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A strongly convergent extragradient method with non-monotone self adaptive step size rule for solving pseudomonotone variational inequalities in a real Hilbert space

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Abstract

In this paper, a new algorithm is proposed to solve pseudo-monotone variational inequalities with the Lipschitz condition in a real Hilbert space. This problem is an exceptionally general mathematical problem in the sense that it consists of a number of the applied mathematical problems as a special instance, such as optimization problems, equilibrium models, fixed point problems, the saddle point problems, and Nash equilibrium point problems. The algorithm is formulated around two algorithms: the extra gradient algorithm and the inertial algorithm. The proposed algorithm uses a new step size rule based on local operator information rather than its Lipschitz constant or any other line search strategy and operates without any knowledge of the operator's Lipschitz constant. It presents the strong convergence of the algorithm. Finally, we conduct a number of numerical experiments to determine the performance and superiority of the described algorithm.

Keywords: Variational Inequalities, Subgradient Extragradient-like Algorithm, Strong Convergence Theorems, Lipschitz Continuity, Pseudomonotone Mapping
2020 MSC: 65Y05, 65K15, 68W10, 47H05, 47H10

1 Introduction

This paper examines the problem of classic variational inequalities [27, 11] and *variational inequality problem* (VIP) for mapping $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ is defined as follows:

$$\text{Find } \omega^* \in \mathcal{A} \text{ in order that } \langle \mathcal{T}(\omega^*), y - \omega^* \rangle \geq 0, \ \forall y \in \mathcal{A} \tag{VIP}$$

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where $\mathcal{A}$ is a non-empty, convex and closed subset of a certain Hilbert space $\mathcal{E}$ and $\langle .,. \rangle$ and $\|.\|$ serve as an inner product and the led the induced norm in $\mathcal{E}$, respectively. Moreover, $\mathbb{R}$, $\mathbb{N}$ are the sets of real numbers and natural numbers, respectively. It is significant to notice that the problem (VIP) is identical to figure out the following problem:

$$\text{Find } \omega^* \in \mathcal{A} \text{ in order that } \omega^* = P_{\mathcal{A}}[\omega^* - \zeta \mathcal{T}(\omega^*)].$$

In order to study the strong convergence, the following conditions are believed to have been met:

($\mathcal{T}1$) The solution set of problem (VIP), denoted by Φ is non-empty;

($\mathcal{T}2$) An operator $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ is said to be pseudo-monotone, i.e.

$$\big\langle \mathcal{T}(y_1), y_2 - y_1 \big\rangle \geq 0 \Longrightarrow \big\langle \mathcal{T}(y_2), y_1 - y_2 \big\rangle \leq 0, \ \forall\, y_1, y_2 \in \mathcal{A};$$

($\mathcal{T}3$) An operator $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ is said to be *Lipschitz continuous* with constant $L > 0$, i.e., there exists $L > 0$ such that

$$\|\mathcal{T}(y_1) - \mathcal{T}(y_2)\| \leq L\|y_1 - y_2\|, \ \forall\, y_1, y_2 \in \mathcal{A};$$

($\mathcal{T}4$) An operator $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ is said to be *sequentially weakly continuous*, i.e., $\{\mathcal{T}(u_n)\}$ converges weakly to $\mathcal{T}(u)$ for every sequence $\{u_n\}$ converges weakly to u.

This variational inequalities was introduced by Stampacchia [27] in 1964. This is an important mathematical construction that brings together several key topics of applied mathematics, such as the problems of network equilibrium, the necessary optimality conditions, the complementarity problems and the systems of nonlinear equations (for more details [7, 8, 9, 10, 4, 15, 29, 24, 21, 20, 22]) and others in [25, 14, 19, 26, 18, 23, 17, 33, 32]. Korpelevich [12] and Antipin [1] set up the following extragradient algorithm.

$$\begin{cases} u_0 \in \mathcal{A}, \\ y_n = P_{\mathcal{A}}[u_n - \zeta \mathcal{T}(u_n)], \\ u_{n+1} = P_{\mathcal{A}}[u_n - \zeta \mathcal{T}(y_n)]. \end{cases} \tag{1.1}$$

Recently, the subgradient extragradient algorithm was presented by Censor et al. [3] for solving the problem (VIP) in a real Hilbert space. Their algorithm takes the form

$$\begin{cases} u_0 \in \mathcal{A}, \\ y_n = P_{\mathcal{A}}[u_n - \zeta \mathcal{T}(u_n)], \\ u_{n+1} = P_{\mathcal{E}_n}[u_n - \zeta \mathcal{T}(y_n)]. \end{cases} \tag{1.2}$$

where

$$\mathcal{E}_n = \{z \in \mathcal{E} : \langle u_n - \zeta \mathcal{T}(u_n) - y_n, z - y_n \rangle \leq 0\}.$$

It is important to remember that the proposed well-established algorithm has two serious shortcomings, the first being the fixed constant step size, which requires the information or approximation of the Lipschitz constant of the associated operator and is only weakly convergent in Hilbert spaces. From a numerical point of view, it could be challenging to use a fixed step size and thus the convergence rate and performance of the algorithm may be affected. The main purpose of this study is to set up an inertial-type algorithm, that is needed to enhance the convergence rate of the iterative sequence. Such algorithms have been formerly formed owing to the oscillator equation with a damping and conservative force restoration. This second-order dynamical system is called a heavy friction ball, which was formerly designed by Polyak in [16].

So there is a crucial question:

"Is it possible to establish a new inertial-like strongly convergent extragradient-type algorithm with a non-monotone variable step size rule"?

In this research, we present an acceptable answer of the raised question, i.e., the gradient algorithm indeed establishes a strong convergence sequence by maintaining variable step size rule for dealing with problem (VIP) combined with pseudo-monotone mappings. Motivated by the works of Censor et al. [3] and Polyak [16], we introduce a new inertial extragradient-type algorithm to figure out the problem (VIP) in the situation of an infinite-dimensional real Hilbert space.

The rest of the paper is given as follows: The section 2 consists of the necessary definitions and fundamental lemmas needed in the article. Section 3 consists of inertial-type iterative scheme and convergence analysis theorem. Section 4 has provided numerical results to explain the performance of the new algorithm and to associate them with other algorithms.

2 Preliminaries

In this section of the text, we have written a number of significant identities and related lemmas and definitions. The *metric projection* $P_{\mathcal{A}}(y_1)$ of $y_1 \in \mathcal{E}$ is defined by

$$P_{\mathcal{A}}(y_1) = \arg\min\{\|y_1 - y_2\| : y_2 \in \mathcal{A}\}.$$

Next, we list some of the important properties of the projection mapping.

Lemma 2.1. [2] Suppose that $P_{\mathcal{A}} : \mathcal{E} \to \mathcal{A}$ is a metric projection. Then, we have

(i) $y_3 = P_{\mathcal{A}}(y_1)$ if and only if

$$\langle y_1 - y_3, y_2 - y_3\rangle \leq 0, \ \forall\, y_2 \in \mathcal{A}.$$

(ii)

$$\|y_1 - P_{\mathcal{A}}(y_2)\|^2 + \|P_{\mathcal{A}}(y_2) - y_2\|^2 \leq \|y_1 - y_2\|^2, \ y_1 \in \mathcal{A}, y_2 \in \mathcal{E}.$$

(iii)

$$\|y_1 - P_{\mathcal{A}}(y_1)\| \leq \|y_1 - y_2\|, \ y_2 \in \mathcal{A}, y_1 \in \mathcal{E}.$$

Lemma 2.2. [30] Let $\{p_n\} \subset [0, +\infty)$ be a sequence satisfying the following inequality

$$p_{n+1} \leq (1 - q_n)p_n + q_n r_n, \ \forall\, n \in \mathbb{N}.$$

Furthermore, $\{q_n\} \subset (0, 1)$ and $\{r_n\} \subset \mathbb{R}$ be two sequences such that

$$\lim_{n\to+\infty} q_n = 0, \ \sum_{n=1}^{+\infty} q_n = +\infty \text{ and } \limsup_{n\to+\infty} r_n \leq 0.$$

Then, $\lim_{n\to+\infty} p_n = 0$.

Lemma 2.3. [13] Suppose that $\{p_n\}$ is a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$p_{n_i} < p_{n_{i+1}} \quad \forall \quad i \in \mathbb{N}.$$

Then, there is a non decreasing sequence $m_k \subset \mathbb{N}$ such that $m_k \to +\infty$ as $k \to +\infty$, and meet the following requirements for numbers $k \in \mathbb{N}$:

$$p_{m_k} \leq p_{m_{k+1}} \quad \text{and} \quad p_k \leq p_{m_{k+1}}.$$

Indeed, $m_k = \max\{j \leq k : p_j \leq p_{j+1}\}$.

Next, we list some of the important identities that were used to prove the convergence analysis.

Lemma 2.4. [2] For any $y_1, y_2 \in \mathcal{E}$ and $\ell \in \mathbb{R}$. Then, the following inequalities are holds.

(i)

$$\|\ell y_1 + (1 - \ell)y_2\|^2 = \ell\|y_1\|^2 + (1 - \ell)\|y_2\|^2 - \ell(1 - \ell)\|y_1 - y_2\|^2.$$

(ii)

$$\|y_1 + y_2\|^2 \leq \|y_1\|^2 + 2\langle y_2, y_1 + y_2\rangle.$$

Lemma 2.5. [28] Assume that $\mathcal{T} : \mathcal{A} \to \mathcal{E}$ is a pseudo-monotone and continuous mapping. Then, ω^* is a solution of the problem (VIP) if and only if ω^* is a solution of the following problem.

$$\text{Find } u \in \mathcal{A} \text{ such that } \langle \mathcal{T}(y), y - u\rangle \geq 0, \ \forall\, y \in \mathcal{A}.$$

3 Main Results

In this section, we introduce inertial-type sub-gradient extragradient algorithm which incorporates the new step size rule and the inertial term as well as provides both strong convergence theorems. The following main result is outlined as follows:

Algorithm 1

Step 0: Let $u_{-1}, u_0 \in \mathcal{A}$, $\alpha > 0$, $\mu \in (0,1)$, $\zeta_0 > 0$ and select a nonnegative real sequence $\{\varphi_n\}$ such that

$$\sum_{n=1}^{+\infty} \varphi_n < +\infty.$$

Moreover, choose $\{\delta_n\} \subset (0,1)$ satisfies the following conditions:

$$\lim_{n\to+\infty} \delta_n = 0 \text{ and } \sum_{n=1}^{+\infty} \delta_n = +\infty.$$

Step 1: Evaluate

$$\Im_n = (1-\delta_n)\big[u_n + \alpha_n(u_n - u_{n-1})\big]$$

where α_n such that

$$0 \le \alpha_n \le \hat{\alpha_n} \quad \text{and} \quad \hat{\alpha_n} = \begin{cases} \min\left\{\alpha, \frac{\epsilon_n}{\|u_n - u_{n-1}\|}\right\} & \text{if} \quad u_n \ne u_{n-1}, \\ \\ \alpha & \text{else}, \end{cases} \tag{3.1}$$

while $\epsilon_n = \circ(\delta_n)$ is a positive sequence, i.e., $\lim_{n\to+\infty} \frac{\epsilon_n}{\delta_n} = 0$.

Step 2: Evaluate

$$y_n = P_{\mathcal{A}}(\Im_n - \zeta\mathcal{T}(\Im_n)).$$

If $\Im_n = y_n$, then STOP and y_n is a solution.

Step 3: Evaluate

$$u_{n+1} = P_{\mathcal{E}_n}(\Im_n - \zeta\mathcal{T}(y_n)).$$

where

$$\mathcal{E}_n = \{z \in \mathcal{E} : \langle \Im_n - \zeta\mathcal{T}(\Im_n) - y_n, z - y_n\rangle \le 0\}.$$

(iii) Compute

$$\zeta_{n+1} = \begin{cases} \min\left\{\zeta_n + \varphi_n, \frac{\mu\|\Im_n - y_n\|^2 + \mu\|u_{n+1} - y_n\|^2}{2\langle \mathcal{T}(\Im_n) - \mathcal{T}(y_n), u_{n+1} - y_n\rangle}\right\} \\ \quad \text{if} \qquad \big\langle \mathcal{T}(\Im_n) - \mathcal{T}(y_n), u_{n+1} - y_n\big\rangle > 0, \\ \zeta_n + \varphi_n \qquad\qquad \textit{otherwise}. \end{cases} \tag{3.2}$$

Set $n = n+1$ and go back to **Step 1**.

Lemma 3.1. Let a sequence $\{\zeta_n\}$ generated by (3.2) is convergent to ζ and also satisfy the following inequality

$$\min\left\{\frac{\mu}{L}, \zeta_0\right\} \le \zeta \le \zeta_0 + P \quad \text{where} \quad P = \sum_{n=1}^{+\infty} \varphi_n.$$

Proof . Due to the Lipschitz continuity of a mapping $\mathcal{T}$ there exists a fixed number $L > 0$. Consider that $\big\langle \mathcal{T}(\Im_n) -$

$\mathcal{T}(y_n), u_{n+1} - y_n\rangle > 0$ such that

$$\begin{aligned}\frac{\mu(\|\Im_n - y_n\|^2 + \|u_{n+1} - y_n\|^2)}{2\langle \mathcal{T}(\Im_n) - \mathcal{T}(y_n), u_{n+1} - y_n\rangle} &\geq \frac{2\mu\|\Im_n - y_n\|\|u_{n+1} - y_n\|}{2\|\mathcal{T}(\Im_n) - \mathcal{T}(y_n)\|\|u_{n+1} - y_n\|}\\ &\geq \frac{2\mu\|\Im_n - y_n\|\|u_{n+1} - y_n\|}{2L\|\Im_n - y_n\|\|u_{n+1} - y_n\|}\\ &\geq \frac{\mu}{L}. \end{aligned} \tag{3.3}$$

By using mathematical induction on the definition of ζ_{n+1}, we have

$$\min\left\{\frac{\mu}{L}, \zeta_0\right\} \leq \zeta_n \leq \zeta_0 + P.$$

Let

$$[\zeta_{n+1} - \zeta_n]^+ = \max\{0, \zeta_{n+1} - \zeta_n\}$$

and

$$[\zeta_{n+1} - \zeta_n]^- = \max\{0, -(\zeta_{n+1} - \zeta_n)\}.$$

From the definition of $\{\zeta_n\}$, we have

$$\sum_{n=1}^{+\infty}(\zeta_{n+1} - \zeta_n)^+ = \sum_{n=1}^{+\infty}\max\{0, \zeta_{n+1} - \zeta_n\} \leq P < +\infty. \tag{3.4}$$

That is, the series $\sum_{n=1}^{+\infty}(\zeta_{n+1} - \zeta_n)^+$ is convergent. Next we need to prove the convergence of $\sum_{n=1}^{+\infty}(\zeta_{n+1} - \zeta_n)^-$. Let $\sum_{n=1}^{+\infty}(\zeta_{n+1} - \zeta_n)^- = +\infty$. Due to the reason that $\zeta_{n+1} - \zeta_n = (\zeta_{n+1} - \zeta_n)^+ - (\zeta_{n+1} - \zeta_n)^-$. Thus, we have

$$\zeta_{k+1} - \zeta_0 = \sum_{n=0}^{k}(\zeta_{n+1} - \zeta_n) = \sum_{n=0}^{k}(\zeta_{n+1} - \zeta_n)^+ - \sum_{n=0}^{k}(\zeta_{n+1} - \zeta_n)^-. \tag{3.5}$$

By allowing $k \to +\infty$ in (3.5), we have $\zeta_k \to -\infty$ as $k \to \infty$. This is a contradiction. Due to the convergence of the series $\sum_{n=0}^{k}(\zeta_{n+1} - \zeta_n)^+$ and $\sum_{n=0}^{k}(\zeta_{n+1} - \zeta_n)^-$ taking $k \to +\infty$ in (3.5), we obtain $\lim_{n\to\infty}\zeta_n = \zeta$. This completes the proof. $\square$

Lemma 3.2. Suppose that an operator $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ meet the conditions ($\mathcal{T}1$)-($\mathcal{T}4$). For $\omega^* \in \Phi \neq \emptyset$, we have

$$\|u_{n+1} - \omega^*\|^2 \leq \|\Im_n - \omega^*\|^2 - \Big(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\Big)\|\Im_n - y_n\|^2 - \Big(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\Big)\|u_{n+1} - y_n\|^2.$$

Proof . It is provides that

$$\begin{aligned}\big\|u_{n+1} - \omega^*\big\|^2 &= \big\|P_{\mathcal{E}_n}[\Im_n - \zeta_n\mathcal{T}(y_n)] - \omega^*\big\|^2\\ &= \big\|P_{\mathcal{E}_n}[\Im_n - \zeta_n\mathcal{T}(y_n)] + [\Im_n - \zeta_n\mathcal{T}(y_n)] - [\Im_n - \zeta_n\mathcal{T}(y_n)] - \omega^*\big\|^2\\ &= \big\|[\Im_n - \zeta_n\mathcal{T}(y_n)] - \omega^*\big\|^2 + \big\|P_{\mathcal{E}_n}[\Im_n - \zeta_n\mathcal{T}(y_n)] - [\Im_n - \zeta_n\mathcal{T}(y_n)]\big\|^2\\ &\quad + 2\big\langle P_{\mathcal{E}_n}[\Im_n - \zeta_n\mathcal{T}(y_n)] - [\Im_n - \zeta_n\mathcal{T}(y_n)], [\Im_n - \zeta_n\mathcal{T}(y_n)] - \omega^*\big\rangle. \end{aligned} \tag{3.6}$$

This is provided that $\omega^* \in \Phi \subset \mathcal{A} \subset \mathcal{E}_n$, we obtain

$$\begin{aligned}&\big\|P_{\mathcal{E}_n}[\Im_n - \zeta_n\mathcal{T}(y_n)] - [\Im_n - \zeta_n\mathcal{T}(y_n)]\big\|^2\\ &\quad + \big\langle P_{\mathcal{E}_n}[\Im_n - \zeta_n\mathcal{T}(y_n)] - [\Im_n - \zeta_n\mathcal{T}(y_n)], [\Im_n - \zeta_n\mathcal{T}(y_n)] - \omega^*\big\rangle\\ &= \big\langle [\Im_n - \zeta_n\mathcal{T}(y_n)] - P_{\mathcal{E}_n}[\Im_n - \zeta_n\mathcal{T}(y_n)], \omega^* - P_{\mathcal{E}_n}[\Im_n - \zeta_n\mathcal{T}(y_n)]\big\rangle \leq 0, \end{aligned} \tag{3.7}$$

that implies that

$$\begin{aligned}&\big\langle P_{\mathcal{E}_n}[\Im_n - \zeta_n \mathcal{T}(y_n)] - [\Im_n - \zeta_n \mathcal{T}(y_n)], [\Im_n - \zeta_n \mathcal{T}(y_n)] - \omega^* \big\rangle \\ &\leq -\big\| P_{\mathcal{E}_n}[\Im_n - \zeta_n \mathcal{T}(y_n)] - [\Im_n - \zeta_n \mathcal{T}(y_n)] \big\|^2. \end{aligned} \tag{3.8}$$

Incorporating the expressions (3.6) and (3.8), we have

$$\begin{aligned}\|u_{n+1} - \omega^*\|^2 &\leq \big\|\Im_n - \zeta_n \mathcal{T}(y_n) - \omega^*\big\|^2 - \big\| P_{\mathcal{E}_n}[\Im_n - \zeta_n \mathcal{T}(y_n)] - [\Im_n - \zeta_n \mathcal{T}(y_n)] \big\|^2 \\ &\leq \|\Im_n - \omega^*\|^2 - \|\Im_n - u_{n+1}\|^2 + 2\zeta_n \big\langle \mathcal{T}(y_n), \omega^* - u_{n+1} \big\rangle. \end{aligned} \tag{3.9}$$

Since ω^* is the solution of problem (VIP), we have

$$\langle \mathcal{T}(\omega^*), y - \omega^* \rangle \geq 0, \text{ for all } y \in \mathcal{A}.$$

Due to the pseudo-monotonicity of $\mathcal{T}$ on $\mathcal{A}$, we get

$$\langle \mathcal{T}(y), y - \omega^* \rangle \geq 0, \text{ for all } y \in \mathcal{A}.$$

By substituting $y = y_n \in \mathcal{A}$, we get

$$\langle \mathcal{T}(y_n), y_n - \omega^* \rangle \geq 0.$$

Thus, we have

$$\big\langle \mathcal{T}(y_n), \omega^* - u_{n+1} \big\rangle = \big\langle \mathcal{T}(y_n), \omega^* - y_n \big\rangle + \big\langle \mathcal{T}(y_n), y_n - u_{n+1} \big\rangle \leq \big\langle \mathcal{T}(y_n), y_n - u_{n+1} \big\rangle. \tag{3.10}$$

By use of (3.9) and (3.10), we get

$$\begin{aligned}\|u_{n+1} - \omega^*\|^2 &\leq \|\Im_n - \omega^*\|^2 - \|\Im_n - u_{n+1}\|^2 + 2\zeta_n \big\langle \mathcal{T}(y_n), y_n - u_{n+1} \big\rangle \\ &\leq \|\Im_n - \omega^*\|^2 - \|\Im_n - y_n + y_n - u_{n+1}\|^2 + 2\zeta_n \big\langle \mathcal{T}(y_n), y_n - u_{n+1} \big\rangle \\ &\leq \|\Im_n - \omega^*\|^2 - \|\Im_n - y_n\|^2 - \|y_n - u_{n+1}\|^2 + 2\big\langle \Im_n - \zeta_n \mathcal{T}(y_n) - y_n, u_{n+1} - y_n \big\rangle. \end{aligned} \tag{3.11}$$

By use of $u_{n+1} = P_{\mathcal{E}_n}[\Im_n - \zeta_n \mathcal{T}(y_n)]$ and ζ_{n+1}, we have

$$\begin{aligned}&2\big\langle \Im_n - \zeta_n \mathcal{T}(y_n) - y_n, u_{n+1} - y_n \big\rangle \\ &= 2\big\langle \Im_n - \zeta_n \mathcal{T}(\Im_n) - y_n, u_{n+1} - y_n \big\rangle + 2\zeta_n \big\langle \mathcal{T}(\Im_n) - \mathcal{T}(y_n), u_{n+1} - y_n \big\rangle \\ &= 2\frac{\zeta_n}{\zeta_{n+1}} \zeta_{n+1} \big\langle \mathcal{T}(\Im_n) - \mathcal{T}(y_n), u_{n+1} - y_n \big\rangle \\ &\leq \frac{\mu \zeta_n}{\zeta_{n+1}} \|\Im_n - y_n\|^2 + \frac{\mu \zeta_n}{\zeta_{n+1}} \|u_{n+1} - y_n\|^2. \end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12), we get

$$\begin{aligned}&\|u_{n+1} - \omega^*\|^2 \\ &\leq \|\Im_n - \omega^*\|^2 - \|\Im_n - y_n\|^2 - \|y_n - u_{n+1}\|^2 + \frac{\zeta_n}{\zeta_{n+1}}\big[\mu\|\Im_n - y_n\|^2 + \mu\|u_{n+1} - y_n\|^2\big] \\ &\leq \|\Im_n - \omega^*\|^2 - \Big(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\Big)\|\Im_n - y_n\|^2 - \Big(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\Big)\|u_{n+1} - y_n\|^2. \end{aligned} \tag{3.13}$$

□

Theorem 3.3. Suppose that $\{u_n\}$ is a sequence formed by Algorithm 1 and meet the conditions ($\mathcal{T}1$)-($\mathcal{T}4$). Then, $\{u_n\}$ strongly converges to $\omega^* \in \Phi$. Moreover, $P_\Phi(0) = \omega^*$.

Proof . It is given that $\zeta_n \to \zeta$, such that $\epsilon \in (0, 1 - \mu)$ and

$$\lim_{n\to\infty} \Big(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\Big) = 1 - \mu > \epsilon > 0.$$

Thus, there is a finite number $n_1 \in \mathbb{N}$ such that

$$\left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right) > \epsilon > 0, \ \forall n \geq n_1. \tag{3.14}$$

Thus, implies that

$$\|u_{n+1} - \omega^*\|^2 \leq \|\Im_n - \omega^*\|^2, \ \forall n \geq n_1. \tag{3.15}$$

It is given in expression (3.1) that

$$\lim_{n\to+\infty} \frac{\alpha_n}{\delta_n}\big\|u_n - u_{n-1}\big\| \leq \lim_{n\to+\infty} \frac{\epsilon_n}{\delta_n}\big\|u_n - u_{n-1}\big\| = 0. \tag{3.16}$$

By the use of definition of $\{\Im_n\}$ and inequality (3.16), we obtain

$$\begin{aligned}
\big\|\Im_n - \omega^*\big\| &= \big\|u_n + \alpha_n(u_n - u_{n-1}) - \delta_n u_n - \alpha_n\delta_n(u_n - u_{n-1}) - \omega^*\big\| \\
&= \big\|(1-\delta_n)(u_n - \omega^*) + (1-\delta_n)\alpha_n(u_n - u_{n-1}) - \delta_n\omega^*\big\| \qquad (3.17)\\
&\leq (1-\delta_n)\big\|u_n - \omega^*\big\| + (1-\delta_n)\alpha_n\big\|u_n - u_{n-1}\big\| + \delta_n\big\|\omega^*\big\| \\
&\leq (1-\delta_n)\|u_n - \omega^*\| + \delta_n M_1, \qquad (3.18)
\end{aligned}$$

where

$$(1-\delta_n)\frac{\alpha_n}{\delta_n}\big\|u_n - u_{n-1}\big\| + \big\|\omega^*\big\| \leq M_1.$$

By using expressions (3.15) with (3.18), we obtain

$$\begin{aligned}
\|u_{n+1} - \omega^*\| &\leq (1-\delta_n)\|u_n - \omega^*\| + \delta_n M_1 \\
&\leq \max\big\{\|u_n - \omega^*\|, M_1\big\} \\
&\ \ \vdots \\
&\leq \max\big\{\|u_0 - \omega^*\|, M_1\big\}.
\end{aligned} \tag{3.19}$$

Thus, we conclude that the $\{u_n\}$ is bounded sequence. Indeed, by expression (3.18) we have

$$\begin{aligned}
\big\|\Im_n - \omega^*\big\|^2 &\leq (1-\delta_n)^2\|u_n - \omega^*\|^2 + \delta_n^2 M_1^2 + 2M_1\delta_n(1-\delta_n)\|u_n - \omega^*\| \\
&\leq \|u_n - \omega^*\|^2 + \delta_n\big[\delta_n M_1^2 + 2M_1(1-\delta_n)\|u_n - \omega^*\|\big] \\
&\leq \|u_n - \omega^*\|^2 + \delta_n M_2,
\end{aligned} \tag{3.20}$$

where

$$\delta_n M_1^2 + 2M_1(1-\delta_n)\|u_n - \omega^*\| \leq M_2$$

for some $M_2 > 0$. By using the expressions (3.13) with (3.20), we have

$$\begin{aligned}
\|u_{n+1} - \omega^*\|^2 &\leq \|u_n - \omega^*\|^2 + \delta_n M_2 \\
&\quad - \left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right)\|\Im_n - y_n\|^2 - \left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right)\|u_{n+1} - y_n\|^2.
\end{aligned} \tag{3.21}$$

The remainder of the facts shall be split into the following two parts:

Case 1: Next, consider that a fixed number $n_2 \in \mathbb{N}$ $(n_2 \geq n_1)$ such that

$$\|u_{n+1} - \omega^*\| \leq \|u_n - \omega^*\|, \ \forall n \geq n_2. \tag{3.22}$$

Thus, above implies that $\lim_{n\to+\infty}\|u_n - \omega^*\|$ exists and let $\lim_{n\to+\infty}\|u_n - \omega^*\| = l$, for some $l \geq 0$. From the expression (3.21), we have

$$\begin{aligned}
&\left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right)\|\Im_n - y_n\|^2 + \left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right)\|u_{n+1} - y_n\|^2 \\
&\leq \|u_n - \omega^*\|^2 + \delta_n M_2 - \|u_{n+1} - \omega^*\|^2.
\end{aligned} \tag{3.23}$$

Due to existence of a limit of sequence $\|u_n - \omega^*\|$ and $\delta_n \to 0$, we deduce that

$$\|\Im_n - y_n\| \to 0 \quad \text{and} \quad \|u_{n+1} - y_n\| \to 0 \quad \text{as} \quad n \to +\infty. \tag{3.24}$$

By the use of expression (3.24), we have

$$\lim_{n\to+\infty} \|\Im_n - u_{n+1}\| \leq \lim_{n\to+\infty} \|\Im_n - y_n\| + \lim_{n\to+\infty} \|y_n - u_{n+1}\| = 0. \tag{3.25}$$

Next, we compute

$$\begin{aligned} \|\Im_n - u_n\| &= \|u_n + \alpha_n(u_n - u_{n-1}) - \delta_n\big[u_n + \alpha_n(u_n - u_{n-1})\big] - u_n\| \\ &\leq \alpha_n\|u_n - u_{n-1}\| + \delta_n\|u_n\| + \alpha_n\delta_n\|u_n - u_{n-1}\| \\ &= \delta_n \frac{\alpha_n}{\delta_n}\|u_n - u_{n-1}\| + \delta_n\|u_n\| + \delta_n^2 \frac{\alpha_n}{\delta_n}\|u_n - u_{n-1}\| \longrightarrow 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.26}$$

The following provides that

$$\lim_{n\to+\infty} \|u_n - u_{n+1}\| \leq \lim_{n\to+\infty} \|u_n - \Im_n\| + \lim_{n\to+\infty} \|\Im_n - u_{n+1}\| = 0. \tag{3.27}$$

The above explanation guarantees that the sequences $\{\Im_n\}$ and $\{y_n\}$ are also bounded. By the use of reflexivity of $\mathcal{E}$ and the boundedness of $\{u_n\}$ guarantees that there exits a subsequence $\{u_{n_k}\}$ in order that $\{u_{n_k}\} \rightharpoonup \hat{u} \in \mathcal{E}$ as $k \to +\infty$. Next, we have to prove that $\hat{u} \in \Phi$.

It is given that

$$y_{n_k} = P_{\mathcal{A}}[w_{n_k} - \zeta\mathcal{T}(w_{n_k})]$$

that is equivalent to

$$\langle w_{n_k} - \zeta\mathcal{T}(w_{n_k}) - y_{n_k}, y - y_{n_k}\rangle \leq 0, \ \forall y \in \mathcal{A}. \tag{3.28}$$

The inequality described above implies that

$$\langle w_{n_k} - y_{n_k}, y - y_{n_k}\rangle \leq \zeta\langle\mathcal{T}(w_{n_k}), y - y_{n_k}\rangle, \ \forall y \in \mathcal{A}. \tag{3.29}$$

Further, we obtain

$$\frac{1}{\zeta}\langle w_{n_k} - y_{n_k}, y - y_{n_k}\rangle + \langle\mathcal{T}(w_{n_k}), y_{n_k} - w_{n_k}\rangle \leq \langle\mathcal{T}(w_{n_k}), y - w_{n_k}\rangle, \ \forall y \in \mathcal{A}. \tag{3.30}$$

Due to boundedness of the sequence $\{w_{n_k}\}$ implies that $\{\mathcal{T}(w_{n_k})\}$ is also bounded. By the use of $\lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0$ and $k \to \infty$ in (3.30), we obtain

$$\liminf_{k\to\infty}\langle\mathcal{T}(w_{n_k}), y - w_{n_k}\rangle \geq 0, \ \forall y \in \mathcal{A}. \tag{3.31}$$

Furthermore, we have

$$\begin{aligned} &\langle\mathcal{T}(y_{n_k}), y - y_{n_k}\rangle \\ &= \langle\mathcal{T}(y_{n_k}) - \mathcal{T}(w_{n_k}), y - w_{n_k}\rangle + \langle\mathcal{T}(w_{n_k}), y - w_{n_k}\rangle + \langle\mathcal{T}(y_{n_k}), w_{n_k} - y_{n_k}\rangle. \end{aligned} \tag{3.32}$$

By the use of $\lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0$ and $\mathcal{T}$ is L-Lipschitz continuous on $\mathcal{E}$ implies that

$$\lim_{k\to\infty} \|\mathcal{T}(w_{n_k}) - \mathcal{T}(y_{n_k})\| = 0. \tag{3.33}$$

which together with (3.32) and (3.33), we obtain

$$\liminf_{k\to\infty}\langle\mathcal{T}(y_{n_k}), y - y_{n_k}\rangle \geq 0, \ \forall y \in \mathcal{A}. \tag{3.34}$$

Consider a sequence of positive numbers $\{\epsilon_k\}$ that is decreasing and converge to zero. For each k, we denote m_k by the smallest positive integer such that

$$\langle\mathcal{T}(w_{n_i}), y - w_{n_i}\rangle + \epsilon_k \geq 0, \ \forall i \geq m_k. \tag{3.35}$$

Due to $\{\epsilon_k\}$ is decreasing and $\{m_k\}$ is increasing.

Case A: If there is a $w_{n_{m_{k_j}}}$ subsequence of $w_{n_{m_k}}$ such that $\mathcal{T}(w_{n_{m_{k_j}}}) = 0$ $(\forall j)$. Let $j \to \infty$, we obtain

$$\langle \mathcal{T}(\hat{u}), y - \hat{u} \rangle = \lim_{j \to \infty} \langle \mathcal{T}(w_{n_{m_{k_j}}}), y - \hat{u} \rangle = 0. \tag{3.36}$$

Hence $\hat{u} \in \mathcal{A}$, therefore we obtain $\hat{u} \in \Phi$.

Case B: If there exits $n_0 \in \mathbb{N}$ such that for all $n_{m_k} \geq n_0$, $\mathcal{T}(w_{n_{m_k}}) \neq 0$. Suppose that

$$\mho_{n_{m_k}} = \frac{\mathcal{T}(w_{n_{m_k}})}{\|\mathcal{T}(w_{n_{m_k}})\|^2}, \ \forall\, n_{m_k} \geq n_0. \tag{3.37}$$

On the basis of the above definition, we shall obtain

$$\langle \mathcal{T}(w_{n_{m_k}}), \mho_{n_{m_k}} \rangle = 1, \ \forall\, n_{m_k} \geq n_0. \tag{3.38}$$

By using expressions (3.35) and (3.38), for all $n_{m_k} \geq n_0$, we have

$$\langle \mathcal{T}(w_{n_{m_k}}), \ y + \epsilon_k \mho_{n_{m_k}} - w_{n_{m_k}} \rangle \geq 0. \tag{3.39}$$

Due to the pseudo-monotonicity of $\mathcal{T}$ for $n_{m_k} \geq n_0$, we have

$$\langle \mathcal{T}(y + \epsilon_k \mho_{n_{m_k}}), \ y + \epsilon_k \mho_{n_{m_k}} - w_{n_{m_k}} \rangle \geq 0. \tag{3.40}$$

For all $n_{m_k} \geq n_0$, we have

$$\langle \mathcal{T}(y), y - w_{n_{m_k}} \rangle \geq \langle \mathcal{T}(y) - \mathcal{T}(y + \epsilon_k \mho_{n_{m_k}}), \ y + \epsilon_k \mho_{n_{m_k}} - w_{n_{m_k}} \rangle - \epsilon_k \langle \mathcal{T}(y), \mho_{n_{m_k}} \rangle. \tag{3.41}$$

Due to $\{w_{n_k}\}$ weakly converges to $\hat{u} \in \mathcal{A}$ through $\mathcal{T}$ is sequentially weakly continuous on the set $\mathcal{A}$, we get $\{\mathcal{T}(w_{n_k})\}$ weakly converges to $\mathcal{T}(\hat{u})$. Suppose that $\mathcal{T}(\hat{u}) \neq 0$, we have

$$\|\mathcal{T}(\hat{u})\| \leq \liminf_{k \to \infty} \|\mathcal{T}(w_{n_k})\|. \tag{3.42}$$

Since $\{w_{n_{m_k}}\} \subset \{w_{n_k}\}$ and $\lim_{k \to \infty} \epsilon_k = 0$, we have

$$0 \leq \lim_{k \to \infty} \|\epsilon_k \mho_{n_{m_k}}\| = \lim_{k \to \infty} \frac{\epsilon_k}{\|\mathcal{T}(w_{n_{m_k}})\|} \leq \frac{0}{\|\mathcal{T}(\hat{u})\|} = 0. \tag{3.43}$$

Next, consider $k \to \infty$ in (3.41), we obtain

$$\langle \mathcal{T}(y), y - \hat{u} \rangle \geq 0, \ \forall\, y \in \mathcal{A}. \tag{3.44}$$

By the use of Minty Lemma 2.5, we infer $\hat{u} \in \Phi$.

By using the Lipschitz-continuity and pseudo-monotonicity of $\mathcal{T}$ implies that the solution set Φ is a closed and convex set. It is given that $\omega^* = P_\Phi(0)$ and by using Lemma 2.1 (ii), we have

$$\langle 0 - \omega^*, y - \omega^* \rangle \leq 0, \ \forall\, y \in \Phi. \tag{3.45}$$

Next, we have to

$$\limsup_{n \to +\infty} \langle \omega^*, \omega^* - u_n \rangle = \lim_{k \to +\infty} \langle \omega^*, \omega^* - u_{n_k} \rangle = \langle \omega^*, \omega^* - \hat{u} \rangle \leq 0. \tag{3.46}$$

By the use of $\lim_{n \to +\infty} \big\|u_{n+1} - u_n\big\| = 0$. Therefore, (3.46) indicates that

$$\begin{aligned} &\limsup_{n \to +\infty} \langle \omega^*, \omega^* - u_{n+1} \rangle \\ &\leq \limsup_{n \to +\infty} \langle \omega^*, \omega^* - u_n \rangle + \limsup_{n \to +\infty} \langle \omega^*, u_n - u_{n+1} \rangle \leq 0. \end{aligned} \tag{3.47}$$

Take into account the expression (3.17), we have

$$
\begin{aligned}
&\big\|\Im_n - \omega^*\big\|^2 \\
&= \big\|u_n + \alpha_n(u_n - u_{n-1}) - \delta_n u_n - \alpha_n \delta_n (u_n - u_{n-1}) - \omega^*\big\|^2 \\
&= \big\|(1-\delta_n)(u_n - \omega^*) + (1-\delta_n)\alpha_n(u_n - u_{n-1}) - \delta_n \omega^*\big\|^2 \\
&\le \big\|(1-\delta_n)(u_n - \omega^*) + (1-\delta_n)\alpha_n(u_n - u_{n-1})\big\|^2 + 2\delta_n\langle -\omega^*, \Im_n - \omega^*\rangle \\
&= (1-\delta_n)^2\big\|u_n - \omega^*\big\|^2 + (1-\delta_n)^2\alpha_n^2\big\|u_n - u_{n-1}\big\|^2 \\
&\quad + 2\alpha_n(1-\delta_n)^2\big\|u_n - \omega^*\big\|\big\|u_n - u_{n-1}\big\| + 2\delta_n\langle -\omega^*, \Im_n - u_{n+1}\rangle + 2\delta_n\langle -\omega^*, u_{n+1} - \omega^*\rangle \\
&\le (1-\delta_n)\big\|u_n - \omega^*\big\|^2 + \alpha_n^2\big\|u_n - u_{n-1}\big\|^2 + 2\alpha_n(1-\delta_n)\big\|u_n - \omega^*\big\|\big\|u_n - u_{n-1}\big\| \\
&\quad + 2\delta_n\big\|\omega^*\big\|\big\|\Im_n - u_{n+1}\big\| + 2\delta_n\langle -\omega^*, u_{n+1} - \omega^*\rangle \\
&= (1-\delta_n)\big\|u_n - \omega^*\big\|^2 + \delta_n\Big[\alpha_n\big\|u_n - u_{n-1}\big\|\frac{\alpha_n}{\delta_n}\big\|u_n - u_{n-1}\big\| \\
&\quad + 2(1-\delta_n)\big\|u_n - \omega^*\big\|\frac{\alpha_n}{\delta_n}\big\|u_n - u_{n-1}\big\| + 2\big\|\omega^*\big\|\big\|\Im_n - u_{n+1}\big\| + 2\langle \omega^*, \omega^* - u_{n+1}\rangle\Big].
\end{aligned} \tag{3.48}
$$

From expressions (3.15) and (3.48) we obtain

$$
\begin{aligned}
&\big\|u_{n+1} - \omega^*\big\|^2 \\
&\le (1-\delta_n)\big\|u_n - \omega^*\big\|^2 + \delta_n\Big[\alpha_n\big\|u_n - u_{n-1}\big\|\frac{\alpha_n}{\delta_n}\big\|u_n - u_{n-1}\big\| \\
&\quad + 2(1-\delta_n)\big\|u_n - \omega^*\big\|\frac{\alpha_n}{\delta_n}\big\|u_n - u_{n-1}\big\| + 2\big\|\omega^*\big\|\big\|\Im_n - u_{n+1}\big\| + 2\langle \omega^*, \omega^* - u_{n+1}\rangle\Big].
\end{aligned} \tag{3.49}
$$

By the use of (3.25), (3.47), (3.49) and applying Lemma 2.2, conclude that $\lim_{n\to+\infty}\big\|u_n - \omega^*\big\| = 0$.

Case 2: Consider that there exist a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|u_{n_i} - \omega^*\| \le \|u_{n_{i+1}} - \omega^*\|,\ \forall\, i \in \mathbb{N}.$$

By using Lemma 2.3 there exists a sequence $\{m_k\} \subset \mathbb{N}$ as $\{m_k\} \to +\infty$ such that

$$\|u_{m_k} - \omega^*\| \le \|u_{m_{k+1}} - \omega^*\| \quad \text{and} \quad \|u_k - \omega^*\| \le \|u_{m_{k+1}} - \omega^*\|, \text{ for all } k \in \mathbb{N}. \tag{3.50}$$

As similar to Case 1, the relation (3.23) implies that

$$
\begin{aligned}
&\Big(1 - \frac{\mu\zeta_{m_k}}{\zeta_{m_k+1}}\Big)\|w_{m_k} - y_{m_k}\|^2 + \Big(1 - \frac{\mu\zeta_{m_k}}{\zeta_{m_k+1}}\Big)\|u_{m_k+1} - y_{m_k}\|^2 \\
&\le \|u_{m_k} - \omega^*\|^2 + \delta_{m_k}M_2 - \|u_{m_k+1} - \omega^*\|^2.
\end{aligned} \tag{3.51}
$$

Due to $\delta_{m_k} \to 0$, we can conclude the following:

$$\lim_{k\to+\infty}\|w_{m_k} - y_{m_k}\| = \lim_{k\to+\infty}\|u_{m_k+1} - y_{m_k}\| = 0. \tag{3.52}$$

It continues from that

$$\lim_{k\to+\infty}\|u_{m_{k+1}} - w_{m_k}\| \le \lim_{k\to+\infty}\|u_{m_{k+1}} - y_{m_k}\| + \lim_{k\to+\infty}\|y_{m_k} - w_{m_k}\| = 0. \tag{3.53}$$

Next, we will determine

$$
\begin{aligned}
\|w_{m_k} - u_{m_k}\| &= \|u_{m_k} + \alpha_{m_k}(u_{m_k} - u_{m_k-1}) - \delta_{m_k}\big[u_{m_k} + \alpha_{m_k}(u_{m_k} - u_{m_k-1})\big] - u_{m_k}\| \\
&\le \alpha_{m_k}\|u_{m_k} - u_{m_k-1}\| + \delta_{m_k}\|u_{m_k}\| + \alpha_{m_k}\delta_{m_k}\|u_{m_k} - u_{m_k-1}\| \\
&= \delta_{m_k}\frac{\alpha_{m_k}}{\delta_{m_k}}\|u_{m_k} - u_{m_k-1}\| + \delta_{m_k}\|u_{m_k}\| + \delta_{m_k}^2\frac{\alpha_{m_k}}{\delta_{m_k}}\|u_{m_k} - u_{m_k-1}\| \longrightarrow 0.
\end{aligned} \tag{3.54}
$$

This leads from that

$$\lim_{k\to+\infty}\|u_{m_k} - u_{m_k+1}\| \le \lim_{k\to+\infty}\|u_{m_k} - w_{m_k}\| + \lim_{k\to+\infty}\|w_{m_k} - u_{m_k+1}\| = 0. \tag{3.55}$$

By using the same reason as in Case 1, that is to say,

$$\limsup_{k\to+\infty}\langle \omega^*, \omega^* - u_{m_k+1}\rangle \le 0. \tag{3.56}$$

By using the expressions (3.49) and (3.50) we obtain

$$\begin{aligned}
&\left\|u_{m_k+1} - \omega^*\right\|^2 \\
&\le (1-\delta_{m_k})\left\|u_{m_k} - \omega^*\right\|^2 + \delta_{m_k}\Big[\alpha_{m_k}\left\|u_{m_k} - u_{m_k-1}\right\|\frac{\alpha_{m_k}}{\delta_{m_k}}\left\|u_{m_k} - u_{m_k-1}\right\| \\
&\quad + 2(1-\delta_{m_k})\left\|u_{m_k} - \omega^*\right\|\frac{\alpha_{m_k}}{\delta_{m_k}}\left\|u_{m_k} - u_{m_k-1}\right\| + 2\left\|\omega^*\right\|\left\|w_{m_k} - u_{m_k+1}\right\| + 2\langle \omega^*, \omega^* - u_{m_k+1}\rangle\Big] \\
&\le (1-\delta_{m_k})\left\|u_{m_{k+1}} - \omega^*\right\|^2 + \delta_{m_k}\Big[\alpha_{m_k}\left\|u_{m_k} - u_{m_k-1}\right\|\frac{\alpha_{m_k}}{\delta_{m_k}}\left\|u_{m_k} - u_{m_k-1}\right\| \\
&\quad + 2(1-\delta_{m_k})\left\|u_{m_k} - \omega^*\right\|\frac{\alpha_{m_k}}{\delta_{m_k}}\left\|u_{m_k} - u_{m_k-1}\right\| + 2\left\|\omega^*\right\|\left\|w_{m_k} - u_{m_k+1}\right\| + 2\langle \omega^*, \omega^* - u_{m_k+1}\rangle\Big].
\end{aligned} \tag{3.57}$$

Thus, above implies that

$$\begin{aligned}
&\left\|u_{m_k+1} - \omega^*\right\|^2 \\
&\le \Big[\alpha_{m_k}\left\|u_{m_k} - u_{m_k-1}\right\|\frac{\alpha_{m_k}}{\delta_{m_k}}\left\|u_{m_k} - u_{m_k-1}\right\| \\
&\quad + 2(1-\delta_{m_k})\left\|u_{m_k} - \omega^*\right\|\frac{\alpha_{m_k}}{\delta_{m_k}}\left\|u_{m_k} - u_{m_k-1}\right\| + 2\left\|\omega^*\right\|\left\|w_{m_k} - u_{m_k+1}\right\| + 2\langle \omega^*, \omega^* - u_{m_k+1}\rangle\Big].
\end{aligned} \tag{3.58}$$

Since $\delta_{m_k} \to 0$, and $\left\|u_{m_k} - \omega^*\right\|$ is a bounded. Thus, expressions (3.56) and (3.58) implies that

$$\|u_{m_k+1} - \omega^*\|^2 \to 0, \text{ as } k \to +\infty. \tag{3.59}$$

This means that

$$\lim_{n\to+\infty}\|u_k - \omega^*\|^2 \le \lim_{n\to+\infty}\|u_{m_k+1} - \omega^*\|^2 \le 0. \tag{3.60}$$

As a consequence $u_n \to \omega^*$. This will conclude the proof of theorem.

□

4 Numerical Illustrations

This section examines three numerical examples to show the efficacy of the proposed algorithms. Any of these numerical experiments provide a detailed understanding of how better control parameters can be chosen. Some of them show the advantages of the proposed algorithms compared to existing ones in the literature.

Example 4.1. First consider the HpHard problem that is consider from [5]. Let $\mathcal{T} : \mathbb{R}^N \to \mathbb{R}^N$ be an operator is defined by

$$\mathcal{T}(u) = Mu + q$$

where $q \in \mathbb{R}^N$ and

$$M = AA^T + B + D$$

where A is an $N \times N$ matrix, B is an $N \times N$ skew-symmetric matrix and D is an $N \times N$ positive definite diagonal matrix. The set $\mathcal{A}$ is taken in the following way:

$$\mathcal{A} = \{u \in \mathbb{R}^N : -100 \le u_i \le 100\}.$$

It is clear that $\mathcal{T}$ is monotone and Lipschitz continuous through $L = \|M\|$. The control condition are taken as follows: (1) Algorithm 2 in [31]: $\zeta_0 = 0.05;, \mu = 0.80, \alpha_n = \frac{1}{100(n+2)}$; (2) Algorithm 1: $\zeta_0 = 0.05, \mu = 0.80, \alpha = 0.60, \epsilon_n = \frac{1}{(n+1)^2}, \delta_n = \frac{1}{100(n+2)}, \varphi_n = \frac{100}{(n+1)^2}$. During this experiment, the initial point is $u_0 = u_1 = (2, 2, \cdots, 2)$ and $D_n = \|\Im_n - y_n\| \le 10^{-3}$. The numerical results of these algorithms are shown in Table 1.

Table 1: Numerical illustrations for both algorithms

	Algorithm 2 in [31]		Algorithm 1	
N	**Number of Iterations**	**Elapsed Time**	**Number of Iterations**	**Elapsed Time**
5	32	0.143745	11	0.092134
20	47	0.293781	11	0.126725
50	213	1.198352	41	0.341284
100	321	2.391837	31	0.763113
200	209	5.373166	51	1.573813

Example 4.2. For second example, consider the quadratic fractional programming problem in the following form [6]:

$$\begin{cases} \min f(u) = \dfrac{u^T Q u + a^T u + a_0}{b^T u + b_0}, \\ \text{subject to } u \in \mathcal{A} = \{u \in \mathbb{R}^4 : b^T u + b_0 > 0\} \end{cases}$$

where

$$Q = \begin{pmatrix} 5 & -1 & 2 & 0 \\ -1 & 5 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 5 \end{pmatrix}, \quad a = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \ a_0 = -2, \quad \text{and} \quad b_0 = 4.$$

It is easy to verify that Q is symmetric and positive definite on $\mathbb{R}^4$ and consequently f is pseudo-convex on $\mathcal{A}$. Hence, ∇f is pseudo-monotone. Using the quotient rule, we obtain

$$\nabla f(u) = \frac{(b^T u + b_0)(2Qu + a) - b(u^T Q + a^T u + a_0)}{(b^T u + b_0)^2}. \tag{4.1}$$

In this point of view, we can set $\mathcal{T} = \nabla f$ in Theorem 3.3. We minimize f over $\mathcal{A} = \{u \in \mathbb{R}^4 : 1 \le u_i \le 10, \ i = 1, 2, 3, 4\}$. This problem has a unique solution $\omega^* = (1, 1, 1, 1)^T \in \mathcal{A}$. The control condition are taken as follows: (1) Algorithm 2 in [31]: $\zeta_0 = 0.25; , \mu = 0.80, \alpha_n = \frac{1}{100(n+2)}$. (2) Algorithm 1: $\zeta_0 = 0.25, \mu = 0.80, \alpha = 0.60, \epsilon_n = \frac{1}{(n+1)^2}, \varphi_n = \frac{100}{(n+1)^2}, \delta_n = \frac{1}{100(n+2)}$. During this experiment, the initial points are different and $D_n = \|\Im_n - y_n\| \le 10^{-4}$. The numerical results of these algorithms are shown in Tables 2-5.

Table 2: Example 4.2: Numerical study of Algorithm 2 in [31] and $u_0 = u_1 = [5, -10, 5, -10]^T$.

Iter (n)	u_1	u_2	u_3	u_4
1	9.73621052092642	0.833094840825805	9.90241332280372	1.04929123705160
2	9.65398088222099	0.999443650015545	9.83074011956143	1.02908763007986
3	9.57174604515717	1.06492890311881	9.75920250955472	1.00888315702824
4	9.48955289901919	1.12923908105825	9.68783224607597	1.00001777899295
5	9.40743194326346	1.19231579167598	9.61664829940396	1.00000002980971
6	9.32539697955624	1.25416309375667	9.54565902708142	1.00000000018713
7	9.24345882537906	1.31478659951554	9.47487029657638	1.00000000011988
8	9.16162656381652	1.37419323423997	9.40428650097328	1.00000020053245
9	9.07990820317225	1.43239065856804	9.33391113866474	1.00000017038491
10	8.99831103926803	1.48938694679281	9.26374713071480	1.00000014610962
⋮	⋮	⋮	⋮	⋮
180	1.00031601484227	0.999707404385613	1.13272909195984	0.999967307570371
181	1.00031215935883	0.999714823249607	1.11242178130774	0.999968045673433
182	1.00030820768938	0.999722106421074	1.09260313929843	0.999968775268394
183	1.00030416718112	0.999729253535189	1.07326478974856	0.999969496171197
184	1.00030004508967	0.999736264410505	1.05439834405683	0.999970208192181
185	1.00029584834516	0.999743139236891	1.03599542429133	0.999970911118532
186	1.00029157777940	0.999749866096390	1.01804760355120	0.999971604705314
187	1.00028704334254	0.999756607981979	1.00055874144751	0.999972257950801
188	1.00002617317537	0.999977974772264	1.00002397019683	0.999992761540142
CPU time is seconds	0.876569			

Table 3: Example 4.2: Numerical study of Algorithm 1 and $u_0 = u_1 = [5, -10, 5, -10]^T$.

Iter (n)	u_1	u_2	u_3	u_4
1	5.55377533757632	-2.90587269134480	8.61905463436968	1.72038166468790
2	4.41040024375028	1	7.45023669240132	1.19425608076334
3	12.3448991907237	-12.6397501083870	13.0658963027246	-13.2192576516318
4	3.05882604768791	-7.92358662619279	7.05918169098723	-2.21603141462490
5	2.13444665314296	-7.44999847685431	6.31036964537031	1.12023327884896
6	1.87015108945082	1.00000000097286	6.02864753785448	0.955002424010157
7	1.46861344878822	1.15019734000242	5.43281104076976	1.00000000001322
8	2.05261045125816	-0.507408676818891	4.10590329281081	0.236813108691433
9	1.43898856735670	0.490171438084139	3.43815944442871	1.28802829936381
10	1.29986269980453	0.213085356892448	2.58372157493742	0.853687511480199
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
60	1.00004416092722	0.999975499519528	1.00001716683660	0.999986491695085
61	1.00003823749786	0.999978633212697	1.00001506391126	0.999988330027726
62	1.00003579248111	0.999980505234820	1.00001322064060	0.999989034148032
63	1.00003668241231	0.999980324093982	1.00001308096942	0.999988723923595
64	1.00004014710912	0.999978308970547	1.00001461600524	0.999987666312513
65	1.00004007642355	0.999977850272896	1.00001542714126	0.999987735088239
66	1.00003584694207	0.999980050168924	1.00001399157339	0.999989050797591
67	1.00003405336513	0.999981402563276	1.00001267126779	0.999989569857342
68	1.00003483208481	0.999981190052566	1.00001263286680	0.999989305033961
CPU time is seconds	0.308771			

Table 4: Example 4.2: Numerical study of Algorithm 2 in [31] and $u_0 = u_1 = [10, -20, 30, -40]^T$.

Iter (n)	u_1	u_2	u_3	u_4
1	8.77637333455425	-0.0495715784342590	9.29094270258555	0.811840145149117
2	8.69495843974052	0.997670568432512	9.21911519294898	0.993720619633674
3	8.61358693872012	1.06067800577058	9.14744366755259	0.999984362800348
4	8.53228969180900	1.12238972162512	9.07596267735889	0.999999968846574
5	8.45108721657438	1.18280447433853	9.00468628126089	1.00000020030451
6	8.36999399345207	1.24192670810562	8.93362315817494	1.00000016812645
7	8.28902146271176	1.29976325291497	8.86277934051403	1.00000014262042
8	8.20817933122734	1.35632219180389	8.79215937750138	1.00000012236522
9	8.12747622756739	1.41161229504697	8.72176691752843	1.00000010618077
10	8.04692006065515	1.46564271291772	8.65160502630982	1.00000009284194
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
166	1.00031401521149	0.999711293014288	1.12206970860473	0.999967693819351
167	1.00031010862601	0.999718641089626	1.10201854920601	0.999968427501599
168	1.00030610969925	0.999725853261276	1.08245168429949	0.999969152580824
169	1.00030202573472	0.999732929257881	1.06336073020762	0.999969868869467
170	1.00029786389858	0.999739869024494	1.04473729773544	0.999970576173761
171	1.00029362989013	0.999746673790074	1.02657308709375	0.999971274140246
172	.00028932639364	0.999753330048160	1.00885960830576	0.999971962732758
173	1.00014344006196	0.999865399336672	1.00024238576235	0.999967034603878
174	1.00001368001678	0.999988955179276	1.00001158885806	0.999996120420831
CPU time is seconds	0.638950			

Table 5: Example 4.2: Numerical study of Algorithm 1 and $u_0 = u_1 = [10, -20, 30, -40]^T$.

Iter (n)	u_1	u_2	u_3	u_4
1	3.62540316591384	-4.50907358998138	6.32280270247068	-0.00324645702473703
2	2.35551832614327	1.00000000005651	5.05780638110935	-0.698371567191649
3	1.71910447671045	0.953160264334062	4.29177164937510	0.999999999525121
4	1.15470777752108	0.895315463559908	3.59372641421376	0.999999983377171
5	1.13296432437644	0.650962900197076	2.75601344318112	0.902777544700774
6	1.23231081634727	0.698699138712538	2.24361021384599	0.932339788593485
7	1.09676723997533	0.938983693943763	1.99524423852901	1.01664900928582
8	1.06998496490214	0.914461755684416	1.67181167392322	0.988172769212471
9	1.10337085979123	0.897210465543632	1.39018023991409	0.981585822559744
10	1.06931881399759	0.948386311174497	1.22427628816323	0.995433685763384
⋮	⋮	⋮	⋮	⋮
55	1.00003232233466	0.999983095332643	1.00001083828621	0.999990015035678
56	1.00003157023173	0.999983454013759	1.00001064467299	0.999990250646380
57	1.00003082551930	0.999983859831028	1.00001036357615	0.999990479905837
58	1.00003052571655	0.999984045978129	1.00001021378732	0.999990569779100
59	1.00003025202721	0.999984177903946	1.00001014388048	0.999990654787794
60	1.00002964901522	0.999984471374164	1.00000997843332	0.999990843128007
61	1.00002905083912	0.999984794582010	1.00000975810614	0.999991027421348
CPU time is seconds	0.280577			

Example 4.3. Suppose that the non-linear complementarity problem of Kojima–Shindo while the feasible set $\mathcal{A}$ is

$$\mathcal{A} = \{u \in \mathbb{R}^4 : 1 \leq u_i \leq 5,\ i = 1, 2, 3, 4\},$$

and the mapping $\mathcal{T} : \mathbb{R}^4 \to \mathbb{R}^4$ is evaluated by

$$\mathcal{T}(u) = \begin{pmatrix} u_1 + u_2 + u_3 + u_4 - 4u_2u_3u_4 \\ u_1 + u_2 + u_3 + u_4 - 4u_1u_3u_4 \\ u_1 + u_2 + u_3 + u_4 - 4u_1u_2u_4 \\ u_1 + u_2 + u_3 + u_4 - 4u_1u_2u_3 \end{pmatrix}.$$

It is easy to see that $\mathcal{T}$ is not monotone on the set $\mathcal{A}$. By using the Monte-Carlo approach [6], it can be shown that $\mathcal{T}$ is pseudo-monotone on $\mathcal{A}$. This problem has a unique solution $u^* = (5, 5, 5, 5)^T$. Actually, in general, it is a very difficult task to check the pseudomonotonicity of any mapping $\mathcal{T}$ in practice. We here employ the Monte Carlo approach according to the definition of pseudo-monotonicity: Generate a large number of pairs of points u and y uniformly in $\mathcal{A}$ satisfying $\mathcal{T}(u)^T(y - u) \geq 0$ and then check if $\mathcal{T}(y)^T(y - u) \geq 0$. The control condition are taken as follows: (1) Algorithm 2 in [31]: $\zeta_0 = 0.15; , \mu = 0.70, \alpha_n = \frac{1}{100(n+2)}$; (2) Algorithm 1: $\zeta_0 = 0.15, \mu = 0.70, \alpha = 0.60, \epsilon_n = \frac{1}{(n+1)^2}, \varphi_n = \frac{100}{(n+1)^2}, \delta_n = \frac{1}{(n+2)}$. During this experiment, the initial points are different and $D_n = \|\Im_n - y_n\| \leq 10^{-3}$. The numerical results of these algorithms are shown in Tables 6-9.

Table 6: Example 4.3: Numerical study of Algorithm 2 in [31] and $u_0 = u_1 = [1, 2, 3, 4]^T$.

Iter (n)	u_1	u_2	u_3	u_4
1	-34.4858669275805	51.8664403130701	74.7007514676511	81.6555636007234
2	-34.7655055354728	52.7053561367472	75.5396672913282	82.4944794244005
3	-68.6885783093673	54.9801618868864	49.8539261662733	48.7706022828314
4	17.3186243752486	21.2315776206067	33.0214233757886	35.6941859283492
5	2.14464032966519	3.41342758196463	6.08695506300084	6.57660634608289
6	2.16077834343589	3.42336001727149	5.00079945477229	5.00277316956197
7	2.17693740637592	3.43332429093693	5.00000861166124	4.99999726228770
8	2.19314690959707	3.44336852111214	4.99999946975626	4.99999956233048
9	2.20940684182298	3.45349275161455	4.99999955360422	4.99999955283736
10	2.22571727346680	3.46369703796100	4.99999959252883	4.99999959253525
⋮	⋮	⋮	⋮	⋮
141	4.86520623565335	4.99999982676687	4.99999982676687	4.99999982676687
142	4.88890753296478	4.99999983165645	4.99999983165645	4.99999983165645
143	4.91260767205413	4.99999983609968	4.99999983609968	4.99999983609968
144	4.93630665260868	4.99999984001939	4.99999984001939	4.99999984001939
145	4.96000446723282	4.99999984294752	4.99999984294752	4.99999984294752
146	4.98370113286901	4.99999999990961	4.99999999990961	4.99999999990961
147	4.99997372263069	5.00000260496717	5.00000260496717	5.00000260496717
148	4.99999974352490	4.99999988910880	4.99999988910880	4.99999988910880
CPU time is seconds	0.658704			

Table 7: Example 4.2: Numerical study of Algorithm 1 and $u_0 = u_1 = [1, 2, 3, 4]^T$.

Iter (n)	u_1	u_2	u_3	u_4
1	-22.4790198515975	45.1962163525125	61.2227868529569	64.3371909274850
2	-22.7250625769091	45.8890698132465	61.9364390512654	65.0528149718569
3	-46.9191743963588	45.3948805004849	49.2573891717001	50.2790369811032
4	4.76470808480016	31.6637577598404	6.57170133017410	0.238787400016093
5	4.82937349335325	4.99939226879006	5.01026963319225	0.366396789829221
6	34.3710564606067	34.4227198682972	34.4353964718562	0.629949296125400
7	45.0594866095192	45.1112124144806	45.1239165658566	1.91644510896386
8	50.4501740621952	50.5022462269367	50.5150352890470	-1.97169151358552
9	50.5314043274895	50.5834595574629	50.5962444601586	-2.00626829664043
10	50.6403242132421	50.6917607820349	50.7043937855899	-5.12057188697462
⋮	⋮	⋮	⋮	⋮
46	5.00534491845521	5.00534300138664	5.00534228567228	4.98393420214921
47	5.00382749328779	5.00382612764508	5.00382561779860	4.98850257752182
48	5.00271869976732	5.00271773349804	5.00271737275274	4.99183705482842
49	5.00190964154940	5.00190896479704	5.00190871213941	4.99426819503386
50	5.00132055408830	5.00132008712633	5.00131991279145	4.99603731806545
51	5.00089307847857	5.00089276321315	5.00089264551241	4.99732055470556
52	5.00058462738082	5.00058442128729	5.00058434434462	4.99824621008654
53	5.00036429098967	5.00036416273020	5.00036411484598	4.99890728946919
54	5.00021838609581	5.00021830930436	5.00021828063514	4.99934498398791
55	5.00014354541464	5.00014349500391	5.00014347618363	4.99956947666068
CPU time is seconds	0.232178			

Table 8: Example 4.3: Numerical study of Algorithm 2 in [31] and $u_0 = u_1 = [-1, -2, 3, 4]^T$.

Iter (n)	u_1	u_2	u_3	u_4
1	-6.20601542924621	40.0843112633838	48.1337650982707	46.6230007187220
2	-6.34498204992797	40.5012111254306	48.5506649603177	47.0399005807689
3	-12.9307982468928	40.0487124082963	47.5003319282679	46.0967916997756
4	-12.8245110997443	40.1793368929725	47.6269383722835	46.2241156086185
5	-33.1769125028857	34.7884817072125	37.8166900961878	37.2122205719181
6	-3.63715262714896	20.5019731319843	-1.76185029487043	2.54576398430683
7	1.04302948168927	5.05452389604395	1.10650198176853	2.54725466388794
8	1.04978285120354	5.00006058170198	1.11279404365063	2.54921548696884
9	1.05658705719584	4.99999076271743	1.11913985722867	2.55121489498328
10	1.06344317268371	4.99998642519517	1.12554051292081	2.55325387465541
⋮	⋮	⋮	⋮	⋮
181	4.59862733756248	5.00000320217875	4.61146934022478	4.99999682258005
182	4.66305529670103	4.99999994825175	4.67571532974269	5.00000005125787
183	4.72840120415464	5.00000000060611	4.74088184600983	4.99999999898082
184	4.79467821219419	4.99999999981385	4.80698200476301	4.99999999983888
185	4.86189966048480	4.99999999985196	4.87402910964953	4.99999999985158
186	4.93007907865535	4.99999999986557	4.94203665454547	4.99999999986557
187	4.99908430943714	4.99994852203121	5.00068438967529	4.99994852203121
188	4.99998796497466	5.00000039194907	5.00001096999677	5.00000039194907
189	4.99999982724761	5.00000000655933	5.00000015919175	5.00000000655933
CPU time is seconds	0.830074			

Table 9: Example 4.2: Numerical study of Algorithm 1 and $u_0 = u_1 = [-1, -2, 3, 4]^T$.

Iter (n)	u_1	u_2	u_3	u_4
1	67.7657143133604	76.7145530819899	119.464591876179	136.782073520191
2	-68.2650687067241	78.1919764133559	120.945690214610	138.263931342072
3	-112.028480196941	102.736419249501	68.7617248229671	59.8025934358457
4	10.0905654923274	7.52762193553885	48.8773488496240	59.7184201585889
5	1.58599071741342	0.271536323838546	5.42220385209038	5.83759718234703
6	1.76489239217402	0.956593352840844	5.01190765954745	4.97568499729916
7	18.9241448354569	-5.26396384315423	18.9282891809255	19.1336306496812
8	19.1267857198680	-5.33218756241530	19.1290503062070	19.3344556330365
9	19.1362728432230	-6.71061680639647	19.1381744023155	19.3158660670749
10	13.8198984213228	-14.2428711989199	13.8195679016292	13.7892140927893
⋮	⋮	⋮	⋮	⋮
39	5.00036606449632	4.99890329629848	5.00036604809619	5.00036452257377
40	5.00029614550570	4.99911277307117	5.00029613224497	5.00029489874547
41	5.00024019269176	4.99928042449310	5.00024018194207	5.00023918201651
42	5.00021568341312	4.99935386554261	5.00021567376480	5.00021477628717
43	5.00022114499712	4.99933751266914	5.00022113510826	5.00022021525583
44	5.00023906491462	4.99928382965716	5.00023905422789	5.00023806015776
45	5.00024676624015	4.99926074961903	5.00024675521250	5.00024572943005
46	5.00023803185504	4.99928690765120	5.00023802122102	5.00023703205387
47	5.00022237747575	4.99933380427636	5.00022236754428	5.00022144372719
48	5.00020993400317	4.99937108540127	5.00020992463042	5.00020905278423
49	5.00020466823032	4.99938686331783	5.00020465909544	5.00020380937576
CPU time is seconds	0.226685			

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