Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 2413–2424
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.22971.2440

New Lacunary sequence spaces defined by fractional difference operator

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(Communicated by Madjid Eshaghi Gordji)

Abstract

In the present paper, we introduce new lacunary strong convergent vector-valued sequence spaces defined by fractional difference operator and Musielak-Orlicz function. We make an effort to study some topological properties and also prove some inclusion relations between these spaces.

Keywords: Lacunary Sequence, Musielak-Orlicz function, fractional difference operator
2020 MSC: Primary 40A05, 40C05, Secondary 40A30, 40F05

1 Introduction and Preliminaries

The notion of difference sequence spaces was introduced by Kızmaz [10], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [5] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. For details about difference sequence spaces see [5, 10] and references therein.

In [1] Baliarsingh defined the fractional difference operator as follows: Let $x = (x_k) \in w$ and α be a real number, then the fractional difference operator $\Delta^{(\alpha)}$ is defined by

$$\Delta^{(\alpha)} x_k = \sum_{i=0}^{k} \frac{(-\alpha)_i}{i!} x_{k-i},$$

where $(-\alpha)_i$ denotes the Pochhammer symbol defined as:

$$(-\alpha)_i = \begin{cases} 1, & \text{if } \alpha = 0 \text{ or } i = 0, \\ \alpha(\alpha+1)(\alpha+2)...(\alpha+i-1), & \text{otherwise.} \end{cases}$$

For Orlicz function we refer [12] and Musielak-Orlicz function see [15, 21]. A Musielak-Orlicz function (M_k) is said to satisfy Δ_2-condition if there exist constants a, $K > 0$ and a sequence $c = (c_k)_{k=1}^{\infty} \in \ell_+^1$ (the positive cone of ℓ^1) such that the inequality $M_k(2u)] \leq K M_k(u) + c_k$ holds for all $k \in \mathbb{N}$ and $u \in R_+$, whenever $M_k(u) \leq \alpha$. For more details about sequence spaces of Musielak-Orlicz function see [17, 18, 19, 20, 22] and references therein.

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Received: March 2021 *Accepted:* August 2021

The space of lacunary strong convergence have been introduced by Freednman et al. [7]. A sequence of positive integers $\theta = (k_r)$ is called "lacunary" if $k_0 = 0, 0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$, as $r \to \infty$. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. the space of lacunary strongly convergent sequences N_θ is defined by Freedman et. al. [7] as follows:

$$N_\theta = \Big\{x = (x_i) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} |x_i - s| = 0, \text{ for some } s\Big\}.$$

The space $|\sigma_1|$ of strongly Cesaro summable sequences is

$$|\sigma_1| = \Big\{x = (x_k) : \text{there exists L such that} \frac{1}{n} \sum_{i=1}^{n} |x_i - L| \to 0, \text{ as } n \to \infty\Big\}.$$

In case, when $\theta = (2^r), N_\theta = |\sigma_1|$. Recently, Bilgin [2] in his paper generalized the concept of lacunary convergence and introduced the space $N_0(A, f)$, as

$$N_0(A, f) = \Big\{x = (x_i) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} f(|A_i(x) - s|) = 0, \text{ for some } s\Big\},$$

where f is a modulus function and $A = (A_i(x))$, $A_i x = \displaystyle\sum_{k=1}^{\infty} a_{ik} x_k$ converges for each i. Later Bilgin [3] generalized lacunary strongly A-convergent sequences with respect to a sequence of modulus function $F = (f_i)$ as follows:

$$N_0(A, F) = \Big\{x = (x_i) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} f_i(|A_i(x) - s|) = 0, \text{ for some } s\Big\}.$$

We write θ for the zero sequences and by $A_i(\Delta^{(\alpha)} x_k)$ we mean

$$A_i(\Delta^{(\alpha)} x_k) = \sum_{k=1}^{\infty} a_{ik} \Big(\frac{(-\alpha)i}{i!} x_{k-i}\Big).$$

Let $A = (a_{ik})$ be an infinite matrix of complex numbers, $\mathcal{M} = (M_i)$ be a Musielak-Orlicz function. In the present paper we define the following sequence spaces:

$$N_\theta(S, A, \Delta^{(\alpha)}, \mathcal{M}) = \Big\{x = (x_k) : x_k \in S \text{ and } \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} M_i\Big(\frac{\|A_i(\Delta^{(\alpha)} x_k) - t_i e_i\|}{\rho^{(i)}}\Big) = 0$$
$$\text{for some } t = (t_1, t_2, ...) \in S, e_i \in \mathbb{C} \text{ and } \rho^{(i)} > 0\Big\}$$

and

$$N^0_\theta(S, A, \Delta^{(\alpha)}, \mathcal{M}) = \Big\{x = (x_k) : x_k \in S \text{ and } \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} M_i\Big(\frac{\|A_i(\Delta^{(\alpha)} x_k)\|}{\rho^{(i)}}\Big) = 0$$
$$\text{for some } \rho^{(i)} > 0\Big\}.$$

If we take $\mathcal{M}(x) = x$, we have

$$N_\theta(S, A, \Delta^{(\alpha)}) = \Big\{x = (x_k) : x_k \in S \text{ and } \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \Big[\frac{\|A_i(\Delta^{(\alpha)} x_k) - t_i e_i\|}{\rho^{(i)}}\Big] = 0$$
$$\text{for some } t = (t_1, t_2, ...) \in S, e_i \in \mathbb{C} \text{ and } \rho^{(i)} > 0\Big\}$$

and

$$N^0_\theta(S, A, \Delta^{(\alpha)}) = \Big\{x = (x_k) : x_k \in S \text{ and } \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \frac{\|A_i(\Delta^{(\alpha)} x_k)\|}{\rho^{(i)}} = 0 \text{ for some } \rho^{(i)} > 0\Big\},$$

where

$$e_i = \begin{cases} 1, & \text{at the i-th place,} \\ 0, & \text{otherwise.} \end{cases}$$

The main purpose of this paper is to introduce new lacunary strong convergent vector valued sequence spaces with the elements chosen from a Banach space $(E, ||.||)$ over the complex field $\mathbb{C}$, with respect to fractional difference operator and Musielak-Orlicz function $\mathcal{M} = (M_i)$. We have studied some topological properties and also prove inclusion relations between the above defined sequence spaces.

2 Topological properties

Theorem 2.1. Let $A = (a_{ik})$ be an infinite matrix of complex numbers, $\mathcal{M} = (M_i)$ be a Musielak-Orlicz function. Then $N_\theta(S, A, \Delta^{(\alpha)}, \mathcal{M})$ and $N_\theta^0(S, A, \Delta^{(\alpha)}, \mathcal{M})$ are linear space over the field of complex number $\mathbb{C}$.

Proof . Suppose that $x = (x_k), y = (y_k) \in N_\theta(S, A, \Delta^{(\alpha)}, \mathcal{M})$ and $(x_k) \stackrel{\Delta^{(\alpha)}}{\longrightarrow} t, (y_k) \stackrel{\Delta^{(\alpha)}}{\longrightarrow} u$, then for some $t = (t_1, t_2, ...), u = (u_1, u_2, ...) \in S,\ \ e_i \in \mathbb{C}$, we have

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} M_i\Big(\frac{\|A_i(\Delta^{(\alpha)}x_k) - t_i e_i\|}{\rho_1^{(i)}}\Big) = 0, \text{ for some } \rho_1^{(i)} > 0$$

and

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} M_i\Big(\frac{\|A_i(\Delta^{(\alpha)}y_k) - u_i e_i\|}{\rho_2^{(i)}}\Big) = 0, \text{ for some } \rho_2^{(i)} > 0.$$

Let $\beta, \gamma \in \mathbb{C}$. Without loss of generality we may assume that there exists $P_1 > 1, P_2 > 1$ such that $|\beta| \le P_1$ and $|\gamma| \le P_2$. Let $\rho^{(i)} = \max(2\rho_1^{(i)}, 2\rho_2^{(i)})$. Then

$$\begin{aligned}
&\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} M_i\Big(\frac{\|A_i(\beta\Delta^{(\alpha)}x_k + \gamma\Delta^{(\alpha)}y_k) - (\beta t_i e_i + \gamma u_i e_i\|)}{\rho^{(i)}}\Big)\\
\le\ & \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} M_i\Big(\frac{\|\beta A_i(\Delta^{(\alpha)}x_k) - \beta t_i e_i\| + \|\gamma A_i(\Delta^{(\alpha)}y_k) - \gamma u_i e_i\|}{\rho^{(i)}}\Big)\\
=\ & \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \frac{1}{2} M_i\Big(\frac{|\beta|\|A_i(\Delta^{(\alpha)}x_k) - t_i e_i\|}{\rho_1^{(i)}}\Big)\\
+\ & \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \frac{1}{2} M_i\Big(\frac{|\gamma|\|A_i(\Delta^{(\alpha)}y_k - u_i e_i)\|}{\rho_2^{(i)}}\Big)\\
\le\ & \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \frac{1}{2} M_i\Big(\frac{P_1\|A_i(\Delta^{(\alpha)}x_k) - t_i e_i\|}{\rho_1^{(i)}}\Big)\\
+\ & \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \frac{1}{2} M_i\Big(\frac{P_2\|A_i(\Delta^{(\alpha)}y_k) - u_i e_i\|}{\rho_2^{(i)}}\Big)\\
\le\ & K_1 \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \frac{1}{2} M_i\Big(\frac{\|A_i(\Delta^{(\alpha)}x_k) - t_i e_i\|}{\rho_1^{(i)}}\Big)\\
+\ & K_2 \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \frac{1}{2} M_i\Big(\frac{\|A_i(\Delta^{(\alpha)}y_k) - u_i e_i\|}{\rho_2^{(i)}}\Big)\\
& \to 0 \text{ as } r \to \infty.
\end{aligned}$$

Therefore, $(\beta x_k + \gamma y_k) \in N_\theta(S, A, \Delta^{(\alpha)}, \mathcal{M})$. This proves that $N_\theta(S, A, \Delta^{(\alpha)}, \mathcal{M})$ is a linear space. Similarly we can prove that $N_\theta^0(S, A, \Delta^{(\alpha)}, \mathcal{M})$ is a linear space. □

Theorem 2.2. Let $A = (a_{ik})$ be an infinite matrix of complex numbers, $\mathcal{M} = (M_i)$ be Musielak-Orlicz function. Then $N_\theta(S, A, \Delta^{(\alpha)}, \mathcal{M})$ and $N_\theta^0(S, A, \Delta^{(\alpha)}, \mathcal{M})$ are normal spaces, when S is normal.

Proof . Let $x = (x_k) \in N_\theta(S, A, \Delta^{(\alpha)}, \mathcal{M})$ and $(x_k) \stackrel{\Delta^{(\alpha)}}{\longrightarrow} t$, where $t = (t_1, t_2, ...) \in S, e_i \in \mathbb{C}$. Let $\|y_k\| \le \|x_k\|$. Then

$$\|A_i(\Delta^{(\alpha)}y_k) - t_i e_i\| \le \|A_i(\Delta^{(\alpha)}x_k) - t_i e_i\|.$$

Since $\mathcal{M} = (M_i)$ is an increasing,

$$\frac{1}{h_r} \sum_{i\in I_r} M_i\Big(\frac{\|A_i(\Delta^{(\alpha)}y_k) - t_i e_i\|}{\rho^{(i)}}\Big) \le \frac{1}{h_r} \sum_{i\in I_r} M_i\Big(\frac{\|A_i(\Delta^{(\alpha)}x_k) - t_i e_i\|}{\rho^{(i)}}\Big)$$

Consequently, $y = (y_k) \in N_\theta(S, A, \Delta^{(\alpha)}, \mathcal{M})$. This completes the proof of the theorem. Similarly, we can prove that $N_\theta^0(S, A, \Delta^{(\alpha)}, \mathcal{M})$ is normal space. □

Theorem 2.3. The spaces $N_\theta(S, A, \Delta^{(\alpha)}, \mathcal{M})$ and $N^0_\theta(S, A, \Delta^{(\alpha)}, \mathcal{M})$ are paranormed spaces, with respect to the paranorm
$\|x\|_{\Delta^{(\alpha)}} =$

$$\inf\Big\{\rho^{(i)} > 0 : M_i\Big(\frac{\|a_{i0}x_1\|}{\rho^{(i)}}\Big) + \sup_{r\geq 1}\frac{1}{h_r}\sum_{i\in I_r} M_i\Big(\frac{\|A_i(\Delta^{(\alpha)}x_k)\|}{\rho^{(i)}}\Big) \leq 1,\ \ \rho^{(i)} \geq 0\Big\}.$$

Proof . It is easy to prove, so we omit the details. □

3 Relation between the spaces $N_\theta(S, A, \Delta^{(\alpha)})$ and $N_\theta(S, A, \Delta^{(\alpha)}, \mathcal{M})$

In this section we study relation between $N_\theta(S, A, \Delta^{(\alpha)})$ and $N_\theta(S, A, \Delta^{(\alpha)}, \mathcal{M})$.

Theorem 3.1. Let $A = (a_{ik})$ be an infinite matrix of complex numbers, $\mathcal{M} = (M_i)$ be Musielak-Orlicz function satisfying Δ_2 condition. If $x = (x_k)$ is $\Delta^{(\alpha)}$-lacunary strong (A)-convergent to s, with respect to $\mathcal{M}$ and $(S, \|.\|)$ is a normal Banach space, then $N_\theta(S, A, \Delta^{(\alpha)}) \subset N_\theta(S, A, \Delta^{(\alpha)}, \mathcal{M})$.

Proof . Let $x = (x_k) \in N_\theta(S, A, \Delta^{(\alpha)})$ and $(x_k) \xrightarrow{\Delta^{(\alpha)}} s$, where $t = (t_1, t_2, ...) \in S, e_i \in \mathbb{C}$. Then

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}\Big(\frac{\|A_i(\Delta^{(\alpha)}x_k) - t_ie_i\|}{\rho}\Big) = 0 \ \text{ for some } \ \rho > 0.$$

We define two sequences $y = (y_k)$ and $z = (z_k)$ such that

$$u_i(\|A_i(\Delta^{(\alpha)}y_k) - t_ie_i\|) = \begin{cases} u_i(\|A_i(\Delta^{(\alpha)}x_k) - t_ie_i\|), & \text{if } u_i(\|A_i(\Delta^{(\alpha)}x_k) - t_ie_i\|) > 1,\\ \theta, & \text{if } u_i(\|A_i(\Delta^{(\alpha)}x_k) - t_ie_i\|) \leq 1,\end{cases}$$

and

$$u_i(\|A_i(\Delta^{(\alpha)}z_k) - t_ie_i\|) = \begin{cases} \theta, & \text{if } (\|A_i(\Delta^{(\alpha)}x_k) - t_ie_i\|) > 1,\\ (\|A_i(\Delta^{(\alpha)}x_k) - t_ie_i\|), & \text{if } (\|A_i(\Delta^{(\alpha)}x_k) - t_ie_i\|) \leq 1.\end{cases}$$

Hence

$$(\|A_i(\Delta^{(\alpha)}x_k) - t_ie_i\|) = (\|A_i(\Delta^{(\alpha)}y_k) - t_ie_i\|) + (\|A_i(\Delta^{(\alpha)}z_k) - t_ie_i\|).$$

Now,

$$(\|A_i(\Delta^{(\alpha)}y_k) - t_ie_i\|) \leq (\|A_i(\Delta^{(\alpha)}x_k) - t_ie_i\|)$$

and

$$(\|A_i(\Delta^{(\alpha)}z_k) - t_ie_i\|) \leq (\|A_i(\Delta^{(\alpha)}x_k) - t_ie_i\|).$$

Since $N_\theta(S, A, \Delta^{(\alpha)})$ is normal, $y = (y_k),\ z = (z_k) \in N_\theta(S, A, \Delta^{(\alpha)})$. Let $\sup_i M_i(2) = T$. Then

$$\frac{1}{h_r}\sum_{i\in I_r} M_i\Big(\frac{\|A_i(\Delta^{(\alpha)}x_k)-t_ie_i\|}{\rho^{(i)}}\Big)$$

$$\begin{aligned}
&= \frac{1}{h_r}\sum_{i\in I_r} M_i\Big(\frac{\|A_i(\Delta^{(\alpha)}y_k)-t_ie_i\|+\|A_i(\Delta^{(\alpha)}z_k)-t_ie_i\|}{\rho^{(i)}}\Big)\\
&\leq \frac{1}{h_r}\sum_{i\in I_r}\frac{1}{2}M_i\Big(2\frac{\|A_i(\Delta^{(\alpha)}y_k)-t_ie_i\|}{\rho^{(i)}}\Big)\\
&+ \frac{1}{2}M_i\Big(2\frac{\|A_i(\Delta^{(\alpha)}z_k)-t_ie_i\|}{\rho^{(i)}}\Big)\\
&< \frac{1}{2}\frac{1}{h_r}\sum_{i\in I_r}K_1\Big(\frac{\|A_i(\Delta^{(\alpha)}y_k)-t_ie_i\|}{\rho^{(i)}}\Big)M_i(2)\\
&+ \frac{1}{2}\frac{1}{h_r}\sum_{i\in I_r}K_2\Big(\frac{\|A_i(\Delta^{(\alpha)}z_k)-t_ie_i\|}{\rho^{(i)}}\Big)M_i(2)\\
&\leq \frac{1}{2}\frac{1}{h_r}\sum_{i\in I_r}K_1\Big(\frac{\|A_i(\Delta^{(\alpha)}y_k)-t_ie_i\|}{\rho^{(i)}}\Big)\sup M_i(2)\\
&+ \frac{1}{2}\frac{1}{h_r}\sum_{i\in I_r}K_2\Big(\frac{\|A_i(\Delta^{(\alpha)}z_k)-t_ie_i\|}{\rho^{(i)}}\Big)\sup M_i(2)\\
&\to 0 \text{ as } r\to\infty.
\end{aligned}$$

Hence $x=(x_k)\in N_\theta(S,A,\Delta^{(\alpha)},\mathcal{M})$. This completes the proof of the theorem.

□

Theorem 3.2. Let $A=(a_{ik})$ be an infinite matrix of complex numbers, $\mathcal{M}=(M_i)$ be Musielak-Orlicz function satisfying Δ_2-condition. If

$$\lim_{u\to\infty}\inf_i \frac{M_i(\frac{v}{\rho^{(i)}})}{\frac{v}{\rho^{(i)}}}>0 \text{ for some } \rho^{(i)}>0,$$

then $N_\theta(S,A,\Delta^{(\alpha)})=N_\theta(S,A,\Delta^{(\alpha)},\mathcal{M})$.

Proof . If $\lim_{v\to\infty}\inf_i \frac{M_i(\frac{v}{\rho^{(i)}})}{\frac{v}{\rho^{(i)}}}>0$ for some $\rho^{(i)}>0$, then there exists a number $\zeta>0$ such that

$$M_i(\frac{v}{\rho^{(i)}})\geq\zeta(\frac{v}{\rho^{(i)}}) \text{ for all } v>0 \text{ and some } \rho^{(i)}>0.$$

Let $x=(x_k)\in N_\theta(S,A,\Delta^{(\alpha)},\mathcal{M})$ and $(x_k)\stackrel{\Delta^{(\alpha)}}{\longrightarrow}t$, where $t=(t_1,t_2,...)\in S, e_i\in\mathbb{C}$. Then clearly

$$\begin{aligned}
\frac{1}{h_r}\sum_{i\in I_r}M_i\Big(\frac{\|A_i(\Delta^{(\alpha)}x_k)-t_ie_i\|}{\rho^{(i)}}\Big)\Big] &\geq \frac{1}{h_r}\sum_{i\in I_r}u_i\Big[\zeta\Big(\frac{\|A_i(\Delta^m_n x_k)-t_ie_i\|}{\rho^{(i)}}\Big)\\
&= \zeta\frac{1}{h_r}\sum_{i\in I_r}\Big(\frac{\|A_i(\Delta^{(\alpha)}x_k)-t_ie_i\|}{\rho^{(i)}}\Big)
\end{aligned}$$

Hence $x=(x_k)\in N_\theta(S,A,\Delta^{(\alpha)})$. This completes the proof. □

4 Inclusion relation between the spaces $|\Delta^{(\alpha)}\sigma_1(A)|$ and $N_\theta(S,A,\Delta^{(\alpha)})$

A sequence $x=(x_k)$ is said to be $\Delta^{(\alpha)}$-lacunary strong (A)-convergent with respect to a Musielak-Orlicz fumction $\mathcal{M}=(M_k)$ if there is a number $t=(t_1,t_2,...)\in E$, such that $x=(x_k)\in N_\theta(S,A,\Delta^{(\alpha)},\mathcal{M})$.

We have generalized the strongly Cesaro-summable sequence space into $\Delta^{(\alpha)}$-strongly Cesaro-summable vector-valued sequence space as

$$|\Delta^{(\alpha)}\sigma_1(A)| = \Big\{x = x_k : \text{ there exists } L = (L_1, L_2, ...) \in S, e_i \in \mathbb{C} \text{ such that } \frac{1}{n}\sum_{i=1}^{n} \|A_i(\Delta^{(\alpha)}x_k) - L_i e_i\| \to 0\Big\},$$

where $A = (a_{nk})$ is a Cesaro matrix, i.e.,

$$a_{nk} = \begin{cases} \frac{1}{n}, & \text{if } 1 \leq k \leq n, \\ 0, & \text{if } k \geq n. \end{cases}$$

Then it can be shown that $|\Delta^{(\alpha)}\sigma_1(A)|$ is a paranormed space with respect to the paranorm

$$\|x\| = \|x_1\| + \sup_n \Big(\frac{1}{n}\sum_{i=1}^{n} \|A_i(\Delta^{(\alpha)}x_k)\|\Big).$$

In this section of the paper we study relation between the spaces $|\Delta^{(\alpha)}\sigma_1(A)|$ and $N_\theta(S, A, \Delta^{(\alpha)})$.

Theorem 4.1. $|\Delta^{(\alpha)}\sigma_1(A)| \subset N_\theta(S, A, \Delta^{(\alpha)})$ if and only if $\liminf_r q_r > 1$.

Proof . First we assume that $\liminf_r q_r > 1$. Then there exist $\delta > 0$ such that $1 + \delta \leq q_r$ for all $r \geq 1$. Let $x \in |\Delta^{(\alpha)}\sigma_1(A)|$. Then

$$\begin{aligned} \frac{1}{h_r}\sum_{i\in I_r} \|A_i(\Delta^{(\alpha)}x_k)\| &= \frac{1}{h_r}\sum_{i=1}^{k_r} \|A_i(\Delta^{(\alpha)}x_k)\| - \frac{1}{h_r}\sum_{i=1}^{k_{r-1}} \|A_i(\Delta^{(\alpha)}x_k)\| \\ &= \frac{k_r}{h_r}\Big(\frac{1}{k_r}\sum_{i=1}^{k_r} \|A_i(\Delta^{(\alpha)}x_k)\|\Big) - \frac{k_{r-1}}{h_r}\Big(\frac{1}{k_{r-1}}\sum_{i=1}^{k_{r-1}} \|A_i(\Delta^{(\alpha)}x_k)\|\Big). \end{aligned}$$

Now, $h_r = k_r - k_{r-1}$. So we have

$$\frac{k_r}{h_r} = \frac{k_r}{k_r - k_{r-1}} = \frac{q_r}{q_r - 1} = 1 + \frac{1}{q_r - 1} \leq 1 + \frac{1}{\delta} = \frac{\delta + 1}{\delta}.$$

Also

$$\frac{k_{r-1}}{h_r} = \frac{k_{r-1}}{k_r - k_{r-1}} = \frac{1}{q_r - 1} \leq \frac{1}{\delta}.$$

Since $x \in |\Delta^{(\alpha)}\sigma_1(A)|^0$, then

$$\frac{1}{h_r}\sum_{i=1}^{k_r} \|A_i(\Delta^{(\alpha)}x_k)\| \to 0 \text{ and } \frac{1}{h_r}\sum_{i=1}^{k_{r-1}} \|A_i(\Delta^{(\alpha)}x_k)\| \to 0,$$

and hence

$$\frac{1}{h_r}\sum_{i\in I_r} \|A_i(\Delta^{(\alpha)}x_k)\| \to 0$$

i.e., $x = (x_k) \in N_\theta^0(S, A, \Delta^{(\alpha)})$. By linearity, it follows that $|\Delta^{(\alpha)}\sigma_1(A)| \subset N_\theta(S, A, \Delta^{(\alpha)})$.
Next, suppose that $\liminf_r q_r = 1$. Since θ is lacunary we can select a subsequence k_{r_j} of θ such that

$$\frac{k_{r_j}}{k_{r_j - 1}} < 1 + \frac{1}{j} \text{ and } \frac{k_{r_j - 1}}{k_{r_{j-1}}} > j$$

where $r_j \geq r_{j-1} + 2$. Define $x = (x_i)$ by

$$\Delta^{(\alpha)} x_i = \begin{cases} e_i, & \text{if } i \in I_{r_j}, \text{ for some } j = 1, 2, ..., \\ \theta, & \text{otherwise,} \end{cases}$$

where $\|e_i\| = 1$ and let $A = I$, then for any $L = (L_1, L_2, \cdots) \in E, e_i \in \mathbb{C}$,

$$\frac{1}{h_{r_j}} \sum_{i \in I_r} \Big(\frac{\|A_i(\Delta^{(\alpha)} x_k) - L\|}{\rho} \Big) = \frac{\|e_i - L_i e_i\|}{\rho} = \frac{\|1 - L_i\|}{\rho} \text{ for } j = 1, 2, ...$$

and

$$\frac{1}{h_r} \sum_{i \in I_r} \Big(\frac{\|A_i(\Delta^{(\alpha)} x_k)\|}{\rho} \Big) = \frac{\|e_i\|}{\rho} = \frac{1}{\rho}.$$

So, $x = (x_k) \notin N_\theta(S, A, \Delta^{(\alpha)})$. But $x = (x_k)$ is strongly Cesaro-summable, since if t is sufficiently large integer we can find the unique j for which $k_{r_j - 1} < t \leq k_{r_{j+1} - 1}$ and hence
$\frac{1}{t} \sum_{i=1}^{t} \Big(\|A_i(\Delta^{(\alpha)} x_k)\| \Big) < \frac{1}{k_{r_j - 1}} \sum_{i=1}^{t} 1 \leq \frac{1}{k_{r_j - 1}} k_{r_j} \leq \frac{k_{r_j - 1} + h_{r_j}}{k_{r_j - 1}} < \frac{1}{j} + \frac{1}{j} = \frac{2}{j}$, as $t \to \infty$, it follows that also $j \to \infty$. Hence $x = (x_k) \in |\Delta^{(\alpha)} \sigma_1(A)|$. □

Theorem 4.2. $N_\theta(S, A, \Delta^{(\alpha)}) \subset |\Delta^{(\alpha)} \sigma_1(A)|$ iff $\limsup_r q_r < \infty$.

Proof . First we assume that if $\limsup_r q_r < \infty$, there exists $M > 0$ such that $q_r < M$ for all $r \geq 1$. Let $x = (x_k) \in N_\theta(S, A, \Delta^{(\alpha)})$ and $\epsilon > 0$. Then

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{i \in I_r} \Big(\frac{\|A_i(\Delta^{(\alpha)} x_k)\|}{\rho} \Big) = 0 \text{ for some } \rho > 0.$$

Then we can find $R > 0$ and $K > 0$ such that

$$\sup_{j \geq R} \frac{1}{h_j} \sum_{I_j} \Big(\frac{\|A_i(\Delta^{(\alpha)} x_k)\|}{\rho} \Big) < \epsilon$$

and

$$\frac{1}{h_j} \sum_{I_j} \Big(\frac{\|A_i(\Delta^{(\alpha)} x_k)\|}{\rho} \Big) < K \text{ for all } i = 1, 2, ...$$

Then if t is any integer with

$$k_{r-1} \leq t \leq k_r, \text{ where } r > R,$$

then

$$\begin{aligned}
\tfrac{1}{t}\sum_{j=1}^{t}\Big(\frac{\|A_i(\Delta^{(\alpha)}x_k)\|}{\rho}\Big) & \\
&\leq \frac{1}{k_{r-1}}\sum_{i=1}^{k_r}\Big(\frac{\|A_i(\Delta^{(\alpha)}x_k)\|}{\rho}\Big)\\
&= \frac{1}{k_{r-1}}\Big(\sum_{I_1}\Big(\frac{\|A_i(\Delta^{(\alpha)}x_k)\|}{\rho}\Big)+\sum_{I_2}\Big(\frac{\|A_i(\Delta^{(\alpha)}x_k)\|}{\rho}\Big)+\cdots+\sum_{I_r}\Big(\frac{\|A_i(\Delta^{(\alpha)}x_k)\|}{\rho}\Big)\Big)\\
&= \frac{k_1}{k_{r-1}}\frac{1}{h_1}\sum_{I_1}\Big(\frac{\|A_i(\Delta^{(\alpha)}x_k)\|}{\rho}\Big)+\frac{k_2-k_1}{k_{r-1}}\frac{1}{h_2}\sum_{I_2}\Big(\frac{\|A_i(\Delta^{(\alpha)}x_k)\|}{\rho}\Big)+\cdots\\
&+ \frac{k_R-k_{R-1}}{k_{r-1}}\frac{1}{h_R}\sum_{I_R}\Big(\frac{\|A_i(\Delta^{(\alpha)}x_k)\|}{\rho}\Big)+\frac{k_{R+1}-k_R}{k_{r-1}}\frac{1}{h_{R+1}}\sum_{I_{R+1}}\Big(\frac{\|A_i(\Delta^{(\alpha)}x_k)\|}{\rho}\Big)\\
&+ \cdots+\frac{k_r-k_{r-1}}{k_{r-1}}\frac{1}{h_r}\sum_{I_r}\Big(\frac{\|A_i(\Delta^{(\alpha)}x_k)\|}{\rho}\Big)\\
&\leq \frac{k_R}{k_{r-1}}\sup_{i\geq 1}\frac{1}{h_i}\sum_{I_i}\Big(\frac{\|A_i(\Delta^{(\alpha)}x_k)\|}{\rho}\Big)+\frac{k_r-k_R}{k_{r-1}}\frac{1}{h_r}\sum_{I_r}\Big(\frac{\|A_i(\Delta^{(\alpha)}x_k)\|}{\rho}\Big)\\
&< K\frac{k_R}{k_{r-1}}+\epsilon\Big(q_r-\frac{k_R}{k_{r-1}}\Big)\\
&< K\frac{k_R}{k_{r-1}}+\epsilon q_r\\
&< K\frac{k_R}{k_{r-1}}+\epsilon M.
\end{aligned}$$

Since $k_{r-1}\to\infty$ as $r\to\infty$, it follows that

$$\frac{1}{t}\sum_{j=1}^{t}\Big(\frac{\|A_i(\Delta^{(\alpha)}x_k)\|}{\rho}\Big)\to 0$$

and hence $x=(x_k)\in|\Delta^{(\alpha)}\sigma_1(A)|$.

Next, suppose that $\limsup_r q_r=\infty$. We construct a sequence in $N_\theta(S,A,\Delta^{(\alpha)})$ that is not Cesaro $\Delta^{(\alpha)}$-summable. By the idea of Freedman [7] we can construct a subsequence k_{r_j} of the lacunary sequence $\theta=(k_r)$ such that $q_{r_j}>j$, and then define a bounded difference sequence $x=(x_i)$ by

$$\Delta^{(\alpha)}x_i=\begin{cases} e_i, & \text{if } k_{r_j-1}<i<2k_{r_j-1},\\ \theta, & \text{otherwise,}\end{cases}$$

where $\|e_i\|=1$. Let $A=I$ and $\rho=1$. Then

$$\frac{1}{h_{r_j}}\sum_{I_{r_j}}\Big(\|A_i(\Delta^{(\alpha)}x_k)\|\Big)=\frac{2k_{r_j-1}-k_{r_j-1}}{k_{r_j}-k_{r_j-1}}=\frac{k_{r_j-1}}{k_{r_j}-k_{r_j-1}}<\frac{1}{j-1}$$

and if $r\neq r_j$,

$$\frac{1}{h_{r_j}}\sum_{I_{r_j}}\Big(\|A_i(\Delta^{(\alpha)}x_k)\|\Big)=0.$$

Thus $x=(x_k)\in N_\theta(E,A,\Delta^{(\alpha)})$. For the above sequence and for $i=1,2,...,k_{r_j}$

$$\begin{aligned}
\frac{1}{k_{r_j}}\sum_{i}\Big(\|A_i(\Delta^{(\alpha)}x_k)-e_i\|\Big) &> \frac{1}{k_{r_j}}(2k_{r_j-1}-k_{r_j-1})\\
&= 1-\frac{2}{q_{r_j}}>1-\frac{2}{j},
\end{aligned}$$

this converges to 1, but for $i = 1, 2, ..., 2k_{r_j - 1}$

$$\frac{2}{k_{r_{j-1}}} \sum_i \Big(\|A_i(\Delta^{(\alpha)} x_k)\| \Big) \geq \frac{k_{r_j - 1}}{2k_{r_j - 1}} = \frac{1}{2}.$$

It proves that $x = (x_k) \notin |\Delta^{(\alpha)} \sigma_1(A)|$, since any sequence in $|\Delta^{(\alpha)} \sigma_1(A)|$ consisting of θ's and e_i's has an limit only 0 or 1. □

Theorem 4.3. Let θ be a lacunary sequence. Then $|\Delta^{(\alpha)} \sigma_1(A)| = N_\theta(S, A, \Delta^{(\alpha)})$ if and only if $1 \leq \liminf_r q_r \leq \limsup_r q_r < \infty$.

Proof . The proof of the theorem follows from Lemma 4.1 and Lemma 4.2. □

5 Statistical Convergence

Many authors have studied the concept of statistical convergence we may refer to [4, 6, 8, 9, 11, 13, 14, 16, 23, 24, 25] and reference therein.

A sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\epsilon > 0$

$$\lim_{n \to \infty} \frac{1}{n} \Big| \{ k \leq n : |x_k - L| \geq \epsilon \} \Big| = 0.$$

Bilgin [3] also introduced the concept of statistical convergence in $N_0(A, F)$ and proved some inclusion relation.

Let θ be a lacunary sequence and $A = (a_{ik})$ be an infinite matrix of complex numbers. Then a sequence $x = (x_k) \in N_\theta(S, A, \Delta^{(\alpha)}, \mathcal{M})$ is said to be $\Delta^{(\alpha)}$-lacunary (A)-statistically convergent to a number $t = (t_1, t_2, ...) \in S, e_i \in \mathbb{C}$ if for any $\epsilon > 0$,

$$\lim_{r \to \infty} \frac{1}{h_r} |\Delta^{(\alpha)} A_0(\epsilon)| = 0,$$

where

$$\Delta^{(\alpha)} A_0(\epsilon) = \Big\{ i \in I_r : M_i\Big(\frac{\|A_i(\Delta^{(\alpha)} x_k) - t_i e_i\|}{\rho^{(i)}} \Big) \geq \epsilon \Big\}.$$

We denote it as $(x_k) \overset{\Delta^{(\alpha)} - \text{stat}}{\longrightarrow} s$. The vertical bar denotes the cardinality of the set. The set of all $\Delta^{(\alpha)}$-lacunary (A)-statistical convergent sequences is denoted by $\Delta^{(\alpha)} S_\theta(A)$.

In this section we study some relation between the spaces $|\Delta^{(\alpha)} S_\theta(A)|$ and $N_\theta(S, A, \Delta^{(\alpha)}, \mathcal{M})$.

Theorem 5.1. Let $\mathcal{M} = (M_i)$ be Musielak-Orlicz function and (M_i) be pointwise convergent. Then $N_\theta(S, A, \Delta^{(\alpha)}, \mathcal{M}) \subset \Delta^{(\alpha)} S_\theta(A)$ if and only if $\lim_i M_i(\frac{v}{\rho^{(i)}}) > 0$ for some $v > 0, \ \ \rho^{(i)} > 0$.

Proof . Let $\epsilon > 0$ and $x = (x_k) \in N_\theta(S, A, \Delta^{(\alpha)}, \mathcal{M})$. Let $(x_k) \overset{\Delta^{(\alpha)}}{\longrightarrow} t$, where $t = (t_1, t_2, ...) \in S, \quad e_i \in \mathbb{C}$. Since $\lim_i M_i(\frac{v}{\rho}) > 0$, there exists a number $c > 0$ such that

$$M_i(\frac{v}{\rho}) \geq c \ \text{ for } \ v > \epsilon.$$

Let

$$I_r^1 = \Big\{ i \in I_r : \Big[M_i\Big(\frac{\|A_i(\Delta^{(\alpha)} x_k) - t_i e_i\|}{\rho^{(i)}} \Big) \Big] \geq \epsilon \Big\}.$$

Then

$$\begin{aligned} \frac{1}{h_r} \sum_{i \in I_r} \Big[M_i\Big(\frac{\|A_i(\Delta^{(\alpha)} x_k) - t_i e_i\|}{\rho^{(i)}} \Big) \Big] \quad &\geq \quad \frac{1}{h_r} \sum_{i \in I_r^1} \Big[M_i\Big(\frac{\|A_i(\Delta^{(\alpha)} x_k) - t_i e_i\|}{\rho^{(i)}} \Big) \Big] \\ &\geq \quad c \frac{1}{h_r} |\Delta^{(\alpha)} A_0(\epsilon)|. \end{aligned}$$

Hence it follows that $x=(x_k)\in \Delta^{(\alpha)}S_\theta(A)$.
Conversely, let us assume that the condition does not hold good. Then there is a number $v>0$ such that $\lim_i M_i(\frac{v}{\rho})=0$ for some $\rho>0$. Now, we select a lacunary sequence $\theta=(k_r)$ such that $M_i(\frac{v}{\rho})<2^{-r}$ for any $i>k_r$.
Let $A=I$, define the sequence $x=(x_k)$ by putting

$$\Delta^{(\alpha)}x_i=\begin{cases} v, & \text{if } k_{r-1}<i\leq \frac{k_r+k_{r-1}}{2},\\ \theta, & \text{if } \frac{k_r+k_{r-1}}{2}<i\leq k_r.\end{cases}$$

Therefore,

$$\begin{aligned}\frac{1}{h_r}\sum_{i\in I_r}M_i\Big(\frac{\|\Delta^{(\alpha)}x_i\|}{\rho^{(i)}}\Big) &= \frac{1}{h_r}\sum_{k_{r-1}<i\leq \frac{k_r+k_{r-1}}{2}}M_i\Big(\frac{v}{\rho^{(i)}}\Big)\\ &< \frac{1}{h_r}\frac{1}{2^{r-1}}\Big[\frac{k_r+k_{r-1}}{2}-k_{r-1}\Big]\\ &= \frac{1}{2^r}\to 0 \text{ as } r\to\infty.\end{aligned}$$

Thus, we have $x=(x_k)\in N^0_\theta(S,A,\Delta^{(\alpha)},\mathcal{M})$. But

$$\begin{aligned}\lim_{r\to\infty}\frac{1}{h_r}\Big|\Big\{i\in I_r: M_i\Big(\frac{\|\Delta^{(\alpha)}x_i\|}{\rho^{(i)}}\Big)\geq\epsilon\Big\}\Big| &= \lim_{r\to\infty}\frac{1}{h_r}\Big|\Big\{i\in (k_{r-1},\frac{k_r+k_{r-1}}{2}): M_i\Big(\frac{v}{\rho^{(i)}}\Big)\geq\epsilon\Big\}\Big|\\ &= \lim_{r\to\infty}\frac{1}{h_r}\frac{k_r-k_{r-1}}{2}=\frac{1}{2}.\end{aligned}$$

So $x=(x_k)\notin \Delta^{(\alpha)}S_\theta(A)$. □

Theorem 5.2. Let $\mathcal{M}=(M_i)$ be Musielak-Orlicz function. Then $\Delta^{(\alpha)}S_\theta(A)\subset N_\theta(S,A,\Delta^{(\alpha)},\mathcal{M})$ if and only if $\sup_v\sup_i M_i(\frac{v}{\rho})<\infty$.

Proof . Let $x=(x_k)\in \Delta^{(\alpha)}S_\theta(A)$ and $(x_k)\overset{\Delta^{(\alpha)}-\text{stat}}{\longrightarrow} t$. Suppose $h(v)=\sup_i M_i(\frac{v}{\rho})$ and $h=\sup_v h(v)$. Let

$$I^2_r=\Big\{i\in I_r: M_i\Big(\frac{\|A_i(\Delta^{(\alpha)}x_k)-t_ie_i\|}{\rho^{(i)}}\Big)<\epsilon\Big\}.$$

Now, $M_i(v)\leq h$ for all i, $v>0$. So

$$\begin{aligned}\frac{1}{h_r}\sum_{i\in I_r}\Big[M_i\Big(\frac{\|A_i(\Delta^{(\alpha)}x_k)-t_ie_i\|}{\rho^{(i)}}\Big)\Big] &= \frac{1}{h_r}\sum_{i\in I^1_r}\Big[M_i\Big(\frac{\|A_i(\Delta^{(\alpha)}x_k)-t_ie_i\|}{\rho^{(i)}}\Big)\Big]\\ &+ \frac{1}{h_r}\sum_{i\in I^2_r}\Big[M_i\Big(\frac{\|A_i(\Delta^{(\alpha)}x_k)-t_ie_i\|}{\rho^{(i)}}\Big)\Big]\\ &\leq h\frac{1}{h_r}|\Delta^{(\alpha)}A_0(\epsilon)|+h(\epsilon).\end{aligned}$$

Hence, as $\epsilon\to 0$, it follows that $x=(x_k)\in N_\theta(S,A,\Delta^{(\alpha)},\mathcal{M})$.
Conversely, suppose that

$$\sup_v\sup_i M_i(\frac{v}{\rho})=\infty.$$

Then we have

$$0<v_1<v_2<...<v_{r-1}<v_r<...,$$

so that $M_{k_r}(\frac{v_r}{\rho})\geq h_r$ for $r\geq 1$. Let $A=I$. We set a sequence $x=(x_i)$ by

$$\Delta^{(\alpha)}x_i=\begin{cases} v_r, & \text{if } i=k_r \text{ for some } r=1,2,...,\\ \theta, & \text{otherwise.}\end{cases}$$

Then

$$\lim_{r\to\infty}\frac{1}{h_r}\Big|\Big\{i\in I_r:\Big[M_i\Big(\frac{\|\Delta^{(\alpha)}x_i\|}{\rho^{(i)}}\Big)\Big]\geq\epsilon\Big\}\Big|=\lim_{r\to\infty}\frac{1}{h_r}=0.$$

Hence $(x_k)\overset{\Delta^{(\alpha)}-\text{stat}}{\longrightarrow}0$ and hence $x=(x_k)\in\Delta^{(\alpha)}S_\theta(A)$.
But

$$\begin{aligned}\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}\Big[M_i\Big(\frac{\|A_i(\Delta^{(\alpha)}x_i)-t_ie_i\|}{\rho^{(i)}}\Big)\Big] &= \lim_{r\to\infty}\frac{1}{h_r}\Big[M_{k_r}\Big(\frac{v_r-t_ie_i\|}{\rho^{(i)}}\Big)\Big]\\ &\geq \lim_{r\to\infty}\frac{1}{h_r}h_r=1.\end{aligned}$$

So, $x=(x_k)\notin N_\theta(S,A,\Delta^{(\alpha)},\mathcal{M})$. □

Conclusion and Future works: We have introduced here some new lacunary strong convergent vector valued sequence spaces defined by fractional difference operator and Musielak-Orlicz function. We have studied their topological properties and proved some inclusion relations between these newly defined spaces.

In future we have to study some new lacunary strong convergent vector valued sequence spaces defined by fractional difference operator of order (α,β) and also try to study these sequence spaces over n-normed spaces.

Acknowledgement

I would like to thanks reviewers for their valuable suggestions and comments which improve the presentation of the paper.

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