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Analytical solutions for time-fractional Swift–Hohenberg equations via a modified integral transform technique

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Abstract

In this work, the fractional natural transform decomposition method (FNTDM) is employed to obtain approximate analytical solutions for some time-fractional versions of the nonlinear Swift-Hohenberg (S-H) equation with the fractional derivatives taken in the sense of Caputo. The S-H equation models problems arising from fluid dynamics and describes temperature dynamics of thermal convection as well as complex pattern formation processes in liquid surfaces bounded along a horizontally well-conducting boundary. To explore the applicability, simplicity, and efficiency of the FNTDM, numerical simulations are provided for each of the considered problems to demonstrate the behavior of the obtained approximate solutions for different values of the fractional parameter index. The obtained simulations show a similar resemblance with those in existing related literature and further confirm the applicability of the considered method to even complex problems arising in diverse fields of applied mathematics and physics.

Keywords: Fractional Swift-Hohenberg equation, Caputo derivative, Natural transform method, Adomian decomposition method
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1 Introduction

The theory of fractional calculus (that is, calculus of non-integer or arbitrary order differential and integral operators) is one of the indispensable areas of modern mathematical analysis with extensive range of important applications in diverse research areas of applied mathematics [6, 7, 11, 12, 14, 16, 17, 18, 19, 20, 23, 29, 30, 40, 35, 36, 37, 52]. It generalizes the concept of classical (integer order) calculus in the sense that it allows fractional values for the order of differentiation and integration. Generally, fractional calculus encompasses various studies involving fractional order (ordinary and partial) differential and integral operators/equations. Significant advantages of fractional calculus that necessitate its wide applicability in the mathematical modeling of various physical phenomena are embedded in the nonlocality properties of the operators, which cannot be adequately captured when modeling within the framework of classical calculus. By nonlocality properties, we refer to features which allow future states of a given dynamical system to not only depend on its present state, but also on all past states as well. This makes it possible for the system's response to be influenced by all previous responses at any given moment. In many studies, the nonlocality

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properties have been shown to include global information as well as memory and hereditary effects which are elegantly incorporated in the modeling of real world problems via suitable extensions of the classical differential and integral operators to those of fractional orders.

Following their immense important significance, fractional differential and integral operators are now being used to reformulate and study interesting dynamical properties of various new and existing models arising in fluid flow and turbulence, mathematical epidemiology, financial systems, electric networks and oscillatory systems, reaction-diffusion and anomalous processes, optical fibers and plasma, climate change, porous structures and several other physical phenomena (see, for instance [17, 18, 19, 20, 23, 35, 36, 37, 48, 51] and the references therein). Essentially, the formulated models are often posed in the forms of systems of nonlinear differential equations with fractional order derivatives. Unlike the classical differential and integral operators, there abound different types of fractional order differential and integral operators such as the Riesz, Hadamard, Grünwald–Letnikov, Erdélyi–Kober, Riemann-Liouville, Caputo (see [11, 14, 29, 30, 40]) types as well as the recently introduced Caputo-Fabrizio [12] and Atangana-Baleanu [6] types . However, recent related works indicate that the Caputo fractional derivative appears to be the most widely used type of fractional differential operator. Its definition borders around the convolution of the local derivative of a given function with the power law function as its kernel. For this operator, the derivative of a constant function is zero and the initial conditions for the Caputo fractional differential equation can be properly defined and handled in a similar manner as that of the classical ordinary derivative. In recent years, there have been several studies on nonlinear fractional partial differential equations (PDEs). This have yielded a wealth of interesting works on nonlinear mathematical models arising in diverse fields of applied mathematics, engineering and finance within the framework of the Caputo fractional derivative (see, for instance [5, 15, 51] as well as some of the earlier cited references).

Generally, obtaining exact solutions of nonlinear PDEs is more difficult than those of linear PDEs. However, the case of nonlinear fractional PDEs is even much more complex as there exist no known straightforward method for finding their exact or analytical solutions with respect to given initial and/or boundary conditions. Due to this general difficulty, many authors have introduced effective numerical techniques as well as analytical methods for obtaining approximate and series-type solutions for the aforementioned class of problems. Some well studied numerical solution techniques for fractional differential equations include, the meshless method [23], Riccati transformation method [8], reproducing kernel Hilbert space method [7] and finite element method [25, 26]. On the other hand, examples of some analytical techniques that have been extensively used to solve nonlinear fractional order PDEs include variational iteration method (VIM) [2], homotopy analysis method (HAM) [4, 22, 34], residual power series method (RPSM) [41], differential transform method (DTM) [46], Laplace transform method (LPM) [49], natural transform method (NTM) [43], Sumudu transform method (STM) [45] and many more. In other to further extend these analytical methods, some authors have devised new hybrid techniques by suitable combination of some of the mentioned analytical techniques with each other to obtain even more efficient hybrid methods. Some of such hybrid methods include the $q-$homotopy analysis transform method ($q-$HATM) [37], modified reduced differential transform method (MRDTM), fractional natural transform decomposition method (FNDTM) [1] and several others (see, for instance [3, 10, 18, 19, 24] among others). Particularly, the FNDTM is a unique combination of the NTM and the ADM. For this method, the solution of the considered problem assumes the form of a decomposition series while the nonlinear term is decomposed by using Adomian polynomials. Generally, analytical solution methods are essentially based on obtaining succeeding solution components of the series from given initial conditions by using properly defined iterative processes yielded by the associated recurrence relations.

In this paper, the FNTDM is applied to obtain series-type solutions of some one dimensional fractional versions of the Swift-Hohenberg (S-H) equation whose general classical form given by

$$\frac{\partial u}{\partial t} = \eta u - (1-\nabla^2)^2 u + N(u),\ x \in \mathbb{R},\ t > 0, \tag{1.1}$$

was first introduced in 1977 by J. Swift and P. Hohenberg [50] as a universal model for describing fluid velocity and temperature dynamics of thermal convection. Here, $u = u(x,t)$ is a scalar function defined on the line or the plane, η is a bifurcation parameter and $N(u)$ is a nonlinear term. Apart from its use in the study of the impact of noise on bifurcations, spatiotemporal chaos and defect dynamics, it has also been used to model patterns in both simplistic (e.g., Rayleig-Bénard convection) and complicated liquids and biological substances (e.g., brain tissues) [43]. There are diverse important applications of the S-H equation in biological and chemical systems, laser physics, hydrodynamics, magneto-convection as well as in liquid-crystal light-valve experiment, see [13, 21, 31, 38, 39]. It plays a central role in complex pattern formation phenomena arising in fluid layers confined between horizontal well-conducting boundaries.

More precisely, the fractional versions of the S-H equation considered are

$$ {}^{\mathtt{C}}D_{\tau}^{\varrho}\Psi(\varsigma,\tau)+\frac{\partial^4\Psi(\varsigma,\tau)}{\partial\varsigma^4}+2\frac{\partial^2\Psi(\varsigma,\tau)}{\partial\varsigma^2}+(1-\alpha)\Psi(\varsigma,\tau)+\Psi^3(\varsigma,\tau)=0, \tag{1.2} $$

$$ {}^{\mathtt{C}}D_{\tau}^{\varrho}\Psi(\varsigma,\tau)+\frac{\partial^4\Psi(\varsigma,\tau)}{\partial\varsigma^4}-\eta\frac{\partial^3\Psi(\varsigma,\tau)}{\partial\varsigma^3}+2\frac{\partial^2\Psi(\varsigma,\tau)}{\partial\varsigma^2}-\gamma\Psi(\varsigma,\tau)-2\Psi^2(\varsigma,\tau)+\Psi^3(\varsigma,\tau)=0, \tag{1.3} $$

and

$$ {}^{\mathtt{C}}D_{\tau}^{\varrho}\Psi(\varsigma,\tau)+\frac{\partial^4\Psi(\varsigma,\tau)}{\partial\varsigma^4}+2\frac{\partial^2\Psi(\varsigma,\tau)}{\partial\varsigma^2}+(1-\alpha)\Psi(\varsigma,\tau)-\Psi^l(\varsigma,\tau)+\left(\frac{\partial\Psi(\varsigma,\tau)}{\partial\varsigma}\right)^l=0, \tag{1.4} $$

where $(\varsigma,\tau)\in$, $\alpha\in\mathbb{R}$, $l\geq 0$ $0<\varrho\leq 1$ is the fractional parameter index and η and γ are dispersive and bifurcation parameters, respectively. Some important approximate results have been obtained for various versions of the S-H equation. For instance, using the HAM, Akyildiz et al. [4] obtained analytical solutions for the classical version of the S-H equation 1.2. The HPM and DTM were employed by Khan et al. [27] to obtain analytical solutions for the time-fractional S-H equation 1.2. The fractional S-H equations 1.2-1.4 were considered in [56] using the Laplace Adomian decomposition method (LADM). Vishal and others obtained approximate analytical solutions of time-fractional versions of nonlinear S-H equation while the fractional variational iteration method (FVIM) with modified Riemann–Liouville derivative was employed by Merdan [33] to obtain approximate solutions of time-fractional S-H equation.

In concluding this section of the paper, we outline the arrangement of the remaining part of this paper as follows: In Section **??** we present basic definitions of the fractional operator related to this work. We also provide a brief discussion on the natural transform operator as well as on some of its important properties. By considering a general fractional initial value problem with time-fractional derivative in the sense of Caputo, we present the basic solution principle of the FNDTM in Section 3. In Section 4, we exhibit the effectiveness of the considered method by present the main results on the application of the FNDTM for three variant forms of the fraction S-H equation. Finally, we summarize our findings via the conclusion provided in Section 5.

2 Preliminaries

Here, we recall some important definitions and results that are sufficient for this work (see [11, 30, 40] for more details). We also introduce the natural transform operator [9, 32] as well as some of its important properties.

Definition 2.1. A real function $\Psi(\tau)$, $\tau>0$, is said to be in the space C_μ ($\mu\in\mathbb{R}$) if there exists a real number $p(>\mu)$ such that $\Psi(\tau)=\tau^p\psi(\tau)$, where $\psi(\tau)\in C[0,\infty)$. It is said to be in the space C_μ^n if $\Psi^{(n)}\in C_\mu$, $n\in\mathbb{N}$.

Definition 2.2. The Riemann-Liouville fractional derivative of a function $\Psi\in C_{-1}^n$ ($n\in\mathbb{N}$) is defined as

$$ {}^{\mathtt{RL}}D_{\tau}^{\varrho}\Psi(\tau)=\begin{cases}\dfrac{d^n\Psi(\tau)}{d\tau^n}, & \varrho=n,\\[2mm] \dfrac{1}{\Gamma(n-\varrho)}\dfrac{d^n}{d\tau^n}\displaystyle\int_0^t(\tau-\zeta)^{n-\varrho-1}\Psi(\zeta)d\zeta, & n-1<\varrho\leq n,\end{cases} \tag{2.1} $$

where $\Gamma(\cdot)$ is the well known Gamma function.

Definition 2.3. The Riemann-Liouville fractional integral of order $\varrho\geq 0$ of a function $\Psi\in C_\mu$ ($\mu\geq -1$), is defined as

$$ I_{\tau}^{\varrho}\Psi(\tau)=\begin{cases}\Psi(\tau), & \varrho=0,\\[2mm] \dfrac{1}{\Gamma(\varrho)}\displaystyle\int_0^t(\tau-\zeta)^{\varrho-1}\Psi(\zeta)d\zeta, & \varrho>0,\ \tau>0.\end{cases} \tag{2.2} $$

Definition 2.4. The fractional derivative of order $\varrho\geq 0$ in the Caputo sense is defined as

$$ {}^{\mathtt{C}}D_{\tau}^{\varrho}\Psi(\tau)=\begin{cases}\dfrac{d^n\Psi(\tau)}{d\tau^n}, & \varrho=n,\\[2mm] \dfrac{1}{\Gamma(n-\varrho)}\displaystyle\int_0^t(\tau-\zeta)^{n-\varrho-1}D_\zeta^n\Psi(\zeta)d\zeta, & n-1<\varrho\leq n.\end{cases} \tag{2.3} $$

Lemma 2.5. For $n-1 < \varrho \leq n \in \mathbb{N}$ and $\Psi \in C_\mu^n$ $(\mu \geq -1)$, the relations

$$^{\mathtt{C}}D_\tau^\varrho[I_\tau^\varrho \Psi(\tau)] = \Psi(\tau),\ \tau > 0, \quad \text{and} \quad I_\tau^\varrho[^{\mathtt{C}}D_\tau^\varrho \Psi(\tau)] = \Psi(\tau) - \sum_{k=0}^{n-1} \Psi^+(0)\frac{\tau^k}{k!},\ \tau > 0,$$

are satisfied by the Riemann-Liouville integral and Caputo derivative.

Definition 2.6. The natural transform ($\mathbb{NT}^+$) of the function $\Psi \in \Delta$, denoted by $\mathbb{NT}^+[\Psi(\tau)]$, over the set

$$\Delta = \left\{ \Psi(\tau) : \exists\ M, \nu_1, \nu_2 > 0, |\Psi(\tau)| < M \exp\left(\frac{|t|}{\nu_i}\right),\ if\ t \in (-1)^i \times [0,\infty) \right\},$$

is defined by the integral

$$\mathbb{NT}^+[\Psi(\tau)] = \int_0^\infty \exp(-st)\Psi(\omega t)dt, \tag{2.4}$$

where $s > 0$ and $\omega > 0$ denote the natural transform variables.

Remark 2.7. The natural transform admits the following properties:

(i) $\mathbb{NT}^+[1] = \frac{1}{s}$; $\quad \mathbb{NT}^+[\tau] = \frac{\omega}{s^2}$; $\quad \mathbb{NT}^+[\tau^\varrho] = \frac{\Gamma(\varrho+1)\omega^\varrho}{s^{\varrho+1}},\ \varrho > -1$; $\quad \mathbb{NT}^+\left[\frac{\tau^{n-1}}{(n-1)!}\right] = \frac{\omega^{n-1}}{s^n}, n = 1, 2, \cdots.$

(ii) $\mathbb{NT}^+[\alpha\Psi_1(\tau) + \beta\Psi_2(\tau)] = \alpha\mathbb{NT}^+[\Psi_1(\tau)] + \beta\mathbb{NT}^+[\Psi_2(\tau)]$ where α and β are non-zero constants.

(iii) $\mathbb{NT}^+[\Psi^n(\tau)] = \frac{s^n}{\omega^n}\mathbb{NT}^+[\Psi(\tau)] - \sum_{k=0}^{n-1} \frac{s^{n-k-1}}{\omega^{n-k}}\Psi^{(k)}(0),\ n \in \mathbb{N}.$

Definition 2.8. The inverse $\mathbb{NT}^+$ is defined by

$$\mathbb{NT}^{-1}[\mathbb{NT}^+[\Psi(\tau)]] = \Psi(\tau) = \frac{1}{2i\pi}\int_{\delta-i\infty}^{\delta+i\infty} \exp\left(\frac{s\tau}{\omega}\right)\mathfrak{T}^+(s,\omega)ds, \tag{2.5}$$

where δ is a real constant and the integral is taken along $s = \delta$ in the complex plane $s = x + iy$.

3 Basic solution principle of the FNTDM

In this section, we discuss the basic solution principle of the FNTDM. Firstly, we state the following important result.

Theorem 3.1. [28] If $\mathbb{NT}^+[\Psi(\varsigma,\tau)]$ is the $\mathbb{NT}^+$ of $\Psi(\varsigma,\tau)$, then the $\mathbb{NT}^+$ of the Riemann-Liouville fractional integral of $\Psi(\varsigma,\tau)$ of order ϱ, is given by

$$\mathbb{NT}^+[I_\tau^\varrho \Psi(\varsigma,\tau)] = \frac{\omega^\varrho}{s^\varrho}\mathbb{NT}^+[\Psi(\varsigma,\tau)]. \tag{3.1}$$

Theorem 3.2. [28] Let $n-1 < \varrho \leq n$ with $n \in \mathbb{N}$ and $\mathbb{NT}^+[\Psi(\varsigma,\tau)]$ be the $\mathbb{NT}^+$ of the function $\Psi(\varsigma,\tau)$. Then the $\mathbb{NT}^+$ with respect to the Riemann-Liouville and Caputo fractional derivatives of the function $\Psi(\varsigma,\tau)$ is given by

$$\mathbb{NT}^+[^{\Xi(\varrho)}D_\tau^\varrho \Psi(\varsigma,\tau)] = \begin{cases} \frac{s^\varrho}{\omega^\varrho}\mathbb{NT}^+[\Psi(\varsigma,\tau)] - \sum_{k=0}^{n-1}\frac{s^k}{\omega^{\varrho-k}}\left[\frac{\partial^{\varrho-k-1}\Psi(\varsigma,\tau)}{\partial\tau^{\varrho-k-1}}\right]_{\tau=0}, & if\ \Xi(\varrho) = \mathtt{RL}, \\ \\ \frac{s^\varrho}{\omega^\varrho}\mathbb{NT}^+[\Psi(\varsigma,\tau)] - \sum_{k=0}^{n-1}\frac{s^{\varrho-k-1}}{\omega^{\varrho-k}}\left[\frac{\partial^k\Psi(\varsigma,\tau)}{\partial\tau^k}\right]_{\tau=0}, & if\ \Xi(\varrho) = \mathtt{C}. \end{cases} \tag{3.2}$$

Next, to discuss the solution methodology of the FNTDM, we consider the following general fractional initial value problem with time-fractional derivative given in the sense of Caputo:

$$\begin{cases} {}^{\mathrm{C}}D_\tau^\varrho \Psi(\varsigma,\tau) + \mathscr{L}[\Psi(\varsigma,\tau)] + \mathscr{N}[\Psi(\varsigma,\tau)] = \mathbb{F}(\varsigma,\tau), \ (\varsigma,\tau) \in [0,1]\times[0,T], \quad n-1<\varrho\le n, \\ \Psi^{(n)}(\varsigma,0) = \Psi_n(\varsigma), \quad n=0,1,2,\cdots,n-1, \end{cases} \tag{3.3}$$

where $\mathscr{L}$ and $\mathscr{N}$ represent linear and nonlinear partial differential operators, respectively, and $\mathbb{F}(\varsigma,\tau)$ is a non-homogeneous term. An application of $\mathbb{NT}^+$ on (3.3) gives

$$\mathbb{NT}^+[{}^{\mathrm{C}}D_\tau^\varrho \Psi(\varsigma,\tau)] + \mathbb{NT}^+[\mathscr{L}[\Psi(\varsigma,\tau)]] + \mathbb{NT}^+[\mathscr{N}[\Psi(\varsigma,\tau)]] = \mathbb{NT}^+[\mathbb{F}(\varsigma,\tau)].$$

Moreover, the differentiation property (3.2) for the Caputo derivative implies

$$\begin{aligned} \mathbb{NT}^+[\Psi(\varsigma,\tau)] =& \frac{\omega^\varrho}{s^\varrho}\sum_{k=0}^{n-1}\frac{s^{\varrho-k-1}}{\omega^{\varrho-k}}[{}^{\mathrm{C}}D^k\Psi(\varsigma,\tau)]_{\tau=0} + \frac{\omega^\varrho}{s^\varrho}\mathbb{NT}^+[\mathbb{F}(\varsigma,\tau)] \\ &- \frac{\omega^\varrho}{s^\varrho}\mathbb{NT}^+[\mathscr{L}[\Psi(\varsigma,\tau)] + \mathscr{N}[\Psi(\varsigma,\tau)]] \end{aligned} \tag{3.4}$$

Applying the inverse $\mathbb{NT}^+$ on (3.4) gives

$$\begin{aligned} \Psi(\varsigma,\tau) =& \mathbb{NT}^{-1}\left[\frac{\omega^\varrho}{s^\varrho}\sum_{k=0}^{n-1}\frac{s^{\varrho-k-1}}{\omega^{\varrho-k}}[{}^{\mathrm{C}}D^k\Psi(\varsigma,\tau)]_{\tau=0} + \frac{\omega^\varrho}{s^\varrho}\mathbb{NT}^+[\mathbb{F}(\varsigma,\tau)]\right] \\ &- \mathbb{NT}^{-1}\left[\frac{\omega^\varrho}{s^\varrho}\mathbb{NT}^+[\mathscr{L}[\Psi(\varsigma,\tau)] + \mathscr{N}[\Psi(\varsigma,\tau)]]\right]. \end{aligned} \tag{3.5}$$

As in the ADM, the FNTDM defines the solution $\Psi(\varsigma,\tau)$ of the general fractional IVP (3.3) as an infinite series of the form

$$\Psi(\varsigma,\tau) = \sum_{n=0}^{\infty}\Psi_n(\varsigma,\tau), \tag{3.6}$$

while the nonlinear term appearing in (3.3) is represented by the infinite series of the Adomian polynomials [55]

$$\mathscr{N}[\Psi(\varsigma,\tau)] = \sum_{k=0}^{\infty}A_k \quad \text{with} \quad A_k = \frac{1}{k!}\left[\frac{d^k}{d\lambda^k}\left[\mathscr{N}\sum_{k=0}^{\infty}\left(\lambda^k\Psi_k(\varsigma,\tau)\right)\right]\right]_{\lambda=0}, \quad k=0,1,2,\cdots. \tag{3.7}$$

Substituting (3.6)-(3.7) into (3.5) yields

$$\sum_{\infty}^{n=0}\Psi_n(\varsigma,\tau) = K(\varsigma,\tau) - \mathbb{NT}^{-1}\left[\frac{\omega^\varrho}{s^\varrho}\mathbb{NT}^+\left[\mathscr{L}\sum_{n=0}^{\infty}\Psi_n(\varsigma,\tau)\right] + \sum_{n=0}^{\infty}A_n\right], \tag{3.8}$$

where the term

$$K(\varsigma,\tau) \equiv \mathbb{NT}^{-1}\left[\frac{\omega^\varrho}{s^\varrho}\sum_{k=0}^{n-1}\frac{s^{\varrho-k-1}}{\omega^{\varrho-k}}[{}^{\mathrm{C}}D^k\Psi(\varsigma,\tau)]_{\tau=0} + \frac{\omega^\varrho}{s^\varrho}\mathbb{NT}^+[\mathbb{F}(\varsigma,\tau)]\right]$$

arises from the prescribed initial data and the non-homogeneous term. From (3.8), we equate terms on both sides to obtain the first few solution components as

$$\begin{aligned} \Psi_0(\varsigma,\tau) &= K(\varsigma,\tau), \\ \Psi_1(\varsigma,\tau) &= -\mathbb{NT}^{-1}\left[\frac{\omega^\varrho}{s^\varrho}\mathbb{NT}^+\left[\mathscr{L}\Psi_0(\varsigma,\tau)\right] + A_0\right] \\ \Psi_2(\varsigma,\tau) &= -\mathbb{NT}^{-1}\left[\frac{\omega^\varrho}{s^\varrho}\mathbb{NT}^+\left[\mathscr{L}\Psi_1(\varsigma,\tau)\right] + A_1\right]. \\ \Psi_3(\varsigma,\tau) &= -\mathbb{NT}^{-1}\left[\frac{\omega^\varrho}{s^\varrho}\mathbb{NT}^+\left[\mathscr{L}\Psi_2(\varsigma,\tau)\right] + A_2\right], \end{aligned} \tag{3.9}$$

which yields the general recurrence relation as

$$\begin{cases} \Psi_0(\varsigma,\tau) = K(\varsigma,\tau), \\ \Psi_{n+1}(\varsigma,\tau) = -\mathbb{NT}^{-1}\left[\frac{\omega^\varrho}{s^\varrho}\mathbb{NT}^+\left[\mathscr{L}\Psi_n(\varsigma,\tau)\right] + A_n\right], \ n\ge 0. \end{cases} \tag{3.10}$$

Finally, the approximate FNTDM solution is given by

$$\Psi(\varsigma,\tau) = \sum_{n=0}^{\infty} \Psi_n(\varsigma,\tau) = \Psi_0(\varsigma,\tau) + \Psi_1(\varsigma,\tau) + \Psi_2(\varsigma,\tau) + \cdots . \tag{3.11}$$

3.1 Convergence analysis

We present the uniqueness and convergence of the FNTDM.

Theorem 3.3. (**Uniqueness Result**) The FNTDM solution for the fractional IVP 3.3 is unique if $0 < \varepsilon < 1$ holds where

$$\varepsilon = (\gamma_1 + \gamma_2)\frac{\tau^{\varrho}}{\Gamma(\varrho+1)}.$$

Proof . Let $\mathbf{X} = (C[J], \|\cdot\|)$ be a Banach space of continuous mapping defined on $J = [0,T)$ with norm $\|\Psi(\varsigma,\tau)\| = \max_{\tau\in J}|\Psi(\varsigma,\tau)|$. We introduce the mapping $\Pi : \mathbf{X} \to \mathbf{X}$ with

$$\Psi_{n+1}(\varsigma,\tau) = \Psi_0(\varsigma,\tau) + \mathbb{NT}^{-1}\left[\frac{\omega^{\varrho}}{s^{\varrho}}\mathbb{NT}^{+}\left[\mathscr{L}\Psi_n(\varsigma,\tau) + \mathscr{N}\Psi_n(\varsigma,\tau)\right]\right],\ n \geq 0. \tag{3.12}$$

Suppose that $\|\mathscr{L}\Psi_n(\varsigma,\tau) - \mathscr{L}\Psi_n^*(\varsigma,\tau)\| \leq \gamma_1\|\Psi_n(\varsigma,\tau) - \Psi_n^*(\varsigma,\tau)\|$ and $\|\mathscr{N}\Psi_n(\varsigma,\tau) - \mathscr{N}\Psi_n^*(\varsigma,\tau)\| \leq \gamma_2\|\Psi_n(\varsigma,\tau) - \Psi_n^*(\varsigma,\tau)\|$ where γ_1 and γ_2 are Lipschitz constants, then

$$\begin{aligned}
&\|\Pi[\Psi(\varsigma,\tau)] - \Pi[\Psi^*(\varsigma,\tau)]\| \\
&= \max_{\tau\in J}\left|\mathbb{NT}^{-1}\left[\frac{\omega^{\varrho}}{s^{\varrho}}\mathbb{NT}^{+}\Big[\mathscr{L}[\Psi_n(\varsigma,\tau)] - \mathscr{L}[\Psi_n^*(\varsigma,\tau)]\Big] + \frac{\omega^{\varrho}}{s^{\varrho}}\mathbb{NT}^{+}\Big[\mathscr{N}[\Psi_n(\varsigma,\tau)] - \mathscr{N}[\Psi_n^*(\varsigma,\tau)]\Big]\right]\right| \\
&\leq \max_{\tau\in J}\left[\gamma_1\mathbb{NT}^{-1}\left[\frac{\omega^{\varrho}}{s^{\varrho}}\mathbb{NT}^{+}|\Psi_n(\varsigma,\tau) - \Psi_n^*(\varsigma,\tau)|\right] + \gamma_2\mathbb{NT}^{-1}\left[\frac{\omega^{\varrho}}{s^{\varrho}}\mathbb{NT}^{+}|\Psi_n(\varsigma,\tau) - \Psi_n^*(\varsigma,\tau)|\right]\right] \\
&\leq (\gamma_1+\gamma_2)\max_{\tau\in J}\left[\mathbb{NT}^{-1}\left[\frac{\omega^{\varrho}}{s^{\varrho}}\mathbb{NT}^{+}|\Psi_n(\varsigma,\tau) - \Psi_n^*(\varsigma,\tau)|\right]\right] \\
&\leq (\gamma_1+\gamma_2)\left[\mathbb{NT}^{-1}\left[\frac{\omega^{\varrho}}{s^{\varrho}}\mathbb{NT}^{+}\|\Psi_n(\varsigma,\tau) - \Psi_n^*(\varsigma,\tau)\|\right]\right] \\
&\leq (\gamma_1+\gamma_2)\frac{\tau^{\varrho}}{\Gamma(\varrho+1)}\|\Psi_n(\varsigma,\tau) - \Psi_n^*(\varsigma,\tau)\|.
\end{aligned} \tag{3.13}$$

Under the condition that $0 < \varepsilon < 1$, the mapping Π is a contraction. Hence, the Banach fixed point theorem asserts that there exists a unique solution to the fractional IVP (3.3). The proof is complete. □

Theorem 3.4. (**Convergence Result**) The FNTDM solution of the fractional IVP 3.3 is convergent.

Proof . Let $\Psi_k(\varsigma,\tau) = \sum_{r=0}^{k}\Psi_r(\varsigma,\tau)$. We prove that $\Psi_k(\varsigma,\tau)$ is a Cauchy sequence in $\mathbf{X}$. To this end, we have

$$\begin{aligned}
&\|\Psi_k(\varsigma,\tau) - \Psi_n(\varsigma,\tau)\| \\
&\quad = \max_{\tau\in J}|\Psi_k(\varsigma,\tau) - \Psi_n(\varsigma,\tau)| = \max_{\tau\in J}\left|\sum_{r=n+1}^{k}\Psi_r(\varsigma,\tau)\right| \\
&\quad \leq \max_{\tau\in J}\left|\mathbb{NT}^{-1}\left[\frac{\omega^{\varrho}}{s^{\varrho}}\mathbb{NT}^{+}\left[\sum_{r=n+1}^{k}(\mathscr{L}[\Psi_{r-1}(\varsigma,\tau)] + \mathscr{N}[\Psi_{r-1}(\varsigma,\tau)])\right]\right]\right| \\
&\quad = \max_{\tau\in J}\left|\mathbb{NT}^{-1}\left[\frac{\omega^{\varrho}}{s^{\varrho}}\mathbb{NT}^{+}\left[\sum_{r=n}^{k-1}(\mathscr{L}[\Psi_r(\varsigma,\tau)] + \mathscr{N}[\Psi_r(\varsigma,\tau)])\right]\right]\right| \\
&\quad \leq \max_{\tau\in J}\left|\mathbb{NT}^{-1}\left[\frac{\omega^{\varrho}}{s^{\varrho}}\mathbb{NT}^{+}\left[(\mathscr{L}[\Psi_{k-1}(\varsigma,\tau)] - \mathscr{L}[\Psi_{n-1}(\varsigma,\tau)] + \mathscr{N}[\Psi_{k-1}(\varsigma,\tau)] - \mathscr{N}[\Psi_{n-1}(\varsigma,\tau)])\right]\right]\right| \\
&\quad \leq \max_{\tau\in J}\left|\mathbb{NT}^{-1}\left[\frac{\omega^{\varrho}}{s^{\varrho}}\mathbb{NT}^{+}\left[(\mathscr{L}[\Psi_{k-1}(\varsigma,\tau)] - \mathscr{L}[\Psi_{n-1}(\varsigma,\tau)])\right]\right]\right|
\end{aligned}$$

$$
\begin{aligned}
&+\max_{\tau\in J}\left|\mathbb{NT}^{-1}\left[\frac{\omega^{\varrho}}{s^{\varrho}}\mathbb{NT}^{+}\left[(\mathscr{N}[\Psi_{k-1}(\varsigma,\tau)]-\mathscr{N}[\Psi_{n-1}(\varsigma,\tau)])\right]\right]\right|\\
\leq&\gamma_1\max_{\tau\in J}\left|\mathbb{NT}^{-1}\left[\frac{\omega^{\varrho}}{s^{\varrho}}\mathbb{NT}^{+}\left[\Psi_{k-1}(\varsigma,\tau)-\Psi_{n-1}(\varsigma,\tau)\right]\right]\right|\\
&+\gamma_2\max_{\tau\in J}\left|\mathbb{NT}^{-1}\left[\frac{\omega^{\varrho}}{s^{\varrho}}\mathbb{NT}^{+}\left[\Psi_{k-1}(\varsigma,\tau)-\Psi_{n-1}(\varsigma,\tau)\right]\right]\right|.
\end{aligned}
$$

This implies

$$
\|\Psi_k(\varsigma,\tau)-\Psi_n(\varsigma,\tau)\|=\leq(\gamma_1+\gamma_2)\frac{\tau^{\varrho}}{\Gamma(\varrho+1)}\|\Psi_{k-1}(\varsigma,\tau)-\Psi_{n-1}(\varsigma,\tau)\|. \tag{3.14}
$$

Let $k=n+1$, then

$$
\begin{aligned}
\|\Psi_{n+1}(\varsigma,\tau)-\Psi_n(\varsigma,\tau)\|\leq&\gamma\|\Psi_n(\varsigma,\tau)-\Psi_{n-1}(\varsigma,\tau)\|\leq\gamma^2\|\Psi_{n-1}(\varsigma,\tau)-\Psi_{n-2}(\varsigma,\tau)\|\\
&<\cdots\leq\gamma^n\|\Psi_1(\varsigma,\tau)-\Psi_0(\varsigma,\tau)\|,
\end{aligned}
$$

where $\gamma=(\gamma_1+\gamma_2)\frac{\tau^{\varrho}}{\Gamma(\varrho+1)}$. Similarly, we easily see that

$$
\begin{aligned}
\|\Psi_k(\varsigma,\tau)-\Psi_n(\varsigma,\tau)\|\leq&\|\Psi_{n+1}(\varsigma,\tau)-\Psi_n(\varsigma,\tau)\|+\|\Psi_{n+2}(\varsigma,\tau)-\Psi_{n+1}(\varsigma,\tau)\|\\
&+\cdots+\|\Psi_k(\varsigma,\tau)-\Psi_{k-1}(\varsigma,\tau)\|\\
\leq&\left[\gamma^n+\gamma^{n+1}+\cdots+\gamma^{m-1}\right]\|\Psi_1(\varsigma,\tau)-\Psi_0(\varsigma,\tau)\|\\
\leq&\gamma^n\left(\frac{1-\gamma^{m-n}}{1-\gamma}\right)\max_{\tau\in J}|\Psi_1(\varsigma,\tau)|.
\end{aligned}
$$

Now $0<\gamma<1$ implies $0<\gamma^{m-n}<1$ and hence

$$
\|\Psi_k(\varsigma,\tau)-\Psi_n(\varsigma,\tau)\|\leq\frac{\gamma^n}{1-\gamma}\max_{\tau\in J}|\Psi_1(\varsigma,\tau)|.
$$

But $|\Psi_1(\varsigma,\tau)|\leq\infty$ (since $\Psi(\varsigma,\tau)$ is bounded). Thus $\|\Psi_k(\varsigma,\tau)-\Psi_n(\varsigma,\tau)\|\to 0$ when $n\to 0$. Hence, $\Psi_k(\varsigma,\tau)$ is a Cauchy sequence in $\mathbf{X}$, therefore the series is convergent. □

4 Implementation and Results

4.1 Problem I

Let $0<\varrho\leq 1,\ \alpha\in\mathbb{R},\ \tau>0$. Consider nonlinear time-fractional S-H equation [27, 54, 56]

$$
\begin{cases}
{}^{\mathrm{C}}D_{\tau}^{\varrho}\Psi(\varsigma,\tau)+\dfrac{\partial^4\Psi(\varsigma,\tau)}{\partial\varsigma^4}+2\dfrac{\partial^2\Psi(\varsigma,\tau)}{\partial\varsigma^2}+(1-\alpha)\Psi(\varsigma,\tau)+\Psi^3(\varsigma,\tau)=0,\\
\Psi(\varsigma,0)=\dfrac{1}{10}\sin\left(\dfrac{\pi\varsigma}{l}\right).
\end{cases} \tag{4.1}
$$

In view of the earlier discussed approach, we assume an infinite series solution for (4.1) in the form $\Psi(\varsigma,\tau)=\sum_{n=0}^{\infty}\Psi_n(\varsigma,\tau)$ while the nonlinear term $\mathscr{N}[\Psi(\varsigma,\tau)]=\Psi^3(\varsigma,\tau)$ defined by the Adomian polynomials. By virtue of the FNTDM solution steps leading to (3.8), we have

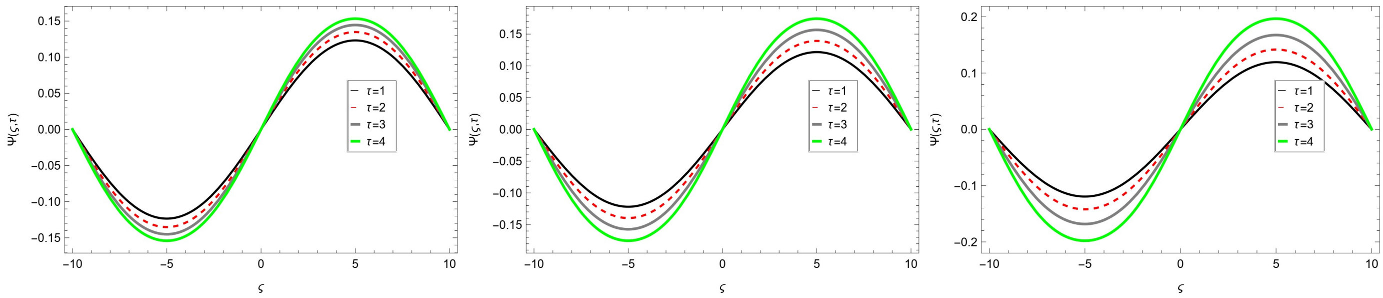
$$
\begin{aligned}
\sum_{n=0}^{\infty}\Psi_n(\varsigma,\tau)=&\mathbb{NT}^{-1}\left[\frac{\Psi(\varsigma,0)}{s}\right]-\mathbb{NT}^{-1}\left[\frac{\omega^{\varrho}}{s^{\varrho}}\mathbb{NT}^{+}\left[\sum_{n=0}^{\infty}\frac{\partial^4\Psi_n(\varsigma,\tau)}{\partial\varsigma^4}+2\sum_{k=0}^{\infty}\frac{\partial^2\Psi_n(\varsigma,\tau)}{\partial\varsigma^2}\right.\right.\\
&\left.\left.+(1-\alpha)\sum_{n=0}^{\infty}\Psi_n(\varsigma,\tau)+\sum_{n=0}^{\infty}A_n\right]\right]\\
=&\frac{1}{10}\sin\left(\frac{\pi\varsigma}{l}\right)-\mathbb{NT}^{-1}\left[\frac{\omega^{\varrho}}{s^{\varrho}}\mathbb{NT}^{+}\left[\sum_{n=0}^{\infty}\frac{\partial^4\Psi_n(\varsigma,\tau)}{\partial\varsigma^4}+2\sum_{k=0}^{\infty}\frac{\partial^2\Psi_n(\varsigma,\tau)}{\partial\varsigma^2}\right.\right.\\
&\left.\left.+(1-\alpha)\sum_{n=0}^{\infty}\Psi_n(\varsigma,\tau)+\sum_{n=0}^{\infty}A_n\right]\right],
\end{aligned} \tag{4.2}
$$

from which we obtain the recursive relation

$$\begin{cases} \Psi_0(\varsigma,\tau) = \dfrac{1}{10}\sin\left(\dfrac{\pi\varsigma}{l}\right), \\ \Psi_{n+1}(\varsigma,\tau) = -\mathbb{NT}^{-1}\left[\dfrac{\omega^{\varrho}}{s^{\varrho}}\mathbb{NT}^{+}\left[\dfrac{\partial^4\Psi_n(\varsigma,\tau)}{\partial\varsigma^4} + 2\dfrac{\partial^2\Psi_n(\varsigma,\tau)}{\partial\varsigma^2} + (1-\alpha)\Psi_n(\varsigma,\tau)\right] + A_n\right],\ n \geq 0. \end{cases} \tag{4.3}$$

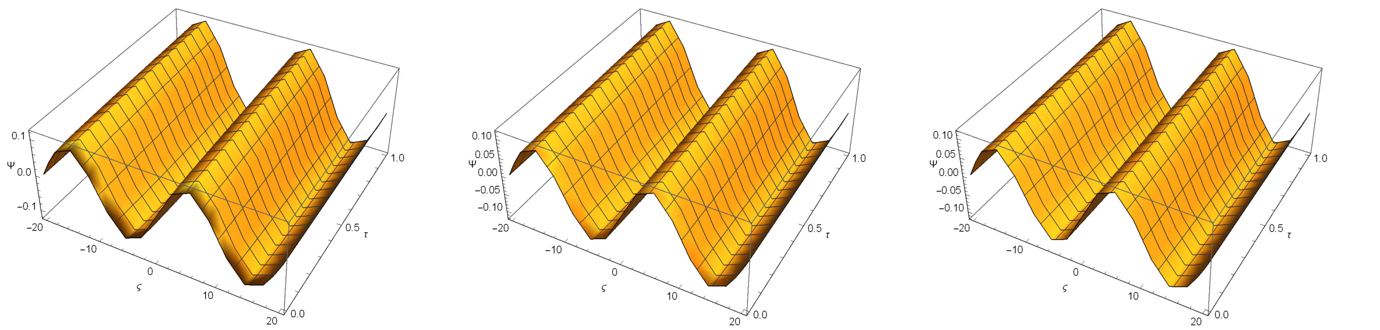
Here, we have that

$$A_0 = \Psi_0^3, \quad A_1 = 3\Psi_0^2\Psi_1, \quad A_2 = 3\Psi_0^2\Psi_2 + 3\Psi_0\Psi_1^2, \cdots. \tag{4.4}$$



(a) $\varrho = 0.5$ (b) $\varrho = 0.75$ (c) $\varrho = 1$

Figure 1: Nature of 3rd-order approximate FNTDM solution for Problem **I** at distinct values of τ when $\alpha = 1$ and $l = 10$.



(a) $\varrho = 0.5$ (b) $\varrho = 0.75$ (c) $\varrho = 1$

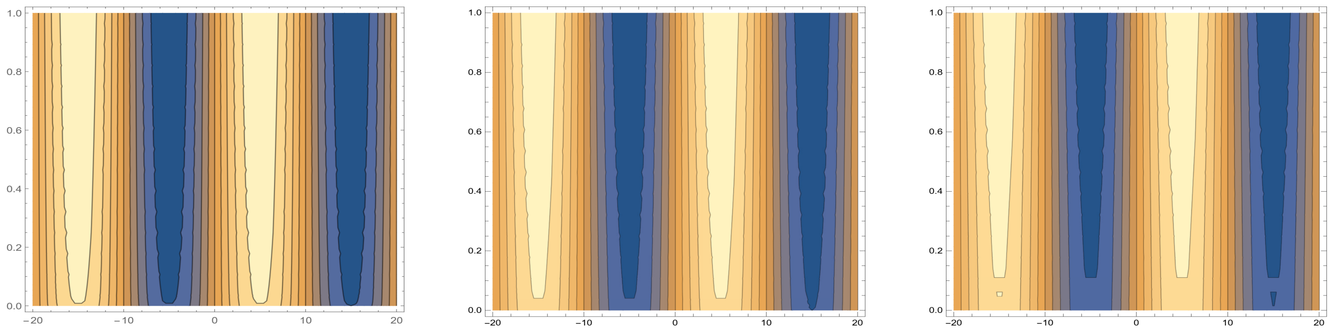
Figure 2: Surface plots for 3rd-order approximate FNTDM solution for Problem **I** at distinct values of ϱ when $\alpha = 1$ and $l = 10$.

In view of (4.3) and (4.4), some of the solution iterates are obtained as

$$\begin{aligned} \Psi_0(\varsigma,\tau) =& \frac{1}{10}\sin\left(\frac{\pi\varsigma}{l}\right), \\ \Psi_1(\varsigma,\tau) =& -\mathbb{NT}^{-1}\left[\frac{\omega^{\varrho}}{s^{\varrho}}\mathbb{NT}^{+}\left[\frac{\partial^4\Psi_0(\varsigma,\tau)}{\partial\varsigma^4} + 2\frac{\partial^2\Psi_0(\varsigma,\tau)}{\partial\varsigma^2} + (1-\alpha)\Psi_0(\varsigma,\tau)\right] + A_0\right], \\ =& -\frac{\tau^{\varrho}\sin\left(\frac{\pi\varsigma}{l}\right)\left(l^4\sin^2\left(\frac{\pi\varsigma}{l}\right) + 100\left(-\left((\alpha-1)l^4\right) - 2\pi^2l^2 + \pi^4\right)\right)}{1000l^4\Gamma(\varrho+1)}, \\ \Psi_2(\varsigma,\tau) =& -\mathbb{NT}^{-1}\left[\frac{\omega^{\varrho}}{s^{\varrho}}\mathbb{NT}^{+}\left[\frac{\partial^4\Psi_1(\varsigma,\tau)}{\partial\varsigma^4} + 2\frac{\partial^2\Psi_1(\varsigma,\tau)}{\partial\varsigma^2} + (1-\alpha)\Psi_1(\varsigma,\tau)\right] + A_1\right], \\ =& \sin\left(\frac{\pi\varsigma}{l}\right)\Bigg(3l^8\sin^5\left(\frac{\pi\varsigma}{l}\right) + 300l^4\left(-\left((\alpha-1)l^4\right) - 2\pi^2l^2 + \pi^4\right)\sin^3\left(\frac{\pi\varsigma}{l}\right) \\ &+1000l^4\left(-\left((\alpha-1)l^4\right) - 6\pi^2l^2 + 21\pi^4\right)\sin^2\left(\frac{\pi\varsigma}{l}\right) + 100000\left((\alpha-1)l^4 + 2\pi^2l^2 - \pi^4\right)^2 \\ &-12000\pi^2\left(5\pi^2 - l^2\right)l^4\cos^2\left(\frac{\pi\varsigma}{l}\right)\Bigg)\frac{\tau^{2\varrho}}{1000000l^8\Gamma(2\varrho+1)}. \end{aligned}$$

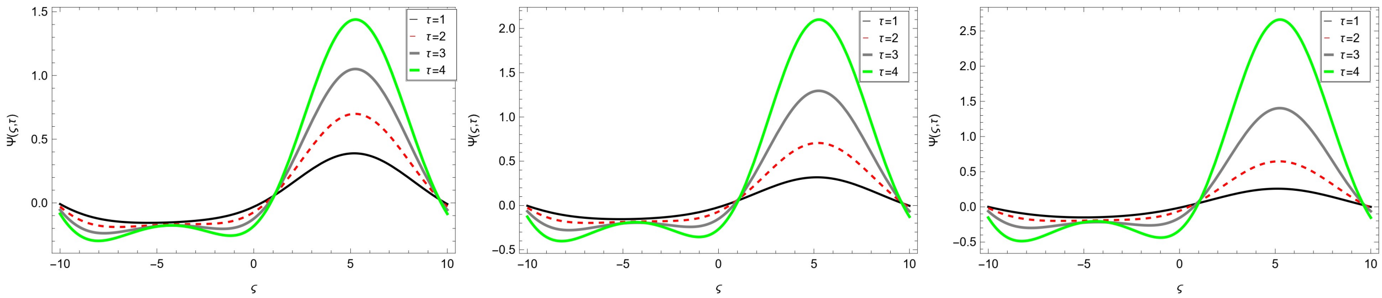
Moreover, the remaining iterates for $n \geq 3$ can be generated in the same manner. According to (3.10), the FNTDM series solution of **Problem I** is obtained as

$$
\begin{aligned}
\Psi(\varsigma,\tau) =& \frac{1}{10}\sin\left(\frac{\pi\varsigma}{l}\right) - \frac{\tau^{\varrho}\sin\left(\frac{\pi\varsigma}{l}\right)\left(l^4\sin^2\left(\frac{\pi\varsigma}{l}\right)+100\left(-\left((\alpha-1)l^4\right)-2\pi^2l^2+\pi^4\right)\right)}{1000l^4\Gamma(\varrho+1)}, \\
&+\sin\left(\frac{\pi\varsigma}{l}\right)\left(3l^8\sin^5\left(\frac{\pi\varsigma}{l}\right)+300l^4\left(-\left((\alpha-1)l^4\right)-2\pi^2l^2+\pi^4\right)\sin^3\left(\frac{\pi\varsigma}{l}\right)\right. \\
&+1000l^4\left(-\left((\alpha-1)l^4\right)-6\pi^2l^2+21\pi^4\right)\sin^2\left(\frac{\pi\varsigma}{l}\right)+100000\left((\alpha-1)l^4+2\pi^2l^2-\pi^4\right)^2 \\
&\left.-12000\pi^2\left(5\pi^2-l^2\right)l^4\cos^2\left(\frac{\pi\varsigma}{l}\right)\right)\frac{\tau^{2\varrho}}{1000000l^8\Gamma(2\varrho+1)}+\cdots
\end{aligned}
\tag{4.5}
$$



(a) $\varrho = 0.5$ (b) $\varrho = 0.75$ (c) $\varrho = 1$

Figure 3: Contour surfaces for 3rd-order approximate FNTDM solution for Problem **I** at distinct values of ϱ with $\alpha = 1$ and $l = 10$.



(a) $\varrho = 0.5$ (b) $\varrho = 0.75$ (c) $\varrho = 1$

Figure 4: Nature of approximate FNTDM solution for **Problem II** at distinct values of τ with $\eta = 1$, $\gamma = 0.5$ and $l = 10$.

4.2 Problem II

Consider nonlinear fractional S-H equation [53, 56, 43]

$$
\begin{cases}
{}^{C}D_{\tau}^{\varrho}\Psi(\varsigma,\tau)+\dfrac{\partial^4\Psi(\varsigma,\tau)}{\partial\varsigma^4}-\eta\dfrac{\partial^3\Psi(\varsigma,\tau)}{\partial\varsigma^3}+2\dfrac{\partial^2\Psi(\varsigma,\tau)}{\partial\varsigma^2}-\gamma\Psi(\varsigma,\tau)-2\Psi^2(\varsigma,\tau)+\Psi^3(\varsigma,\tau)=0, \\
\Psi(\varsigma,\tau)=\dfrac{1}{10}\sin\left(\dfrac{\pi\varsigma}{l}\right),
\end{cases}
\tag{4.6}
$$

where $0<\varrho\leq 1$, η and γ are dispersive and bifurcations real parameters, respectively. As in Problem 4.1 above, the FNTDM yields

$$\begin{cases}\Psi_0(\varsigma,\tau)=\dfrac{1}{10}\sin\left(\dfrac{\pi\varsigma}{l}\right),\\ \Psi_{n+1}(\varsigma,\tau)=-\mathbb{NT}^{-1}\left[\dfrac{\omega^{\varrho}}{s^{\varrho}}\mathbb{NT}^{+}\left[\dfrac{\partial^4\Psi_n(\varsigma,\tau)}{\partial\varsigma^4}-\eta\dfrac{\partial^3\Psi_n(\varsigma,\tau)}{\partial\varsigma^3}+2\dfrac{\partial^2\Psi_n(\varsigma,\tau)}{\partial\varsigma^2}-\gamma\Psi_n(\varsigma,\tau)\right.\right.\\ \qquad\left.\left.-2A_n+B_n\right]\right],\ n\geq 0.\end{cases}\tag{4.7}$$

as the recursive relation for **Problem II** from which all iterations for $n\geq 0$ can be obtained. Here A_n and B_n are components of the Adomian polynomials representing the nonlinear terms $\mathscr{N}_1[\Psi(\varsigma,\tau)]=\Psi^2(\varsigma,\tau)$ and $\mathscr{N}_2[\Psi(\varsigma,\tau)]=\Psi^3(\varsigma,\tau)$, respectively, with

$$\begin{aligned}A_0&=\Psi_0^2, & B_0&=\Psi_0^3,\\ A_1&=2\Psi_0\Psi_1, & B_1&=3\Psi_0^2\Psi_1,\\ A_2&=2\Psi_0\Psi_2+\Psi_1^2, & B_2&=3\Psi_0^2\Psi_2+3\Psi_0\Psi_1^2,\end{aligned}\tag{4.8}$$

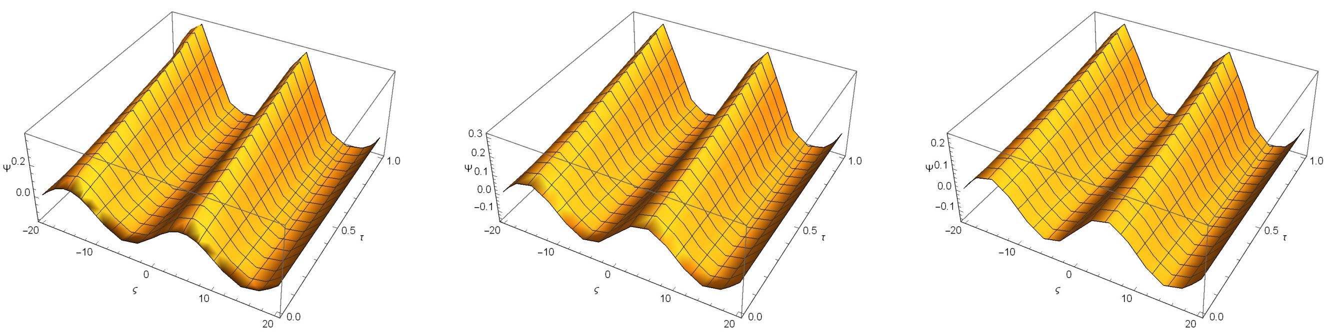
and so on. By substituting (4.8) into the recurrence system (4.7), some of the solution iterates are explicitly obtained as

$$\begin{aligned}\Psi_0(\varsigma,\tau)=&\frac{1}{10}\sin\left(\frac{\pi\varsigma}{l}\right),\\ \Psi_1(\varsigma,\tau)=&-\mathbb{NT}^{-1}\left[\frac{\omega^{\varrho}}{s^{\varrho}}\mathbb{NT}^{+}\left[\frac{\partial^4\Psi_0(\varsigma,\tau)}{\partial\varsigma^4}-\eta\frac{\partial^3\Psi_0(\varsigma,\tau)}{\partial\varsigma^3}+2\frac{\partial^2\Psi_0(\varsigma,\tau)}{\partial\varsigma^2}-\gamma\Psi_0(\varsigma,\tau)-2A_0+B_0\right]\right],\\ =&-\frac{\tau^{\varrho}\left(\sin\left(\frac{\pi\varsigma}{l}\right)\left(l^4\sin^2\left(\frac{\pi\varsigma}{l}\right)-20l^4\sin\left(\frac{\pi\varsigma}{l}\right)+100\left(-\gamma l^4-2\pi^2l^2+\pi^4\right)\right)+100\pi^3\eta l\cos\left(\frac{\pi\varsigma}{l}\right)\right)}{1000l^4\Gamma(\varrho+1)},\\ \Psi_2(\varsigma,\tau)=&-\mathbb{NT}^{-1}\left[\frac{\omega^{\varrho}}{s^{\varrho}}\mathbb{NT}^{+}\left[\frac{\partial^4\Psi_1(\varsigma,\tau)}{\partial\varsigma^4}-\eta\frac{\partial^3\Psi_1(\varsigma,\tau)}{\partial\varsigma^3}+2\frac{\partial^2\Psi_1(\varsigma,\tau)}{\partial\varsigma^2}-\gamma\Psi_1(\varsigma,\tau)-2A_1+B_1\right]\right],\\ =&-\Bigg(600\pi^3\eta l^5\cos^3\left(\frac{\pi\varsigma}{l}\right)+20000\pi^3\eta l\cos\left(\frac{\pi\varsigma}{l}\right)\left(\gamma l^4+l^4\sin\left(\frac{\pi\varsigma}{l}\right)+2\pi^2l^2-\pi^4\right)\\ &+400\pi^2l^4\left(3\left(5\pi^2-l^2\right)\sin\left(\frac{\pi\varsigma}{l}\right)+20\left(l^2-2\pi^2\right)\right)\cos^2\left(\frac{\pi\varsigma}{l}\right)-\frac{1}{8}\sin\left(\frac{\pi\varsigma}{l}\right)\Big(80000\gamma^2l^8\\ &+48000\gamma l^8\sin\left(\frac{\pi\varsigma}{l}\right)-1600\gamma l^8-600l^8\sin\left(\frac{\pi\varsigma}{l}\right)+200l^8\sin\left(\frac{3\pi\varsigma}{l}\right)+3l^8\cos\left(\frac{4\pi\varsigma}{l}\right)\\ &+3209l^8+320000\pi^2\gamma l^6+128000\pi^2l^6\sin\left(\frac{\pi\varsigma}{l}\right)-4800\pi^2l^6+9600\pi^3\eta l^5\sin\left(\frac{2\pi\varsigma}{l}\right)\\ &-160000\pi^4\gamma l^4-160000\pi^4l^4\sin\left(\frac{\pi\varsigma}{l}\right)+329600\pi^4l^4-80000\pi^6\eta^2l^2-320000\pi^6l^2\\ &+4l^4\left((400\gamma-803)l^4+1200\pi^2l^2-2400\pi^4\right)\cos\left(\frac{2\pi\varsigma}{l}\right)+80000\pi^8\Big)\Bigg)\frac{\tau^{2\varrho}}{100000l^8\Gamma(2\varrho+1)}.\end{aligned}$$

The remaining iterates for $n\geq 3$ can be obtained in the same manner. Moreover, the FNTDM series solution is obtained as

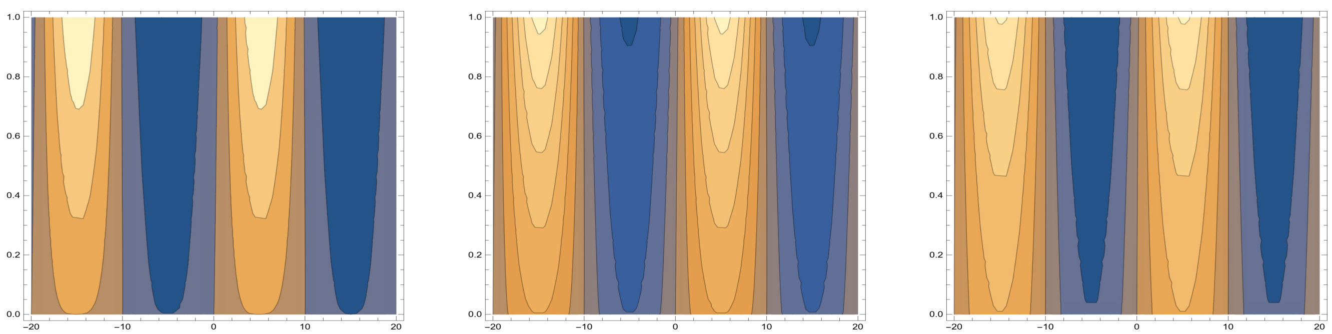
$$\begin{aligned}\Psi(\varsigma,\tau)=&\frac{1}{10}\sin\left(\frac{\pi\varsigma}{l}\right)\\ &-\frac{\tau^{\varrho}\left(\sin\left(\frac{\pi\varsigma}{l}\right)\left(l^4\sin^2\left(\frac{\pi\varsigma}{l}\right)-20l^4\sin\left(\frac{\pi\varsigma}{l}\right)+100\left(-\gamma l^4-2\pi^2l^2+\pi^4\right)\right)+100\pi^3\eta l\cos\left(\frac{\pi\varsigma}{l}\right)\right)}{1000l^4\Gamma(\varrho+1)}\\ &-\Bigg(600\pi^3\eta l^5\cos^3\left(\frac{\pi\varsigma}{l}\right)+20000\pi^3\eta l\cos\left(\frac{\pi\varsigma}{l}\right)\left(\gamma l^4+l^4\sin\left(\frac{\pi\varsigma}{l}\right)+2\pi^2l^2-\pi^4\right)\\ &+400\pi^2l^4\left(3\left(5\pi^2-l^2\right)\sin\left(\frac{\pi\varsigma}{l}\right)+20\left(l^2-2\pi^2\right)\right)\cos^2\left(\frac{\pi\varsigma}{l}\right)-\frac{1}{8}\sin\left(\frac{\pi\varsigma}{l}\right)\\ &\times\Bigg(80000\gamma^2l^8+48000\gamma l^8\sin\left(\frac{\pi\varsigma}{l}\right)-1600\gamma l^8-600l^8\sin\left(\frac{\pi\varsigma}{l}\right)+200l^8\sin\left(\frac{3\pi\varsigma}{l}\right)\end{aligned}$$

$$
\begin{aligned}
&+3l^8\cos\left(\frac{4\pi\varsigma}{l}\right)+3209l^8+320000\pi^2\gamma l^6+128000\pi^2l^6\sin\left(\frac{\pi\varsigma}{l}\right)\\
&-4800\pi^2l^6+9600\pi^3\eta l^5\sin\left(\frac{2\pi\varsigma}{l}\right)-160000\pi^4\gamma l^4-160000\pi^4l^4\sin\left(\frac{\pi\varsigma}{l}\right)\\
&+329600\pi^4l^4-80000\pi^6\eta^2l^2-320000\pi^6l^2+4l^4\left((400\gamma-803)l^4+1200\pi^2l^2\right.\\
&\left.\left.-2400\pi^4\right)\cos\left(\frac{2\pi\varsigma}{l}\right)+80000\pi^8\right)\Big)\\
&\times\frac{\tau^{2\varrho}}{100000l^8\Gamma(2\varrho+1)}+\cdots.
\end{aligned}
\tag{4.9}
$$



(a) $\varrho = 0.5$ (b) $\varrho = 0.75$ (c) $\varrho = 1$

Figure 5: Surface plots for 3rd-order approximate FNTDM solution for **Problem II** at distinct values of ϱ with $\eta = 1$, $\gamma = 0.5$ and $l = 10$.



(a) $\varrho = 0.5$ (b) $\varrho = 0.75$ (c) $\varrho = 1$

Figure 6: Contour surfaces for 3rd-order approximate FNTDM solution for **Problem II** at distinct values of ϱ with $\gamma = 0.5$ and $l = 10$.

4.3 Problem III

Let $0 < \varrho \leq 1,\ \alpha \in \mathbb{R}$. Consider nonlinear fractional S-H equation [43]

$$
\begin{cases}
{}^{\mathrm{C}}D_\tau^\varrho\Psi(\varsigma,\tau)+\dfrac{\partial^4\Psi(\varsigma,\tau)}{\partial\varsigma^4}+2\dfrac{\partial^2\Psi(\varsigma,\tau)}{\partial\varsigma^2}+(1-\alpha)\Psi(\varsigma,\tau)-\Psi^l(\varsigma,\tau)+\left(\dfrac{\partial\Psi(\varsigma,\tau)}{\partial\varsigma}\right)^l=0,\\
\Psi(\varsigma,0)=\exp(\varsigma).
\end{cases}
\tag{4.10}
$$

As in Problem 1 above, the FNTDM yields

$$\begin{cases} \Psi_0(\varsigma,\tau) = \exp(\varsigma), \\ \Psi_{n+1}(\varsigma,\tau) = -\mathbb{NT}^{-1}\left[\frac{\omega^{\varrho}}{s^{\varrho}}\mathbb{NT}^{+}\left[\frac{\partial^4\Psi_n(\varsigma,\tau)}{\partial\varsigma^4} + 2\frac{\partial^2\Psi_n(\varsigma,\tau)}{\partial\varsigma^2} + (1-\alpha)\Psi_n(\varsigma,\tau) \right.\right. \\ \qquad\qquad \left.\left. - A_n + B_n\right]\right], \ n \geq 0. \end{cases} \tag{4.11}$$

as the recursive relation from which all iterations for $n \geq 0$ can be obtained. Here A_n and B_n are components of the Adomian polynomials representing the nonlinear terms $\mathscr{N}_1[\Psi(\varsigma,\tau)] = \Psi^l(\varsigma,\tau)$ and $\mathscr{N}_2[\Psi(\varsigma,\tau)] = \left(\frac{\partial\Psi(\varsigma,\tau)}{\partial\varsigma}\right)^l$, respectively, with

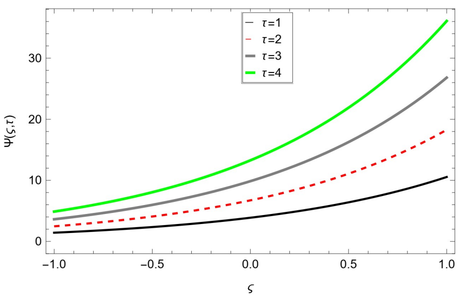
$$\begin{aligned} A_0 &= \Psi_0^l, & B_0 &= \left(\frac{\partial\Psi_0}{\partial\varsigma}\right)^l, \\ A_1 &= l\Psi_0^{l-1}\Psi_1, & B_1 &= l\left(\frac{\partial\Psi_0}{\partial\varsigma}\right)^{l-1}\frac{\partial\Psi_1}{\partial\varsigma}, \\ A_2 &= \frac{1}{2}\left(l(l-1)\Psi_0^{l-2}\Psi_1^2 + 2l\Psi_0^{l-1}\Psi_2\right), & B_2 &= \frac{1}{2}\left(l(l-1)\left(\frac{\partial\Psi_0}{\partial\varsigma}\right)^{l-2}\left(\frac{\partial\Psi_1}{\partial\varsigma}\right)^2 + 2l\left(\frac{\partial\Psi_0}{\partial\varsigma}\right)^{l-1}\frac{\partial\Psi_2}{\partial\varsigma}\right), \end{aligned} \tag{4.12}$$

and so on. For $l = 2$, we substitute (4.12) into the recurrence system (4.11) to obtain the following few iterations:

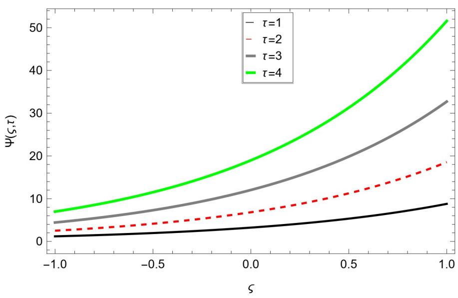
$$\begin{aligned} \Psi_0(\varsigma,\tau) &= \exp(\varsigma), \\ \Psi_1(\varsigma,\tau) &= -\mathbb{NT}^{-1}\left[\frac{\omega^{\varrho}}{s^{\varrho}}\mathbb{NT}^{+}\left[\frac{\partial^4\Psi_0(\varsigma,\tau)}{\partial\varsigma^4} + 2\frac{\partial^2\Psi_0(\varsigma,\tau)}{\partial\varsigma^2} + (1-\alpha)\Psi_0(\varsigma,\tau) - A_0 + B_0\right]\right] = \frac{(\alpha-4)e^{\varsigma}\tau^{\varrho}}{\Gamma(\varrho+1)}, \\ \Psi_2(\varsigma,\tau) &= -\mathbb{NT}^{-1}\left[\frac{\omega^{\varrho}}{s^{\varrho}}\mathbb{NT}^{+}\left[\frac{\partial^4\Psi_1(\varsigma,\tau)}{\partial\varsigma^4} + 2\frac{\partial^2\Psi_1(\varsigma,\tau)}{\partial\varsigma^2} + (1-\alpha)\Psi_1(\varsigma,\tau) - A_1 + B_1\right]\right] = \frac{(\alpha-4)^2e^{\varsigma}\tau^{2\varrho}}{\Gamma(2\varrho+1)}, \\ \Psi_3(\varsigma,\tau) &= -\mathbb{NT}^{-1}\left[\frac{\omega^{\varrho}}{s^{\varrho}}\mathbb{NT}^{+}\left[\frac{\partial^4\Psi_2(\varsigma,\tau)}{\partial\varsigma^4} + 2\frac{\partial^2\Psi_2(\varsigma,\tau)}{\partial\varsigma^2} + (1-\alpha)\Psi_2(\varsigma,\tau) - A_2 + B_2\right]\right] = \frac{(\alpha-4)^3e^{\varsigma}\tau^{3\varrho}}{\Gamma(3\varrho+1)}. \end{aligned}$$

In a similar manner, the remaining solution iterates for $n > 3$ can be obtained. Moreover, the FNTDM series solution is obtained as

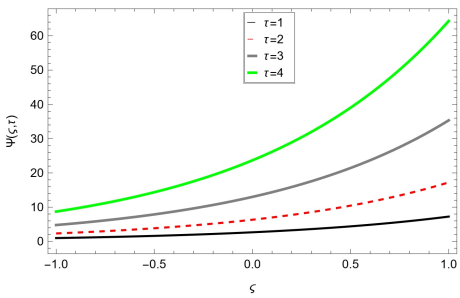
$$\Psi(\varsigma,\tau) = \exp(\varsigma) + \frac{(\alpha-4)e^{\varsigma}\tau^{\varrho}}{\Gamma(\varrho+1)} + \frac{(\alpha-4)^2e^{\varsigma}\tau^{2\varrho}}{\Gamma(2\varrho+1)} + \frac{(\alpha-4)^3e^{\varsigma}\tau^{3\varrho}}{\Gamma(3\varrho+1)} + \cdots = \sum_{k=0}^{\infty}\frac{(\alpha-4)^k e^{\varsigma}\tau^{k\varrho}}{\Gamma(k\varrho+1)}. \tag{4.13}$$



(a) $\varrho = 0.5$

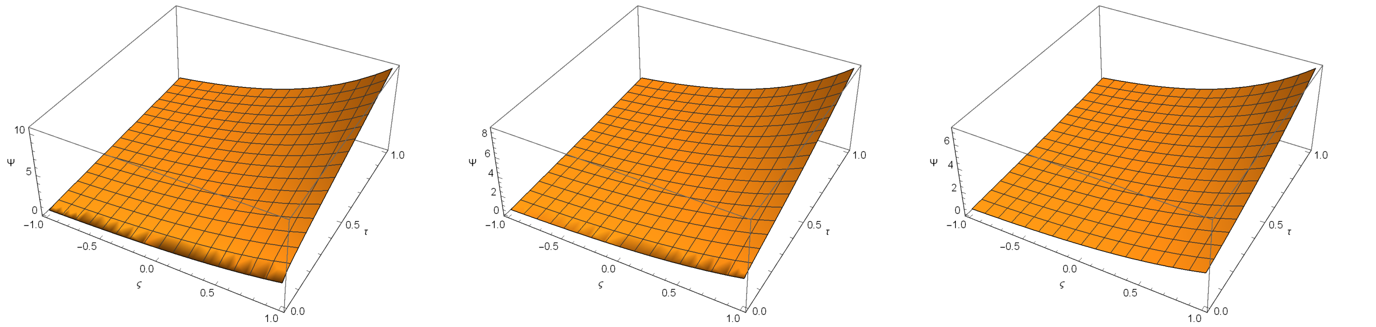


(b) $\varrho = 0.75$



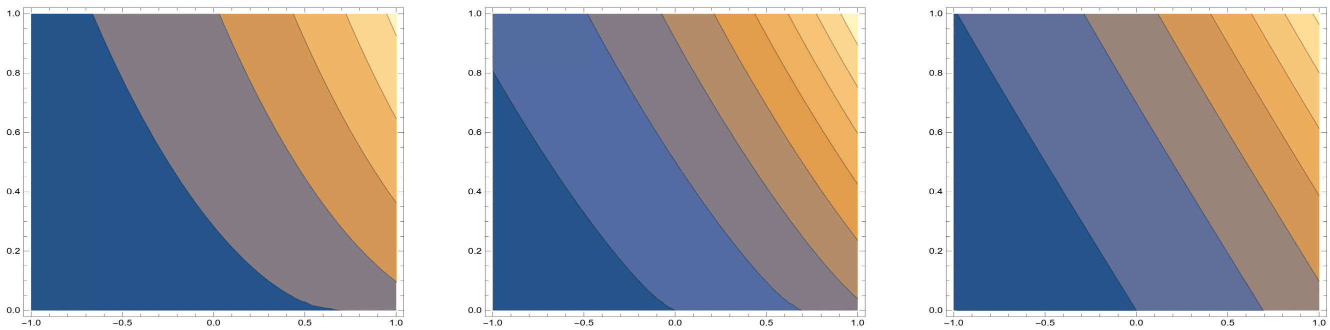
(c) $\varrho = 1$

Figure 7: Behavior of approximate series solution for **Problem III** at distinct values of τ with $\alpha = 5$, and $l = 2$.



(a) $\varrho = 0.5$ (b) $\varrho = 0.75$ (c) $\varrho = 1$

Figure 8: Surface plots for 3rd-order approximate FNTDM solution for **Problem III** at distinct values of ϱ with $\alpha = 5$ and $l = 2$.



(a) $\varrho = 0.5$ (b) $\varrho = 0.75$ (c) $\varrho = 1$

Figure 9: Contour surfaces for 3rd-order approximate FNTDM solution for **Problem III** at distinct values of ϱ with $\alpha = 1$ and $l = 10$.

4.4 Numerical simulations and Discussion

In the present work, the FNTDM formulation is tested upon three time-fractional versions of the Swift-Hohenberg equations. To present the behavior of the obtained results we provide numerical simulations of the third-order iterative solution for each of the considered problem. Furthermore, since the considered problems model important physical phenomena, it is very important to capture the behaviour of their approximate solutions with respect to particular values of the parameters. The numerical plots in Figures 1, 4 and 7 demonstrate the behaviour of the approximate solutions for **Problems I**, **II** and **III**, respectively at distinct values of τ and stated values of α, η, γ and l. For each of the mentioned figures, the respective simulations are considered for different values of and for ϱ, viz., $\varrho = 0.5$, $\varrho = 0.75$ and $\varrho = 1$. The 3D-plots in Figures 2, 5 and 8 and the corresponding contour surface plots in Figures 3, 6 and 9 represent the 3rd order approximate FNTDM solutions for **Problems I**, **II** and **III**, respectively at the stated values of ϱ.

5 Conclusion

In the present work, we obtained approximate series-type solutions of some special fractional order versions of the Swift-Hohenberg equation by utilizing a modified integral transform technique namely, the fraction natural transform decomposition method. The method, which is clearly a very effective combination of the Adomian decomposition method and natural transform, yields series solutions with high exactness and with minimal computations required. In carrying out this work, we first defined the natural transformation of the Caputo fractional operator and employed the considered method to obtain a general recurrence relation for a general fractional order nonlinear partial differential equation. A key advantage of the FNTDM is that it solves the equation directly without any form of discretization, linearization, or perturbation. The reliability, effectiveness and straightforwardness of the considered solution method are attributed to the fact that it has admits a strong ability to yield better convergence region for the solution. Moreover, the considered method generally requires computation of only a few series solution components of the considered problem to attain to the exact solution. This further establishes the strength of method in relation to other existing numerical techniques. Finally, we can reveal that the FNTDM can be considered as a good refinement of the existing numerical techniques and can be employed to study strongly nonlinear mathematical models describing even more complex natural phenomena.

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