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On the set of Gâteaux differentiability of the L^1 norm

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Abstract

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space. In this paper, we establish the set of Gâteaux differentiability for the usual norm of $L^1(\Omega, \mu)$ and the corresponding derivative formulae at each point in this set.

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1 Introduction

The investigation on the sets of differentiability of Lipschitz functions goes back to the classical Rademacher's Theorem [16] of 1919. This theorem states that:

Every Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable almost everywhere.

Regarding the converse of Rademacher's Theorem it has been recently established by Preiss and Speight (cf. [14]) that: if $n > 1$ then there exists a Lebesgue null set in $\mathbb{R}^n$ containing a point of differentiability of each Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^{n-1}$. A more detailed discussion around the Rademacher's Theorem and some converse versions can be found in [1]. An extension of Rademacher's Theorem to functions defined on separable Banach spaces and valued into Banach spaces was established by Phelps in [11]:

Theorem 1.1 (Phelps). Let E be a separable real Banach space and F a real Banach space with the Radon-Nikodým property (RNP). If $T : G \to F$ is a locally Lipschitz mapping defined on a nonempty open subset G of E. Then T is Gâteaux differentiable in the complement of a Gaussian null subset of G.

Other extensions of Rademacher Theorem to Lipschitz maps between infinite-dimensional Banach spaces have been investigated by several authors e.g. Aronszajn [2], Bongiorno [3], Mankiewicz [10], Phelps [11], Preiss and Zajíček [15], among others). The research on the differentiability on Banach spaces has been a very active field in the last decades. We just quote some few recent examples: In Potapov and Sukachev's work [13], they answer to a question in the theory of Schatten- Von Neumann ideals of whether their norms have same differentiability properties as the norms of their commutative counterparts. On the other hand Lindenstrauss et al. [9] prove that a real valued Lipschitz function

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on an Asplund space has points of Fréchet differentiability. In in [8] Goldys and Gozzi apply the Phelps Theorem in the proof of existence and uniqueness of solutions of second order parabolic Hamilton-Jacobi-Bellman equations in Hilbert spaces, obtaining optimal feedback for first order stochastic PDEs that arise in economic theory, the theory of population dynamics and financial models.

The study of the set of differentiability of Lipschitz functions on Banach spaces has been carried out for instance in Deville et al. [5], Diestel [6], Dore and Maleva [7] among others. For the $L^p(\Omega)$ space where Ω is a bounded domain and $1 < p < \infty$, it is well known (cf. [6]) that the norm $\|f\|_{L^p} = (\int_\Omega |f(w)|^p dw)^{1/p}$ is Fréchet differentiable and moreover its derivative is given by

$$\partial_F \|f\|_{L^p} = \frac{|f|^{p-1} \operatorname{sign}(f)}{\|f\|_{L^p}^{p-1}}, \tag{1.1}$$

for any $f \neq 0$. The case of Fréchet differentiability for $p = 1$ is more pathological. Here we will explicitly find the set of Gâteaux differentiability for the $L^1(\Omega, \mu)$ norm, where $(\Omega, \mathcal{M}, \mu)$ is a measure space. The characterization of this set is given in Theorem 3.3 of Section 3.

2 Preliminaries

In this section we briefly give a basic background in our context. We start with the two most classical notions of differentiability on Banach spaces. The most simple one is that of Gâteaux differentiability. Let X, Y be Banach spaces and $\Omega \subseteq X$ an open set. A function $f : \Omega \to Y$ is *Gâteaux differentiable (G-differentiable)* at a point $a \in \Omega$, if

$$\lim_{t \to 0} \frac{f(a + th) - f(a)}{t} = u(h)$$

exists for all $h \in X$, and u defines a bounded linear operator from X into Y. The operator u is called the Gâteaux derivative of f at the point a and is denoted by $\partial_G f(a)$.

A function $f : \Omega \to Y$ is *Fréchet differentiable (F-differentiable)* at a, if there is a bounded linear operator u such that

$$f(a + h) - f(a) - u(h) = r(h)$$

where $\lim_{h \to 0} \frac{\|r(h)\|_Y}{\|h\|_X} = 0$. The operator u is called the Fréchet derivative of f at the point a and is denoted by $\partial_F f(a)$.

It is not difficult to see that, if f is F-differentiable at a then f is G-differentiable at a and in that case $\partial_F f(a) = \partial_G f(a)$. For a given function $\varphi : X \to Y$, we say that B is the set of G-differentiability or F-differentiability of φ, if respectively for every element of the set B the function φ is G-differentiable or F-differentiable.

On the other hand, a set is called a G_δ *set* if it can be expressed as a countable intersection of open sets. This type of set is fundamental in the definition of the *F-Asplund and G-Asplund* spaces, since these respectively are Banach spaces in which all convex and continuous function defined in an open and convex subset is F-differentiable (resp. G-differentiable) in a dense and G_δ set. The following theorem shows important examples of G-Asplund spaces.

Theorem 2.1 (Mazur). Every separable Banach space is a G-Asplund space.

Moreover, it is well known that the set of all F-Asplund spaces is strictly contained in the set of all G-Asplund spaces. In fact, as proved in Deville et al. [5], the usual norm of $\ell^1(\mathbb{R})$ is not F-differentiable at any point of the space $\ell^1(\mathbb{R})$. Then $\ell^1(\mathbb{R})$ is not a F-Asplund space; but $\ell^1(\mathbb{R})$ is a separable Banach space and therefore a G-Asplund space by Mazur's theorem.

As we have quoted before, the L^p norm is F-differentiable at any point in the case $1 < p < \infty$ and its corresponding derivative is given by (1.1). Moreover, since the dual of $L^p(\Omega)$ is $L^q(\Omega)$, for $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, and they are separable Banach spaces, by Theorem 2.12 in Phelps [12] the space $L^p(\Omega)$ is a F-Asplund space for $1 < p < \infty$.

Remark 2.2. In the special case of the $L^1(\Omega, \mathcal{M}, \mu)$ norm we observe the following regarding the set of G-differentiability. It is well known that(cf. [4]) $L^1(\Omega, \mathcal{M}, \mu)$ is separable provided μ is σ-finite and the σ-algebra $\mathcal{M}$ is countably generated. Since $\mathbb{R}$ has the RNP, then the $L^1(\Omega, \mu)$ norm satisfies the conditions of Phelps's Theorem. Therefore, the Gaussian measure of the complement of the G-differentiability set is zero and by Mazur's Theorem this G-differentiability set is dense and a G_δ set.

In the next section we will give an explicit description of the set of G-differentiability for the usual norm of $L^1(\Omega, \mathcal{M}, \mu)$ without assuming the condition of separability.

3 The G-differentiability set for the $L^1(\Omega,\mu)$ norm

Given a measure space $(\Omega,\mathcal{M},\mu)$, we denote by $\mathcal{L}^1(\Omega,\mu)$ the space of the real valued integrable functions over Ω. On $\mathcal{L}(\Omega,\mu)$ one defines the pseudonorm $\varphi(f)=\int_\Omega |f(x)|d\mu(x)$. One also defines on $\mathcal{L}(\mu)$ an equivalence relation $\mathcal{R}$ by, $f\mathcal{R}g \iff f=g\,\mu-a.e.$ The equivalence class of f is denoted by $\tilde{f}$. Thus, the quotient $\mathcal{L}(\Omega)/\mathcal{R}$ defines the classical Banach space $L^1(\Omega,\mu)$ endowed with the norm $\varphi(\tilde{f})=\int_\Omega |f(x)|d\mu(x)$.

We also recall recall that the signature $\operatorname{sign}(x)$ of a real number x, is defined by $\operatorname{sign}(x)=\frac{x}{|x|}$ if $x\neq 0$, and $\operatorname{sign}(x)=0$ if $x=0$.

In order to prepare the proof of the main theorem we first establish a mild lemma which will help to obtain a pointwise formulae for the Gâteaux derivative.

Lemma 3.1. Let $f,\, h:\Omega\to\mathbb{R}$ be functions and $x\in\Omega$ such that $h(x)\neq 0$. Then

$$\lim_{t\to 0}\frac{|f(x)+th(x)|-|f(x)|}{t}, \tag{3.1}$$

exists if and only if $f(x)\neq 0$.
Moreover, if that is the case

$$\lim_{t\to 0}\frac{|f(x)+th(x)|-|f(x)|}{t}=\operatorname{sign}(f(x))h(x). \tag{3.2}$$

Proof . If the limit (3.1) exists and $f(x)=0$, then the limit

$$\lim_{t\to 0}\frac{|th(x)|}{t}=\lim_{t\to 0}\frac{|t||h(x)|}{t}$$

does not exists, which is a contradiction, hence $f(x)\neq 0$.

We now assume $f(x)\neq 0$. Since $h(x)\neq 0$, we can choose $\delta=\frac{|f(x)|}{2|h(x)|}$ and observe that for $|t|<\delta$ we have

$$\operatorname{sign}(f(x)+th(x))=\operatorname{sign}(f(x)). \tag{3.3}$$

Hence

$$\begin{aligned}\frac{|f(x)+th(x)|-|f(x)|}{t} &= \frac{\operatorname{sign}(f(x)+th(x))(f(x)+th(x))-\operatorname{sign}(f(x))f(x)}{t}\\ &= \frac{\operatorname{sign}(f(x))th(x)}{t}\\ &= \operatorname{sign}(f(x))h(x).\end{aligned}$$

Therefore

$$\lim_{t\to 0}\frac{|f(x)+th(x)|-|f(x)|}{t}=\operatorname{sign}(f(x))h(x).$$

□

We now consider a measure space $(\Omega,\mathcal{M},\mu)$. As a consequence of the lemma above we obtain the following corollary.

Corollary 3.2. Let $f,h:\Omega\to\mathbb{R}$ be measurable functions such that $h(x)\neq 0$ μ-a.e. Then

$$\lim_{t\to 0}\frac{|f(x)+th(x)|-|f(x)|}{t}\text{ exists }\mu-a.e \tag{3.4}$$

if and only if $f(x)\neq 0$ μ-a.e. Moreover, if that is the case the formula (3.2) holds.

Proof . We suppose that $h(x)\neq 0$ for all $x\in A$, with $\mu(A^c)=0$ and first assume that the limit (3.4) exists μ-a.e. Hence, $f(x)\neq 0$ for every $x\in A$ by Lemma 3.1, thus $f(x)\neq 0$ μ-a.e.
On the other hand, if $f(x)\neq 0$ μ-a.e. Then there exists a set $D\in\mathcal{M}$ such that $\mu(D^c)=0$ and $f(x)\neq 0$ for all $x\in D$.

By taking $x \in A \cap D$ we have that $f(x), h(x) \neq 0$. Then by Lemma 3.1 the limit (3.4) exists for every $x \in A \cap D$. Therefore the limit (3.4) exists μ-a.e.
The last part of the statement clearly follows from the last part of Lemma 3.1 by applying the formula (3.2) in a set Γ where we have $f(x), h(x) \neq 0$ and $\mu(\Gamma^c) = 0$. □ We are now ready to present our main result. We consider a measure space $(\Omega, \mathcal{M}, \mu)$ and the following condition:

(A1) For every $I \in \mathcal{M}$ with $\mu(I) = \infty$, there exists a set $F \subset I$ with $F \in \mathcal{M}$ such that

$$0 < \mu(F) < \infty.$$

Every σ-finite measure that is not the trivial null measure satisfies the condition (A1). The counting measure on the power set of $\mathbb{R}$ satisfies the condition (A1) and is not σ-finite. On the other hand, every non-atomic measure also satisfies the condition (A1).

The following theorem gives a precise description of the G-differentiability set for the usual $L^1(\Omega, \mu)$ norm and the corresponding formulae for the Gâteaux derivative.

Let G be the subset of $L^1(\Omega, \mu)$ consisting of equivalence classes $\tilde{f}$ such that $f \neq 0$ μ-a.e., for some representative f of the class $\tilde{f}$. It is clear that if $\tilde{f} \in G$, any representative of the class $\tilde{f}$ satisfies the condition defining G.

Theorem 3.3. Let $(\Omega, \mathcal{M}, \mu)$ be a measure space satisfying the condition (A1). Let B be the subset of $L^1(\Omega, \mu)$ above defined. Then the set of G-differentiability of the norm φ is G. Moreover, the Gâteaux derivative of φ for $\tilde{f} \in G$ and the direction $h \in L^1(\Omega, \mu)$ is given by

$$\partial_G \varphi(\tilde{f})(h) = \int_\Omega \operatorname{sign}(f(x)) h(x) d\mu(x), \tag{3.5}$$

where f is any representative of the class $\tilde{f}$.

Proof . We assume $\tilde{f} \in G$ and that f is a representative of the class $\tilde{f}$. We start by observing that for any $h \in L^1(\Omega, \mu)$ the right hand side of the formula (3.5) is independent of the representative in the class $\tilde{f}$.

We are now going to prove the G-differentiability of φ at $\tilde{f}$. Let $h \in L^1(\Omega, \mu) \setminus \{0\}$, an application of the Lebesgue dominated convergence Theorem gives us

$$\begin{aligned}
\lim_{t \to 0} \frac{\varphi(f + th) - \varphi(f)}{t} &= \lim_{t \to 0} \frac{\int\limits_\Omega |f(x) + th(x)| d\mu(x) - \int\limits_\Omega |f(x)| d\mu(x)}{t} \\
&= \lim_{t \to 0} \frac{\int\limits_\Omega \left(|f(x) + th(x)| - |f(x)|\right) d\mu(x)}{t} \\
&= \int\limits_\Omega \lim_{t \to 0} \frac{(|f(x) + th(x)| - |f(x)|)}{t} d\mu(x).
\end{aligned}$$

Since $h(x) \neq 0$ μ-a.e. and $f(x) \neq 0$ μ-a.e., by applying Corollary 3.2 we have

$$\lim_{t \to 0} \frac{|f(x) + th(x)| - |f(x)|}{t} = \operatorname{sign}(f(x)) h(x)$$

μ-a.e., and

$$\lim_{t \to 0} \frac{\varphi(f + th) - \varphi(f)}{t} = \int_\Omega \operatorname{sign}(f(x)) h(x) d\mu(x).$$

We now assume that $\tilde{f} \in G^c$ and we will show that φ does not have G-derivative in $\tilde{f}$. For the set $Z_1 = \{x \in \Omega : f(x) = 0\}$ we have $\mu(Z_1) > 0$. If $\mu(Z_1) = \infty$ by condition (A1) we can choose a subset Z of Z_1 such that $0 < \mu(Z) < \infty$.

We consider $h = 1_Z$, thus $h \in L^1(\Omega, \mu) \setminus 0$. Since $h = 0$ on Z^c, for $t \in \mathbb{R}$ we have $|f + th| - |f| = |f| - |f| = 0$ on Z^c. Hence, for the calculation of the G-derivative of φ at f in the direction h, is enough to consider the integration on Z. For $t \neq 0$ we have

$$\begin{aligned}
\frac{\varphi(f+th)-\varphi(f)}{t} &= \frac{\left(\int_Z |f(x)+th(x)|d\mu(x) - \int_Z |f(x)|d\mu(x)\right)}{t} \\
&= \frac{\int_Z |t| d\mu(x)}{t} \\
&= \frac{|t|}{t}\mu(Z) \\
&= \operatorname{sign}(t)\mu(Z).
\end{aligned}$$

Hence, we have $(\partial_G^+\varphi)f(h) = \mu(Z)$ and $(\partial_G^+\varphi)f(h) = -\mu(Z)$. Therefore the G-derivative of φ at f does not exists.

It is clear that the formula (3.5) defines a bounded linear operator from $L^1(\Omega, \mu)$ into $\mathbb{R}$ for every $\tilde{f}$ in G, and this concludes the proof of the theorem. □

Remark 3.4. (1) In the case $\Omega = \mathbb{N}$ and the counting measure, Theorem 3.3 absorbs a more classical result on the Gâteaux differentiability of the $\ell^1(\mathbb{N})$-norm, namely that the set of Gâteaux differentiability of the $\ell^1(\mathbb{N})$-norm is the set $G \subset \ell^1(\mathbb{N})$ of sequences $(x_n)_n$ such that $x_n \neq 0$ for all n, and the Gâteaux derivative of φ at x and in the direction h is given by

$$\partial_G\varphi(x)(h) = \sum_n \operatorname{sign}(x_n)h_n.$$

(2) The condition (A1) allows L^1 to be be non-separable. Therefore, Theorem 3.3 gives a description of the set of differentiability beyond the scope of Phelps's and Mazur's theorem.

In the special separable case, as a consequence of Theorem 3.3 and Phelps Theorem 1.1 we have:

Corollary 3.5. Let $(\Omega, \mathcal{M}, \mu)$ be a measure space satisfying the condition (A1) and such that $L^1(\Omega, \mathcal{M}, \mu)$ is separable. Then, the set of Gâteaux differentiability of the L^1-norm is G, G is a G_δ set and G^c is a Gaussian null set.

As quoted before $L^1(\Omega, \mathcal{M}, \mu)$ is separable provided μ is σ-finite and the σ-algebra $\mathcal{M}$ is countably generated. If the measure μ is not the trivial null measure then μ satisfies the condition (A1).

4 Conclusions

We have established the Gâteaux differentiability of the L^1 norm for measure spaces $(\Omega, \mathcal{M}, \mu)$ satisfying the condition (A1) and giving the corresponding formulae for the Gâteaux derivative. The condition (A1) is quite general and absorbs a large class of measures including every non-trivial σ-finite measure. Also non-atomic measures satisfy the condition (A1). The sharpness of this condition will be explore in a future work as well as the study of the value of the Gaussian measure on L^1 for the set of Gâteaux differentiability of the L^1 norm in the non-separable case.

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References

[1] G. Alberti, M. Csörnyei and D.Preiss, *Differentiability of Lipschitz functions, structure of null sets, and other problems*, Proc. Int. Cong. Math., New Delhi. 2010, pp. 1379–1394.

[2] N. Aronszajn, *Differentiability of Lipschitzian mappings between Banach spaces*, Studia Math. **57** (1976), 147–190.

[3] D. Bongiorno, *Metric differentiability of Lipschitz maps*, J. Aust. Math. Soc. **96** (2014), no. 1, 25–35.

[4] D. Cohn, Measure theory, Birkhauser, Basel, 2013.

[5] R. Deville, G. Godefroy and V. Zizler, *Smoothness and Renormings in Banach Spaces*, 1st Edition, Longman Scientific andTechnical, Essex, England, 1993.

[6] J. Diestel, Geometry of Banach spaces: Selected topics, Springer-Verlag, 1975.

[7] M. Doré and O. Maleva, *A universal differentiability set in Banach spaces with separable dual*, J. Funct. Anal. **261** (2011), no. 6, 1674–1710.

[8] B. Goldys and F. Gozzi, *Second order parabolic Hamilton-Jacobi-Bellman equations in Hilbert spaces and stochastic control: $L^2\mu$ approach*, Stoch. Process. Appl. **116** (2006), no. 12, 1932–1963.

[9] J. Lindenstrauss, D. Preiss and J. Tišer, *Fréchet differentiability of Lipschitz functions via a variational principle*, J. Eur. Math. Soc. **12** (2010), no. 2, 385–412.

[10] P. Mankiewicz, *On the differentiability of Lipschitz mappings in Fréchet spaces*, Studia Math. 45 (1973), no. , 15–29.

[11] R. Phelps, *Gaussian null sets and differentiability of Lipschitz map on Banach spaces*, Pacific J. Math. **77** (1978), no. 2, 523–531.

[12] R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, Vol. 1364 of Lecture Notes in Mathematics, Springer Berlin Heidelberg, Berlin, Heidelberg, 1989.

[13] D. Potapov and F. Sukochev, *échet differentiability of Sp norms*, Adv. Math. **262** (2014), 436–475.

[14] D. Preiss and G. Speight, *Differentiability of Lipschitz functions in Lebesgue null sets*, Invent. Math. **199** (2014), no. 2, 517–559.

[15] D. Preiss and L. Zajíček, *Directional derivatives of Lipschitz functions*, Israel J. Math. **125** (2001), 1–27.

[16] H. Rademacher, *Über partielle und totale differenzierbarkeit von Funktionen mehrerer Variabeln und über die Transformation der Doppelintegrale*, Math. Annal. **79** (1919), no. 4, 340–359.