hypothesis, let $x^i_0 \in X, 1 \leq i \leq n$, such that condition (2.2) holds. Since $T_0(X^n) \subseteq g(X)$, we can define $x^i_1 \in X,\ 1 \leq i \leq n$ such that

$$gx^i_1 = T_0(x^i_0, x^{i+1}_0, \dots x^n_0, x^1_0, \dots x^{i-1}_0).$$

Again since $T_0(X^n) \subseteq g(X)$, there exists $x^i_2 \in X,\ 1 \leq i \leq n$ such that

$$gx^i_2 = T_1(x^i_1, x^{i+1}_1, \dots x^n_1, x^1_1, \dots x^{i-1}_1).$$

Continuing this process, we can construct sequences $\{x^i_r\},\ 1 \leq i \leq n$, such that

$$gx^i_{r+1} = T_r(x^i_r, x^{i+1}_r, \dots x^n_r, x^1_r, \dots x^{i-1}_r), \tag{2.3}$$

for all $r \geq 0$. Now, by mathematical induction, we show that

$$\begin{aligned} &gx^i_r \preceq gx^i_{r+1}\ (\textit{if } i \textit{ is odd}),\\ &gx^i_r \succeq gx^i_{r+1}\ (\textit{if } i \textit{ is even}),\ 1 \leq i \leq n, \end{aligned} \tag{2.4}$$

for all $r \geq 0$. To show this, since (2.2) holds in view of

$$gx^i_1 = T_0(x^i_0, x^{i+1}_0, \dots x^n_0, x^1_0, \dots x^{i-1}_0),\ 1 \leq i \leq n,$$

we have

$$\begin{aligned} &gx^i_0 \preceq gx^i_1\ (\textit{if } i \textit{ is odd}),\\ &gx^i_0 \succeq gx^i_1\ (\textit{if } i \textit{ is even}),\ 1 \leq i \leq n, \end{aligned} \tag{2.5}$$

that is, (2.4) holds for $r = 0$. We presume that (2.4) holds for some $r > 0$. Now, by (2.3) and (2.4), we deduce that

$$\begin{aligned} gx^i_{r+1} &= T_r(x^i_r, x^{i+1}_r, \dots x^n_r, x^1_r, \dots x^{i-1}_r)\\ &\preceq T_{r+1}(x^i_{r+1}, x^{i+1}_{r+1}, \dots x^n_{r+1}, x^1_{r+1}, \dots x^{i-1}_{r+1})\\ &= gx^i_{r+2},\ (\textit{if } i \textit{ is odd})\\ gx^i_{r+1} &= T_r(x^i_r, x^{i+1}_r, \dots x^n_r, x^1_r, \dots x^{i-1}_r)\\ &\succeq T_{r+1}(x^i_{r+1}, x^{i+1}_{r+1}, \dots x^n_{r+1}, x^1_{r+1}, \dots x^{i-1}_{r+1})\\ &= gx^i_{r+2},\ (\textit{if } i \textit{ is even}). \end{aligned}$$

Thus we have done. Then, by (2.1), we get

$$\begin{aligned} d(gx^1_r, gx^1_{r+1}) &= d(T_{r-1}(x^1_{r-1}, \dots, x^n_{r-1}), T_r(x^1_r, \dots, x^n_r))\\ &\leq \beta_{r-1,r}[d(gx^1_{r-1}, T_{r-1}(x^1_{r-1}, \dots, x^n_{r-1}))\\ &\quad + d(gx^1_r, T_r(x^1_r, \dots, x^n_r))] + \gamma_{r-1,r} d(gx^1_r, gx^1_{r-1})\\ &= \beta_{r-1,r}[d(gx^1_{r-1}, gx^1_r) + d(gx^1_r, gx^1_{r+1})]\\ &\quad + \gamma_{r-1,r} d(gx^1_r, gx^1_{r-1}). \end{aligned}$$

It follows that

$$(1 - \beta_{r-1,r}) d(gx^1_r, gx^1_{r+1}) \leq (\beta_{r-1,r} + \gamma_{r-1,r}) d(gx^1_r, gx^1_{r-1}).$$

Equivalently,

$$d(gx_r^1, gx_{r+1}^1) \leq \left(\frac{\beta_{r-1,r} + \gamma_{r-1,r}}{1 - \beta_{r-1,r}}\right) d(gx_r^1, gx_{r-1}^1). \tag{2.6}$$

Similarly, we get

$$d(gx_r^2, gx_{r+1}^2) \leq \left(\frac{\beta_{r-1,r} + \gamma_{r-1,r}}{1 - \beta_{r-1,r}}\right) d(gx_r^2, gx_{r-1}^2)$$

$$\vdots$$

$$d(gx_r^n, gx_{r+1}^n) \leq \left(\frac{\beta_{r-1,r} + \gamma_{r-1,r}}{1 - \beta_{r-1,r}}\right) d(gx_r^n, gx_{r-1}^n). \tag{2.7}$$

Adding (2.6)-(2.7), we set

$$\delta_r := \sum_{i=1}^{n} d(gx_r^i, gx_{r+1}^i).$$

Then, we have

$$\begin{aligned}
\delta_r = \sum_{i=1}^{n} d(gx_r^i, gx_{r+1}^i) &\leq \left(\frac{\beta_{r-1,r} + \gamma_{r-1,r}}{1 - \beta_{r-1,r}}\right) [\sum_{i=1}^{n} d(gx_r^i, gx_{r-1}^i)] \\
&= \left(\frac{\beta_{r-1,r} + \gamma_{r-1,r}}{1 - \beta_{r-1,r}}\right) \delta_{r-1} \\
&\leq \left(\frac{\beta_{r-1,r} + \gamma_{r-1,r}}{1 - \beta_{r-1,r}}\right) \\
&\left(\frac{\beta_{r-2,r-1} + \gamma_{r-2,r-1}}{1 - \beta_{r-2,r-1}}\right) \delta_{r-2} \\
&\leq \dots \\
&\leq \prod_{m=0}^{r-1} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1 - \beta_{m,m+1}}\right) \delta_0.
\end{aligned}$$

Let $r, p \in \mathbb{N}$ whit $p > r$, and α and n_α as in Definition 1.24. Then for $r \geq n_\alpha$ also, with repeated use of the triangle inequality, we obtain

$$\begin{aligned}
\sum_{i=1}^{n} d(gx_r^i, gx_{r+p}^i) &\leq \sum_{i=1}^{n} d(gx_r^i, gx_{r+1}^i) + \sum_{i=1}^{n} d(gx_{r+1}^i, gx_{r+2}^i) + \cdots \\
&+ \sum_{i=1}^{n} d(gx_{r+p-1}^i, gx_{r+p}^i) \\
&\leq \prod_{m=0}^{r-1} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1 - \beta_{m,m+1}}\right) \delta_0 + \prod_{m=0}^{r} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1 - \beta_{m,m+1}}\right) \delta_0 \\
&+ \dots + \prod_{m=0}^{r+p-2} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1 - \beta_{m,m+1}}\right) \delta_0 \\
&= \sum_{k=r}^{r+p-1} \prod_{m=0}^{k-1} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1 - \beta_{m,m+1}}\right) \delta_0 \\
&\leq \sum_{k=r}^{r+p-1} \left[\frac{1}{k} \sum_{m=0}^{k-1} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1 - \beta_{m,m+1}}\right)\right]^k \delta_0 \\
&\leq \left(\sum_{k=r}^{r+p-1} \alpha^k\right) \delta_0 \\
&\leq \frac{\alpha^r}{1 - \alpha} \delta_0.
\end{aligned}$$

Now, we prove that $\{gx_r^i\}$ are Cauchy sequences in (X,d). Let $\theta \ll c$ be given. There is a neighborhood of θ such as

$$N_\delta(\theta) = \{y \in E : \ \| y \| < \delta\}$$

where $\delta > 0$, such that $c + N_\delta(\theta) \subseteq IntP$, since $c \in IntP$. Choose $N_1 \in \mathbb{N}$ such that

$$\| -\frac{\alpha^{N_1}}{1-\alpha}\delta_0 \| < \delta.$$

Then

$$-\frac{\alpha^r}{1-\alpha}\delta_0 \in N_\delta(\theta),$$

for all $r \geq N_1$. Hence

$$c - \frac{\alpha^r}{1-\alpha}\delta_0 \in c + N_\delta(\theta) \subseteq IntP.$$

Thus we have

$$\frac{\alpha^r}{1-\alpha}\delta_0 \ll c,$$

for all $r \geq N_1$. Therefore

$$\sum_{i=1}^{n} d(gx_r^i, gx_{r+p}^i) \leq \frac{\alpha^r}{1-\alpha}\delta_0 \ll c,$$

for all $p > r \geq N_1$. So we conclude $\{gx_r^i\}, 1 \leq i \leq n$ are Cauchy in $g(X)$. □ Now, we revise Definitions 1.22 and 1.23.

Definition 2.5. Let (X,d) be a cone metric space. The mappings $g : X \to X$ and $T_m : X^n \to X$ are *compatible*, if for arbitrary $c \in intP$, there exists $m_0 \in \mathbb{N}$ such that

$$d(g(T_m(x_m^i, \ldots, x_m^n, x_m^1, \ldots, x_m^{i-1})), T_m(gx_m^i, \ldots, gx_m^n, gx_m^1, \ldots, gx_m^{i-1})) \ll c,$$

where $1 \leq i \leq n$; only, $m > m_0$ and $\{x_m^i\}, 1 \leq i \leq n$ are sequences in X, such that

$$\lim_{m\to+\infty} T_m(x_m^i, x_m^{i+1}, \ldots, x_m^n, x_m^1, \ldots, x_m^{i-1}) = \lim_{m\to+\infty} gx_{m+1}^i := x^i,$$

for some $x^i \in X$. It is said to be *weakly compatible* if

$$gx^i = T_m(x^i, x^{i+1}, \ldots, x^n, x^1, \ldots, x^{i-1}),$$

implies

$$g(T_m(x^i, x^{i+1}, \ldots, x^n, x^1, \ldots, x^{i-1})) = T_m(gx^i, \ldots, gx^n, gx^1, \ldots, gx^{i-1}),$$

where $1 \leq i \leq n$.

Definition 2.6. Let (X,d) be a cone metric space and $g : X \to X$ and $T_m : X^n \to X$ are given. $\{T_m\}_{m\in\mathbb{N}_0}$ and g are called *reciprocally continuous* if

$$\lim_{m\to+\infty} g(T_m(x_m^i, x_m^{i+1}, \ldots, x_m^n, x_m^1, \ldots, x_m^{i-1})) = gx^i, \ \textit{and}$$
$$\lim_{m\to+\infty} T_m(gx_m^i, gx_m^{i+1}, \ldots, gx_m^n, gx_m^1, \ldots, gx_m^{i-1})$$
$$= \lim_{m\to+\infty} T_m(x^i, x^{i+1}, \ldots, x^n, x^1, \ldots, x^{i-1}),$$

whenever $\{x_m^i\}, 1 \leq i \leq n$ are sequences in X, such that

$$\lim_{m\to+\infty} T_m(x_m^i, x_m^{i+1}, \ldots, x_m^n, x_m^1, \ldots, x_m^{i-1}) = \lim_{m\to+\infty} gx_{m+1}^i := x^i,$$

for some $x^i \in X$ and $1 \leq i \leq n$.

Theorem 2.7. In addition to the assumptions of Lemma 2.4, let $T_m : X^n \to X$ and g be reciprocally continuous and compatible, g is continuous and $g(X) \subseteq X$ be complete. Also, suppose that X has the following properties:

1. if an increasing sequence $x_m \to x$, then $x_m \preceq x$ for all m,
2. if a decreasing sequence $x_m \to x$, then $x \preceq x_m$ for all m.

Then $T_m : X^n \to X$ and g have a n-tuple coincidence point.

Proof . Let $\{x_r^i\}$ are the same sequence which appear in the lemma 2.4. Since $g(X)$ is complete, then there exist $s^i \in X$, with $\lim_{r\to+\infty}\{gx_r^i\} = g(s^i) := x^i, 1 \le i \le n$. By construction we have

$$\lim_{r\to+\infty} g(x_{r+1}^i) = \lim_{r\to+\infty} T_r(x_r^i, x_r^{i+1}, \dots x_r^n, x_r^1, \dots x_r^{i-1}) = x^i,\ 1 \le i \le n.$$

Since $\{T_m\}_{m\in\mathbb{N}_0}$ and g are reciprocally continuous and compatible, we have

$$\lim_{r\to+\infty} T_r(gx_r^i, gx_r^{i+1}, \dots, gx_r^n, gx_r^1, \dots, gx_r^{i-1}) = gx^i, 1 \le i \le n.$$

Also from continuity of g, we have

$$\lim_{r\to+\infty} T_r(x^i, x^{i+1}, \dots, x^n, x^1, \dots, x^{i-1}) = gx^i, 1 \le i \le n.$$

From the non-decreasing sequence $\{gx_r^i\}$ when i is odd, we have $g(x_r^i) \preceq x^i(\textit{if } i \textit{ is odd})$. Also, from the non-decreasing sequence $\{gx_r^i\}$ when i is even, we have $g(x_r^i) \succeq x^i(\textit{if } i \textit{ is even}),\ 1 \le i \le n$. Then applying condition (2.1), we get

$$\begin{aligned} d(T_r(gx_r^i, gx_r^{i+1}, \dots, gx_r^n, gx_r^1, \dots, gx_r^{i-1}), T_m(x^i, x^{i+1}, \dots, x^n, x^1, \dots, x^{i-1})) \\ \le \beta_{r,m}[d(ggx_r^i, T_r(gx_r^i, gx_r^{i+1}, \dots, gx_r^n, gx_r^1, \dots, gx_r^{i-1})) \\ + d(gx^i, T_m(x^i, x^{i+1}, \dots, x^n, x^1, \dots, x^{i-1}))] \\ + \gamma_{r,m} d(gx^i, ggx_r^i). \end{aligned}$$

Let $\theta \ll c$ be given. Choose $N_1, N_2 \in \mathbb{N}$ such that

$$\begin{aligned} \beta_{r,m}[d(ggx_r^i, T_r(gx_r^i, gx_r^{i+1}, \dots, gx_r^n, gx_r^1, \dots, gx_r^{i-1})) \\ + d(gx^i, T_m(x^i, x^{i+1}, \dots, x^n, x^1, \dots, x^{i-1}))] \ll \frac{c}{2} \textit{ for all } r \ge N_1, \end{aligned}$$

as $\beta_{r,m} < 1$, and

$$\gamma_{r,m} d(gx^i, ggx_r^i) \ll \frac{c}{2} \quad \textit{for all } r \ge N_2.$$

Let $N_0 = \max\{N_1, N_2\}$. Then

$$\begin{aligned} \beta_{r,m}[d(ggx_r^i, T_r(gx_r^i, gx_r^{i+1}, \dots, gx_r^n, gx_r^1, \dots, gx_r^{i-1})) \\ + d(gx^i, T_m(x^i, x^{i+1}, \dots, x^n, x^1, \dots, x^{i-1}))] \\ + \gamma_{r,m} d(gx_r^i, gx_r^{i+1}, \dots, gx_r^n, gx_r^1, \dots, gx_r^{i-1}) \ll c, \end{aligned}$$

for all $r \ge N_0$. Hence, $T_r(gx_r^i, gx_r^{i+1}, \dots, gx_r^n, gx_r^1, \dots, gx_r^{i-1})$ converges to $T_m(x^i, x^{i+1}, \dots, x^n, x^1, \dots, x^{i-1})$. Let $\theta \ll c$ be given. We choose $k_1, k_2, k_3 \in \mathbb{N}$ such that

$$d(T_r(x^i, x^{i+1}, \dots, x^n, x^1, \dots, x^{i-1}), T_r(gx_r^i, \dots, gx_r^n, gx_r^1, \dots, gx_r^{i-1}))) \ll \frac{c}{3},$$

for all $r \ge k_1$,

$$d(T_r(gx_r^i, \dots, gx_r^n, gx_r^1, \dots, gx_r^{i-1}), g(T_r(x_r^i, \dots, x_r^n, x_r^1, \dots, x_r^{i-1}))) \ll \frac{c}{3},$$

for all $r \ge k_2$, and

$$d(g(T_r(x_r^i, \dots, x_r^n, x_r^1, \dots, x_r^{i-1}))), gx^i) \ll \frac{c}{3} \textit{ for all } r \ge k_3.$$

Let $k_0 = \max\{k_1, k_2, k_2\}$. Then
$$d(T_r(x^i, x^{i+1}, \ldots, x^n, x^1, \ldots, x^{i-1}), gx^i) \ll c.$$
Since c is arbitrary, we have
$$d(T_r(x^i, x^{i+1}, \ldots, x^n, x^1, \ldots, x^{i-1}), gx^i) \ll \frac{c}{n'} \quad \forall n' \in \mathbb{N}$$
According to the fact that as $n' \to +\infty$, then $\dfrac{c}{n'} \to \theta$, we conclude that
$$\begin{aligned}\frac{c}{n'} - & d(T_r(x^i, x^{i+1}, \ldots, x^n, x^1, \ldots, x^{i-1}),\ gx^i) \\ & \longrightarrow -d(T_r(x^i, x^{i+1}, \ldots, x^n, x^1, \ldots, x^{i-1}),\ gx^i),\ \textit{as}\ n' \to +\infty.\end{aligned}$$
Because P is closed, we get $-d(T_r(x^i, x^{i+1}, \ldots, x^n, x^1, \ldots, x^{i-1}),\ gx^i) \in P$. Thus, $d(T_r(x^i, x^{i+1}, \ldots, x^n, x^1, \ldots, x^{i-1}),\ gx^i) \in P \cap -P$. Hence,
$$d(T_r(x^i, x^{i+1}, \ldots, x^n, x^1, \ldots, x^{i-1}),\ gx^i) = \theta.$$
Therefore, $gx^i = T_r(x^i, x^{i+1}, \ldots, x^n, x^1, \ldots, x^{i-1})$. Thus, $(x^1, \cdots, x^n)$ is a n-tuple coincidence point of $\{T_m\}_{m\in\mathbb{N}_0}$ and g. □

Definition 2.8. For all $x_i, u_i \in X$, $1 \le i \le n$; we say that $(x_i)_{i=1}^n$ is n-tuple comparable whit $(u_i)_{i=1}^n$ if
$$x_i \ge u_{\sigma(i)} (\textit{if } i \textit{ is odd}), x_i \le u_{\sigma(i)} (\textit{if } i \textit{ is even})$$
or
$$x_i \le u_{\sigma(i)} (\textit{if } i \textit{ is odd}), x_i \ge u_{\sigma(i)} (\textit{if } i \textit{ is even}),$$
where
$$\sigma \in \Pi = \left\{\sigma^j | \quad \sigma^j : \mathbf{n} \to \mathbf{n},\ \sigma^j(i) = k,\ k \overset{\text{mod } n}{\equiv} i + j,\ 0 \le k \le n-1\right\}.$$
In the other word, the member of Π are the permutation of $\mathbf{n}$, which preserve order (modulus n).

If in the above, we definition replace x_i and $u_{\sigma(i)}$ with gx_i and $gu_{\sigma(i)}$, we call $(x_i)_{i=1}^n$ is n-tuple comparable with $(u_i)_{i=1}^n$ with respect to g.

Theorem 2.9. Let $(X, d, \preceq)$ be a partially ordered cone metric space. Let g and $\{T_m\}_{m\in\mathbb{N}_0}$ are given. g and $\{T_m\}_{m\in\mathbb{N}_0}$ are w-compatible and satisfy the condition (K). If $\{T_m\}_{m\in\mathbb{N}_0}$ have n-tuple coincidence points comparable with respect to g, then g and $\{T_m\}_{m\in\mathbb{N}_0}$ have a unique n-tuple common fixed point, that is, there exists a unique $(x^1, \cdots, x^n) \in X^n$ such that
$$x^i = g(x^i) = T_m(x^i, x^{i+1}, ..., x^n, x^1, ..., x^{i-1}), \textit{where } 1 \le i \le n.$$
Moreover, the common fixed point of $\{T_m\}_{m\in\mathbb{N}_0}$ and g is of the form $(p, \ldots, p)$ for some $p \in X$.

Proof . From Theorem 2.7, the set of n-tuple coincidence points is non-empty. First, we show that if $(x^1, \cdots, x^n)$ and $(u^1, \cdots, u^n)$ are n-tuple coincidence points then $gx^i = gu^i, 1 \le i \le n$. Since the set of n-tuple coincidence points is n-tuple comparable, applying condition (2.1), we get
$$\begin{aligned} d(gx^1, gu^1) &= d(T_m(x^1, \ldots, x^n), T_{m'}(u^1, \ldots, u^n)) \\ &\le \beta_{m,m'}[d(gx^1, T_m(x^1, \ldots, x^n) + d(gu^1, T_{m'}(y^1, , \ldots, y^n))] \\ &+ \gamma_{m,m'} d(gu^1, gx^1). \end{aligned}$$
Since $\gamma_{m,m'} < 1$, it follows that $d(gx^1, gu^1) = 0$, that is, $gx^1 = gu^1$. Similarly, it can be proved that $gx^i = gu^j$, where $1 \le i, j \le n$. So
$$gx^1 = \cdots = gx^n = gu^1 \cdots = gu^n.$$
Therefore $\{T_m\}_{m\in\mathbb{N}_0}$ and g have a unique n-tuple coincidence point $(gx^1, \cdots, gx^1)$. Now, let $gx^1 = p$. Then we have
$$p = gx^1 = T_m(x^1, \cdots, x^1).$$

By w-compatibility of $\{T_m\}_{m\in\mathbb{N}_0}$ and g, we have

$$gp = ggx^1 = g(T_m(x^1,\cdots,x^1)) = T_m(gx^1,\cdots,gx^1) = T_m(p,\cdots,p).$$

On the other hand, $T_m(gx^1,\cdots,gx^1) = gx^1$. So, $(gp,\ldots,gp)$ is an n-tuple coincidence point of $\{T_m\}_{m\in\mathbb{N}_0}$ and g. So, $gp = gx^1$. Hence,

$$p = gp = T_m(p,\cdots,p).$$

Therefore, $(p,\cdots,p)$ is a unique n-tuple common fixed point of $\{T_m\}_{m\in\mathbb{N}_0}$ and g. □

Example 2.10. Let $X = [0,1]$ and

$$P = \{(x_1,\ldots,x_n)\in\mathbb{R}^n : x^i \geq 0, 1\leq i\leq n\} \subseteq E = \mathbb{R}^n.$$

Define $d(x,y) = (|x-y|,|x-y|)$. Then (X,d) is a partially ordered complete cone metric space. Define $\beta_{m,m'} = \dfrac{1}{n^{2m+1}}, \gamma_{m,m'} = \dfrac{1}{n^m}$ *for all* $m,m'\in\mathbb{N}$, and consider the mappings $g: X\to X$ and $T_m: X^n\to X$ with

$$g(x) = 3nx, \quad T_m(x_1,\cdots,x_n) = \frac{x_1+\cdots+x_n}{n^m}$$

for all $m = 1,2,\ldots;\quad x_1,\cdots,x_n\in X$. By mathematical induction we can easily show that $\{T_m\}_{m\in\mathbb{N}_0}$ and g satisfy the condition (K). For this, we know that the greatest value of the first side in (2.1) is when $m=1, m'\to\infty$. Suppose that for $m=1, m'=k$, we take $T_m(x_1,\cdots,x_n) = U, T_{m'}(y_1,\cdots,y_n) = V$. Then, we have

$$\begin{aligned}(\mid U-V\mid,\mid U-V\mid) \leq{}& \frac{1}{n^3}[(\mid 3nx_1-U\mid,\mid 3nx_1-U\mid)\\ &+(\mid 3ny_1-V\mid,\mid 3ny_1-V\mid)]\\ &+\frac{1}{n}(\mid 3n(y_1-x_1)\mid,\mid 3n(y_1-x_1)\mid).\end{aligned}$$

For $m' = k+1$ we have

$$\begin{aligned}A := (\mid U-\frac{1}{n}V)\mid,\mid U-\frac{1}{n}V)\mid) \leq{}& \frac{1}{n^3}[(\mid 3nx_1-U\mid,\mid 3nx_1-U\mid)\\ &+(\mid 3y_1-\frac{1}{n}V)\mid,\mid 3y_1-\frac{1}{n}V)\mid]\\ &+3(\mid\frac{y_1}{n}-x_1\mid,\mid\frac{y_1}{n}-x_1\mid) := B.\end{aligned}$$

So,

$$\begin{aligned}A &\leq \frac{1}{n}[(\mid U-V\mid,\mid U-V\mid)] + \frac{n-1}{n}\mid U\mid\\ &\leq \frac{1}{n}(\frac{1}{n^3}[(\mid 3nx_1-U\mid,\mid 3nx_1-U\mid)+(\mid 3ny_1-V\mid,\mid 3ny_1-V\mid)])\\ &+\frac{1}{n}(3(\mid y_1-x_1\mid,\mid y_1-x_1\mid))+\frac{n-1}{n}\mid U\mid\leq B.\end{aligned}$$

Since $d(x,y)$ is symmetric, therefore, the role of m,m' can be changed together and a similar result can be reached. Thus, the inequality (2.1) for every m,m' is holds. Moreover, the series

$$\sum_{m=1}^{+\infty}\left(\frac{\beta_{m,m+1}+\gamma_{m,m+1}}{1-\beta_{m,m+1}}\right) = \sum_{m=1}^{+\infty}\frac{n^{m+1}+1}{n^{2m+1}-1},\ n>1$$

is an α-series with $\alpha = \dfrac{1}{2}$. So, all conditions of Theorem 2.7 are satisfied and $(0,\cdots,0)$ are the n-tuple coincident points of g and $\{T_m\}_{m\in\mathbb{N}_0}$. Moreover, using the same mappings in Theorem 2.9, $(0,\cdots,0)$ is the unique n-tuple common fixed point of g and $\{T_m\}_{m\in\mathbb{N}_0}$.

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References

[1] M. Abbas and G. Jungck, *Common fixed point results for noncommuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl. **341** (2008), 416-420.

[2] M. Abbas, M.A. Khan and S. Radenović, *Common coupled fixed point theorems in cone metric spaces for w-compatible mappings*, Appl. Math. Comput. **217** (2010), 195–202.

[3] T. Abdeljawad and E. Karapinar, *Quasicone metric spaces and generalizations of Caristi Kirk's theorem*, Fixed Point Theory Appl. **2009** (2009), Article ID 574387, 1–9.

[4] V. Berinde and M. Borcut, *Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces*, Nonlinear Anal. **74** (2011), no. 15, 4889–4897.

[5] T. G. Bhaskar and V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal. **65** (2006), 1379–1393.

[6] M. Borcut and V. Berinde, *Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces*, Appl. Math. Comput. **218** (10) (2012), 5929–5936.

[7] B. S. Choudhary and A. Kundu, *A coupled coincidence point result in partially ordered metric spaces for compatible mappings*, Nonlinear Anal. **73** (2010), 2524–2531.

[8] S. Dalal, M. A. Khan and S. Chauhan, n-tupled coincidence point theorems in partially ordered metric spaces for compatible mappings, Abstr. Appl. Anal. **2014** (2014), Article ID 614019, 1-8.

[9] S. Hadi Bonab, R. Abazari and A. Bagheri Vakilabad, *Partially ordered cone metric spaces and coupled fixed point theorems via α-series*, Math. Anal. Contemp. Appl. **1** (2019), no. 1, 50–61.

[10] S. Hadi Bonab, R. Abazari1, A. Bagheri Vakilabad and H. Hosseinzadeh, *Coupled fixed point theorems on G-metric spaces via α-series*, Glob. Anal. Discrete Math. **6** (2021), no. 1, 1–12.

[11] S. Hadi Bonab, R. Abazari, A. Bagheri Vakilabad and H. Hosseinzadeh, *Generalized metric spaces endowed with vector-valued metrics and matrix equations by tripled fixed point theorems*, J. Inequal. Appl. **2020** (2020), 1–16.

[12] L. G. Haung and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **332** (2007), 1468–1476.

[13] A. Razani and V. Parvaneh, *Coupled coincidence point results for (ψ, α, β)-weak contractions in partially ordered metric spaces*, J. Appl. Math. **2012** (2012).

[14] N. Hussain, V. Parvaneh and F. Golkarmanesh, *Coupled and tripled coincidence point results under (F, g)-invariant sets in G_b-metric spaces and $G-\alpha$-admissible mappings*, Math. Sci. **9** (2015), no. 1, 11–26.

[15] M. Imdad, A. H. Soliman, B.S. Choudhury and P. Das, *On n-tupled coincidence point results in metric spaces*, J. Oper. **2013** (2013), Article ID 532867, 1–8.

[16] E. Karapinar, *Fixed point theorems in cone Banach spaces*, Fixed Point Theory Appl. **2009** (2009), Article ID 609281, 1–9.

[17] V. Lakshmikantham and L. Cirić, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal. **70** (2009), 4341–4349.

[18] D. Ilić and V. Rakocević, *Common fixed points for maps on cone metric space*, J. Math. Anal. Appl. **341** (2008), 876–882.

[19] V. Parvaneh , S. Hadi Bonab, H. Hosseinzadeh and H. Aydi, *A tripled fixed point theorem in C^*-algebra-valued metric spaces and application in integral equations*, Adv. Math. Phys. **2021** (2021), 1–6.

[20] W. Shatanawi, *Partially ordered cone metric spaces and coupled fixed point results*, Comput. Math. Appl. **60** (2010), 2508–2515.

[21] V. Sihag, C. Vetro and R.K. Vats, *A fixed point theorem in G-metric spaces via α-series*, Q. Math. **37** (2014), 1–6.

[22] R. K. Vats, K. Tas, V. Sihag and A. Kumar, *Triple fixed point theorems via α-series in partially ordered metric spaces*, J. Inequal. Appl. **2014** (2014), 1–12.