Int. J. Nonlinear Anal. Appl. 14 (2023) 1, 2759–2770
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.24121.2673

The numerical solution of time fractional generalized Benjamin-Bona-Mahony equation via the Sinc functions

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(Communicated by Madjid Eshaghi Gordji)

Abstract

The aim of this paper is to analyze a numerical method for solving the time fractional Benjamin-Bona-Mahony (BBM) equation. The time variable has been discretized by using the finite forward difference procedure. The unconditionally stable semi-discrete formula has been proven. Then we apply the Sinc collocation method to approximate the solution of the semi-discrete scheme. The exponential convergence rate of the Sinc method has also been proven. To show the efficiency of the proposed method, two examples were given. Numerical results verified the theoretical results and illustrate the efficiency and accuracy of the method compared with other methods.

Keywords: Fractional Benjamin-Bona-Mahony equation, Unconditionally stable, Sinc-collocation method, Convergence analysis
2020 MSC: 65M12, 65M70, 26A33

1 Introduction

Numerous phenomena in engineering and other fields of science can be modeled via linear and non-linear PDEs. For example, from quantum mechanics nonlinear Klein-Gordon and nonlinear Schrödinger equations, fluid mechanics Navier-Stokes and Euler equations, material science Cahn-Hilliard equation, the Hasegawa-Mima equation [14] which describes turbulence in plasma physics, the Fitzhugh-Nagumo equation [11] that models biological neuron and the Hunter-Saxton equation [29] used to study waves orientation in nematic liquid crystal are just a few well-known samples of nonlinear evolution equations.

Recently, more and more attention has been paid to the development and research of fractional differential equations. Since the fact that the fractional derivatives are nonlocal, researchers found that they are more appropriate for the description of real-life problems with memorial and hereditary properties of various materials and processes than the traditional integer order equations. Fractional differential equations have numerous effective applications to various areas of science and engineering (see [6, 7, 13, 18, 23, 26]).

Benjamin-Bona-Mahony (BBM) equation was firstly introduced by Benjamin et al. [4] for modeling long waves of small amplitude in certain nonlinear dispersive media. This equation can also characterize the hydromagnetic waves in a cold plasma, acoustic waves in inharmonic crystals, and acoustic-gravity waves in compressible fluids [4, 15]. Several numerical techniques including meshfree method [16], Adomian decomposition method [17] and various forms of finite element methods in [8, 9, 10] and [12] have been used for the solution of the BBM equation.

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Received: June 2021 *Accepted:* August 2021

In this work, we consider the time fractional generalized BBM equation in the following form:

$$^{C}D_t^{\alpha}u - u_{xxt} - \gamma u_{xx} + f(u)_x = 0, \qquad x \in \Omega = [a,b], \qquad t \in (0,T], \tag{1.1}$$

$$u(a,t) = u(b,t) = 0, \qquad u(x,0) = \psi(x) \tag{1.2}$$

where $\alpha \in (0,1), f \in \mathcal{C}^2(\mathbb{R})$ and γ is positive constant. Also, the fractional derivative $^{C}D_t^{\alpha}$ is Caputo fractional derivative defined by

$$^{C}D_t^{\alpha}u = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(x,s)}{\partial s}\frac{ds}{(t-s)^{\alpha}}, \qquad \alpha \in (0,1].$$

In recent years, different numerical and analytical schemes have been presented and developed to solve time fractional generalized Benjamin-Bona-Mahony equation such as finite difference method [19], compact difference method [32], extended trial equation method [24] and high-Order Method with a temporal nonuniform mesh [20].

The idea of this paper is to present a computational technique for the solution of the time-fractional Benjamin-Bona-Mahony (gBBM) equation, we first employ a finite difference approximation to discrete the time-fractional derivative (1.1), then we will apply the collocation method based on the Sinc function to get the approximate solution of the semi-discrete scheme.

The Sinc method has been introduced by Frank Stenger [30]. In this method the test functions are defined by the Sinc-function $sinc(x) = \sin(\pi x)/(\pi x)$ is based on the Whittaker-Shannon-Kotel' nikov sampling theorem for entire functions. In recent years, due to the appropriate properties of this function, various problems have been solved using this method such as [1, 3, 5, 27, 31, 33].

The paper is organized as follows. In section 2, Some preparations required for fractional derivative and the Sinc approximation are expressed. In section 3, the time-fractional derivative is discretized and the stability of this method is proven. In section 4, the Sinc collocation method is applied to obtain the approximate solution at each step. In section 5, the convergence analysis of the proposed method is proven. In section 6, some numerical experiments are brought to verify the efficiency of our method. Finally, a brief conclusion is made in section 7.

2 Notation and background

In this section, we will mention some of the definitions and properties that are required in the following sections.

2.1 Fractional Derivatives

Definition 2.1. [25]. Suppose $\alpha \in \mathbb{R}_+$, $-\infty < a < \infty$, $n \in \mathbb{N}$ and $x > 0$. Then, the α-th order Caputo derivative of function $f(x)$ is defined as follows:

$$^{C}D_x^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{f^{(n)}(s)ds}{(x-s)^{(\alpha-n+1)}}, \qquad n-1 < \alpha \leq n. \tag{2.1}$$

Note that for $\alpha = n$, The the Caputo differential operator is the same as the usual differential operator. Some of the properties of Caputo derivative, which are similar to the properties of integer-order derivative, are as follows:

$$^{C}D_x^{\alpha}(c_1 f(x) + c_2 g(x)) = c_1 {}^{C}D_x^{\alpha}f(x) + c_2 {}^{C}D_x^{\alpha}g(x),$$

where c_1 and c_2 are constant. The fractional derivative of polynomial functions by the Caputo operator in Eq. (2.1) can also be obtained as follows:

- $^{C}D_x^{\alpha}c = 0$, c is a constant.
- $^{C}D_x^{\alpha}x^p = \begin{cases} 0, & p \in \mathbb{N}_0, p < \lceil\alpha\rceil, \\ \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)}x^{(p-\alpha)}, & otherwise, \end{cases}$

where the ceiling function $\lceil\alpha\rceil$ is the smallest integer greater than or equal to α, and $\mathbb{N}_0 = \{0, 1, 2, ...\}$.

2.2 Sinc function

In this part, we will mention some of the properties and theorems required for the Sinc method. See references [30] and [21] for more details on this method. Sinc function on the real line $-\infty < x < \infty$ is defined as:

$$Sinc(x) = \begin{cases} \frac{sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Also for each integer j and the mesh size $h > 0$, the sinc basis functions are defined by

$$S(j,h)(x) = Sinc\Big(\frac{x - jh}{h}\Big), \quad j = 0, \pm 1, \pm 2, \ldots.$$

For $h > 0$, the Whittaker cardinal expansion of $f(x)$ is defined as

$$f(x) \approx C(f,h)(x) = \sum_{j=-\infty}^{\infty} f(jh)S(j,h)(x),$$

whenever this series converges. The properties of Whittaker cardinal expansions are given in Stenger [30]. These properties are obtained in the infinite strip D_v of the complex w-plane where $d > 0$

$$D_v = \Big\{ w = u + iv : |v| < d \leq \frac{\pi}{2} \Big\}.$$

To use the sinc approximation on the finite interval (a, b), we use the one-to-one conformal map $w = \phi(z) = \ln(\frac{z-a}{b-z})$, which maps the eye-shaped domain

$$D_E = \Big\{ z = x + iy; |\arg(\frac{z - a}{b - z})| < d \leq \frac{\pi}{2} \Big\},$$

onto the infinite strip D_v.

For the interval (a, b) the basis functions of the sinc are considered as follows

$$S_j(z) = S(j,h) \circ \phi(z) = Sinc\Big(\frac{\phi(z) - jh}{h}\Big). \tag{2.2}$$

Also, the sinc gridpoints on interval (a, b) are denoted by

$$x_k = \phi^{-1}(kh) = \frac{a + be^{kh}}{1 + e^{kh}}, \qquad k = 0, \pm 1, \pm 2, ... \tag{2.3}$$

Definition 2.2. Let $B(D_E)$ denote the class of functions f which are analytic in D_E and satisfy

- $\int_{\psi(u+\Sigma)} |f(z)|dz \to 0, \qquad as \quad u \to \pm\infty$
- $N(f, D_E) \equiv \int_{\partial D_E} |f(z)|dz < \infty,$

where $\Sigma = \{iv : |v| < d \leq \frac{\pi}{2}\}$ and ∂D_E represents the boundary of D_E.

Definition 2.3. Let $L_\eta(D_E)$ be the set of all analytic function g in D_E, for which there exists a constant C such that

$$|g(z)| \leq C \frac{|\rho(z)|^\eta}{[1 + |\rho(z)|]^{2\eta}}, \qquad z \in D_E, \quad 0 < \eta \leq 1. \tag{2.4}$$

where $\rho(z) = e^{\phi(z)}$.

Theorem 2.4. [30] If $\phi' g \in B(D_E)$, and let

$$\sup_{\frac{-\pi}{h} \leq t \leq \frac{\pi}{h}} \left| \Big(\frac{d}{dx}\Big)^l e^{it\phi(x)} \right| \leq C_1 h^{-l}, \qquad x \in (a, b),$$

for $l = 0, 1, \ldots, m$ with C_1 a constant depending only on m and ϕ. If $g \in L_\eta(D_E)$ then taking $h = \sqrt{\pi d/\eta N}$ it follows that

$$\sup_{x\in(a,b)} \left| g^{(l)}(x) - \left(\frac{d}{dx}\right)^l \sum_{j=-N}^{N} g(x_j)S_j(x) \right| \le CN^{(l+1)/2}\exp(-(\pi d\eta N)^{1/2}),$$

where C is a constant depending only on g, d, m, ϕ and η.

Also, to evaluate the derivatives of basis function of Sinc at the nodes x_k, we have [21]:

$$\delta_{jk}^{(0)} = [S(j,h)\circ\phi(x)]|_{x=x_k} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \tag{2.5}$$

$$\delta_{jk}^{(1)} = h\frac{d}{d\phi}[S(j,h)\circ\phi(x)]|_{x=x_k} = \begin{cases} 0, & j = k, \\ \frac{(-1)^{(k-j)}}{k-j}, & j \neq k, \end{cases} \tag{2.6}$$

$$\delta_{jk}^{(2)} = h^2\frac{d^2}{d\phi^2}[S(j,h)\circ\phi(x)]|_{x=x_k} = \begin{cases} \frac{-\pi^2}{3}, & j = k, \\ \frac{-2(-1)^{(k-j)}}{(k-j)^2}, & j \neq k, \end{cases} \tag{2.7}$$

3 Time discretization

We apply a finite difference approximation to discrete the time fractional derivative. Let $t_m = m\Delta t, m = 0, 1, ..., M$ where $\Delta t = T/M$ is the time step. The time fractional derivative $\frac{\partial^\alpha u}{\partial t^\alpha}$ at t_{m+1} is calculated by

$$\begin{aligned} {}^C D_t^\alpha u(x, t_{m+1}) &= \frac{1}{\Gamma(1-\alpha)}\int_0^{t_{m+1}} \frac{\partial u(x,y)}{\partial y}\frac{dy}{(t_{m+1}-y)^\alpha} \\ &= \frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{k=m}\int_{t_k}^{t_{k+1}} \frac{\partial u(x,y)}{\partial y}\frac{dy}{(t_{m+1}-y)^\alpha} \\ &= \frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{k=m}\frac{u(x,t_{k+1})-u(x,t_k)}{\Delta t}\int_{t_k}^{t_{k+1}}\frac{ds}{(t_{m+1}-y)^\alpha} + R_{\Delta t}^{(1)}. \end{aligned} \tag{3.1}$$

By using method in [22] the error $R_{\Delta t}^{(1)}$ is bounded as $R_{\Delta t}^{(1)} \le C(\Delta t)^{(2-\alpha)}$, where C is a constant.
By calculating the integral in (3.1) we have:

$$\begin{aligned} {}^C D_t^\alpha u(x, t_{m+1}) &= \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=0}^{k=m}\lambda_{m-k}\big(u(x,t_{k+1})-u(x,t_k)\big) + R_{\Delta t}^{(1)} \\ &= \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=0}^{k=m}\lambda_k\big(u(x,t_{m-k+1})-u(x,t_{m-k})\big) + R_{\Delta t}^{(1)}, \end{aligned} \tag{3.2}$$

where $\lambda_k = (k+1)^{(1-\alpha)} - k^{(1-\alpha)}$.
Now, by replacing Eq. (3.2) into Eq. (1.1) , we have:

$$\begin{aligned} &\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=0}^{k=m}\lambda_k\big(u(x,t_{m-k+1})-u(x,t_{m-k})\big) \\ &\qquad - \frac{u_{xx}(x,t_{m+1})-u_{xx}(x,t_m)}{\Delta t} - \gamma u_{xx}(x,t_{m+1}) + f(u)_x(x,t_{m+1}) + R_{\Delta t}^{(1)} = 0. \end{aligned} \tag{3.3}$$

Suppose u^m be the numerical estimation to $u(x, t_m)$. Ignoring the error term $R_{\Delta t}^{(1)}$ in (3.3), the semi-discrete scheme can be constructed as follows:

$$u^{m+1} - \mu(1+\gamma\Delta t)u_{xx}^{m+1} + \mu\Delta t f(u^{m+1})_x = (1-\lambda_1)u^m - \mu u_{xx}^m - \sum_{k=1}^{m-1}\big(\lambda_{k+1}-\lambda_k\big)u^{m-k} + \lambda_m u^0, \tag{3.4}$$

$u^0 = u_0(x), \quad u^{m+1}(a) = u^{m+1}(b) = 0, \qquad m = 0, 1, ..., M-1$, where $\mu = (\Delta t)^{\alpha-1}\Gamma(2-\alpha)$.

3.1 Stability analysis

In this section, we discuss the stability of Eq. (3.4) using the following lemma. Firstly, we introduce the functional spaces with standard norms and inner products. The usual L^2-scalar product on the space $L^2(\Omega)$ is defined as:

$$(u,v)=\int_{\Omega} uv d\Omega \tag{3.5}$$

for $u,v\in L^2(\Omega)$, which inducts the L^2-norm

$$\|u\|_2=(u,u)^{\frac{1}{2}}=\Big(\int_{\Omega}(u(x))^2dx\Big)^{\frac{1}{2}}. \tag{3.6}$$

For the difference scheme (3.4), we have the following result.

Lemma 3.1. If $u^m,\quad m=0,1,...,M$ is the solution of (3.4), then

$$\|u^m\|_2^2+s\|u_x^m\|_2^2\le C\big(\|u^0\|_2^2+\|u_x^0\|_2^2\big), \tag{3.7}$$

where C is a constant.

Proof . Our results prove with mathematical induction. Firstly, when $m=0$ by using Eq. (3.4), we have:

$$u^1-su_{xx}^1+\mu\Delta t f(u^1)_x=u^0-\mu u_{xx}^0, \tag{3.8}$$

where $s=\mu(1+\gamma\Delta t)$.
Multiplying the above by u^1 and integrating on Ω then using Green's formula, we get

$$(u^1,u^1)-s(u^1,u_{xx}^1)+\mu\Delta t(u^1,f(u^1)_x)=(u^1,u^0)-\mu(u^1,u_{xx}^0)$$

i.e.

$$\|u^1\|_2^2+s(u_x^1,u_x^1)+\mu\Delta t(u^1,f(u^1)_x)=(u^1,u^0)+\mu(u_x^1,u_x^0),$$

using Young's inequality, we have

$$\|u^1\|_2^2+s\|u_x^1\|_2^2\le C\big(\|u^0\|_2^2+\|u_x^0\|_2^2\big), \tag{3.9}$$

where C is a constant. Suppose now we have proven

$$\|u^j\|_2^2+s\|u_x^j\|_2^2\le C\big(\|u^0\|_2^2+\|u_x^0\|_2^2\big), j=1,2,...,m. \tag{3.10}$$

Multiplying Eq. (3.4) by u^{m+1} and integrating on Ω and then using Green's formula, we get

$$\begin{aligned}\big(u^{m+1},u^{m+1}\big)-s\big(u^{m+1},u_{xx}^{m+1}\big)+\mu\Delta t\big(u^{m+1},f(u^{m+1})_x\big)&=(1-\lambda_1)\big(u^{m+1},u^m\big)\\&-\mu\big(u^{m+1},u_{xx}^m\big)+\sum_{k=1}^{m-1}\big(\lambda_k-\lambda_{k+1}\big)\Big(u^{m+1},u^{m-k}\Big)+\lambda_m\big(u^{m+1},u^0\big)\end{aligned} \tag{3.11}$$

i.e.

$$\begin{aligned}\|u^{m+1}\|_2^2+s\big(u_x^{m+1},u_x^{m+1}\big)+\mu\Delta t\big(u^{m+1},f(u^{m+1})_x\big)&=(1-\lambda_1)\big(u^{m+1},u^m\big)\\&+\mu\big(u_x^{m+1},u_x^m\big)+\sum_{k=1}^{m-1}\big(\lambda_k-\lambda_{k+1}\big)\Big(u^{m+1},u^{m-k}\Big)+\lambda_m\big(u^{m+1},u^0\big),\end{aligned} \tag{3.12}$$

applying Young's inequality, we have:

$$\|u^{m+1}\|_2^2+s\|u_x^{m+1}\|_2^2\le(1-\lambda_1)\|u^m\|_2^2+\mu\|u_x^m\|_2^2+\sum_{k=1}^{m-1}\big(\lambda_k-\lambda_{k+1}\big)\|u^{m-k}\|_2^2+\lambda_m\|u^0\|_2^2. \tag{3.13}$$

On the other hand, according to (3.10), we have

$$\|u^j\|_2^2 \le C\big(\|u^0\|_2^2 + \|u_x^0\|_2^2\big), j = 1, 2, ..., m, \tag{3.14}$$

$$\|u_x^j\|_2^2 \le C\big(\|u^0\|_2^2 + \|u_x^0\|_2^2\big), j = 1, 2, ..., m. \tag{3.15}$$

Hence, by using relations (3.13-3.15) we obtain

$$\|u^{m+1}\|_2^2 + s\|u_x^{m+1}\|_2^2 \le C\big(\|u^0\|_2^2 + \|u_x^0\|_2^2\big), \tag{3.16}$$

this proves the lemma. □ Next, we will prove the stability and convergence results. Let $U^m, (m = 0, 1, ..., M)$ be the exact solution of (3.4), For the time discrete scheme (3.4), we have the following stability result.

Theorem 3.2. The implicit numerical method defined by (3.4) is unconditionally stable.

Proof . Define the error

$$\xi^m(x) = U^m - u^m \tag{3.17}$$

and it satisfies

$$\xi^m - s\xi_{xx}^m + \mu\Delta t f(\xi^m)_x = (1-\lambda_1)\xi^{m-1} - \mu\xi_{xx}^{m-1} - \sum_{k=1}^{m-1}\big(\lambda_{k+1} - \lambda_k\big)\xi^{m-k} + \lambda_m\xi^0, \tag{3.18}$$

with
$\xi^m(a) = \xi^m(b) = 0$. By using Lemma 2 , we conclude

$$\|\xi^m\|_2^2 \le C\big(\|\xi^0\|_2^2 + \|\xi_x^0\|_2^2\big), m = 1, ..., M, \tag{3.19}$$

this proves the theorem. □

4 The Sinc-collocation method

For simplicity, let $\hat{u} = u^{m+1}$ in (3.4), we have:

$$\hat{u} - s\hat{u}_{xx} + \mu\Delta t f(\hat{u})_x = \hat{\tau} \tag{4.1}$$

where $\hat{\tau} = (1-\lambda_1)u^m - \mu u_{xx}^m - \sum_{k=1}^{m-1}\big(\lambda_{k+1} - \lambda_k\big)u^{m-k} + \lambda_m u^0$.
In the sinc collocation method, we take the approximate solution of (4.1) as

$$\hat{u}(x) \approx \hat{u}_N(x) = \sum_{j=-N}^{N} \hat{c}_j S_j(x), \tag{4.2}$$

the unknown coefficients $\hat{c}_j$ in relation (4.2) are determined by collocation method. In addition, to approximate the l-th derivative of the function $\hat{u}(x)$, we have:

$$\hat{u}^{(l)}(x) \approx \sum_{j=-N}^{N} \hat{c}_j \frac{d^l}{dx^l} S_j(x). \tag{4.3}$$

Setting

$$\frac{d^i}{d\phi^i}[S_j(x)] = S_j^{(i)}(x), \qquad 0 \le i \le 2, \tag{4.4}$$

and noting that

$$\frac{d}{dx}[S_j(x)] = S_j^{(1)}(x)\phi'(x), \tag{4.5}$$

$$\frac{d^2}{dx^2}[S_j(x)] = S_j^{(2)}(x)[\phi'(x)]^2 + S_j^{(1)}(x)\phi''(x), \tag{4.6}$$

and

$$\delta_{jk}^{(l)} = h^l \frac{d^l}{d\phi^l}[S_j(x)]_{x=x_k}. \tag{4.7}$$

Substituting each terms of (4.1) with given approximations in (4.2) and (4.3) and collocating at the Sinc points x_k also using relations (4.4-4.7), we can arrive the discrete Sinc-collocation system of nonlinear equations to determining $\{\hat{c}_j\}_{j=-N}^{N}$ as

$$\hat{c}_k - s\left\{\sum_{j=-N}^{N}\hat{c}_j\left(\frac{\delta_{jk}^{(2)}}{h^2}[\phi^{'}(x_k)]^2 + \frac{\delta_{jk}^{(1)}}{h}\phi^{''}(x_k)\right)\right\} + +\mu\Delta t F\left(x_k, \hat{c}_k, \sum_{j=-N}^{N}\hat{c}_j\phi^{'}(x_k)\frac{\delta_{jk}^{(1)}}{h}\right) = \hat{\tau}(x_k), \tag{4.8}$$

where $F(u) = \frac{df(u)}{dx}$, $k = -N, -N+1, ..., N$.
In order to obtain a matrix representation of the equations (4.8), let $\mathbf{I}^{(i)}, i = 0, 1, 2$ be the $r \times r$ Toeplitz matrices ($r = 2N+1$) whose jk- th entry is given by (2.5)-(2.7). Note that the matrix $\mathbf{I}^{(2)}$ and $\mathbf{I}^{(1)}$ are symmetric and skew-symmetric matrices respectively, also $\mathbf{I}^{(0)}$ is identity matrix. We define the $r \times r$ diagonal matrix as follow:

$$\mathbf{D}(g(x))_{ij} = \begin{cases} g(x_i), & i = j, \\ 0, & i \neq j, \end{cases}$$

Thus, the matrix form of the system (4.8) is as follows:

$$\mathbf{A}\widehat{\mathbf{C}} + \mu\Delta t F(\widehat{\mathbf{C}}) = \widehat{\mathbf{G}}, \tag{4.9}$$

where $\widehat{\mathbf{C}}$ and $\widehat{\mathbf{G}}$ are r-vectors and $\mathbf{A}$ is $r \times r$ matrix as:

$$\widehat{\mathbf{C}} = \left(\hat{c}_{-N}, \hat{c}_{-N+1}, \ldots, \hat{c}_N\right)^t,$$
$$\mathbf{A} = \mathbf{I}^{(0)} - s\Big\{\frac{1}{h^2}\mathbf{I}^{(2)}\mathbf{D}(\phi^{'})^2 + \frac{1}{h}\mathbf{I}^{(1)}\mathbf{D}(\phi^{''})\Big\},$$
$$\widehat{\mathbf{G}} = \left(\hat{\tau}_{-N}, \hat{\tau}_{-N+1}, \ldots, \hat{\tau}_N\right)^t,$$

The nonlinear system (4.9) can be solved by means of Newton's method. Finally, approximate solution $\hat{u}_N(x)$ of (4.1) can be obtained by the relation (4.2).

5 Convergence analysis

In this part, we prove that the approximate solution (4.2) converges to the exact solution of (4.1) at an exponential rate. In order to establish a bound of $\|\hat{\mathbf{u}} - \hat{\mathbf{u}}_N\|_\infty$, to do this aim, we suppose that the analytic solution of equation (4.1) at the Sinc points x_k denoted by $\tilde{u}(x)$ and defined by

$$\tilde{u}(x) = \sum_{j=-N}^{N}\hat{u}(x_j)S_j(x). \tag{5.1}$$

In the following theorem, at first we obtain a bound for $\|\hat{\mathbf{u}}_N - \tilde{\mathbf{u}}\|_\infty$.

Theorem 5.1. Suppose $\hat{u}_N(x)$ and $\tilde{u}(x)$ are the same terms defined in (4.2) and (5.1), respectively. then there exists a constant K_1 independent of N such that

$$\|\hat{\mathbf{u}}_N - \tilde{\mathbf{u}}\|_\infty \leq K_1 N^2 \exp(-(\pi\eta d N)^{1/2}). \tag{5.2}$$

Proof . From (5.1) and Cauchy-Schwarz inequality we have:

$|\hat{u}_N(x) - \tilde{u}(x)| = \left|\sum_{j=-N}^{N}\hat{c}_j S_j(x) - \sum_{j=-N}^{N}\hat{u}(x_j)S_j(x)\right| \leq$

$$\left(\sum_{j=-N}^{N}|\hat{c}_j - \hat{u}(x_j)|^2\right)^{\frac{1}{2}}\left(\sum_{j=-N}^{N}|S_j(x)|^2\right)^{\frac{1}{2}} = \kappa, \tag{5.3}$$

note that $\left(\sum_{j=-N}^{N}|S_j(x)|^2\right)^{\frac{1}{2}} \leq k_1$ where k_1 is a constant [30], so

$$\kappa \leq k_1\|\widehat{\mathbf{C}}-\widetilde{\mathbf{C}}\|_2, \tag{5.4}$$

where $\widetilde{\mathbf{C}} = (\hat{u}(x_{-N}), \hat{u}(x_{-N+1}), \ldots, \hat{u}(x_N))^t$ and $\widehat{\mathbf{C}} = \left(\hat{c}_{-N}, \hat{c}_{-N+1}, \ldots, \hat{c}_N\right)^t$.
Now, we must obtain a bound for $\|\widehat{\mathbf{C}}-\widetilde{\mathbf{C}}\|_2$, from (4.9) we have:

$$\mathbf{A}\widetilde{\mathbf{C}} + \mu\Delta t F(\widetilde{\mathbf{C}}) = \widetilde{\mathbf{G}}, \tag{5.5}$$

where

$$\widetilde{\mathbf{G}} = (\tilde{\tau}(x_{-N}), \tilde{\tau}(x_{-N+1}), \ldots, \tilde{\tau}(x_N))^t,$$

by subtracting (5.5) from (4.9) we have

$$\mathbf{A}(\widehat{\mathbf{C}}-\widetilde{\mathbf{C}}) + \mu\Delta t\big(F(\widehat{\mathbf{C}}) - F(\widetilde{\mathbf{C}})\big) = (\widehat{\mathbf{G}}-\widetilde{\mathbf{G}}), \tag{5.6}$$

Now, we find a bound for $\|\widehat{\mathbf{G}}-\widetilde{\mathbf{G}}\|_\infty$. From (4.1) we get

$$\begin{aligned}\big|\hat{\tau}(x_k) - \tilde{\tau}(x_k)\big| \leq \big|\hat{u}(x_k) - \tilde{u}(x_k)\big| + s\big|\hat{u}_{xx}(x_k) - \tilde{u}_{xx}(x_k)\big| + \\ + \mu\Delta t\big|F\big(\hat{u}(x_k), \hat{u}_x(x_k)\big) - F\big(\tilde{u}(x_k), \tilde{u}_x(x_k)\big)\big|,\end{aligned} \tag{5.7}$$

for $k = -N, -N+1, \ldots, N$,
from (5.7) and Theorem 9.19 reference [[28] p.218], we have:

$$\|\widehat{\mathbf{G}}-\widetilde{\mathbf{G}}\|_\infty \leq \|\hat{\mathbf{u}}-\tilde{\mathbf{u}}\|_\infty + s\|\hat{\mathbf{u}}^{''}-\tilde{\mathbf{u}}^{''}\|_\infty + \mu\Delta t\|F^{'}(z)\|_\infty\Big(\|\hat{\mathbf{u}}-\tilde{\mathbf{u}}\|_\infty + \|\hat{\mathbf{u}}^{'}-\tilde{\mathbf{u}}^{'}\|_\infty\Big) \tag{5.8}$$

therefore, by using theorem 2.4 we have:

$$\begin{aligned}\|\widehat{\mathbf{G}}-\widetilde{\mathbf{G}}\|_\infty \leq M_1N^{1/2}\exp(-(\pi\eta dN)^{1/2}) + sM_2N^{3/2}\exp(-(\pi\eta dN)^{1/2}) + \\ + \mu\Delta t M_3\Big(M_4N^{1/2}\exp(-(\pi\eta dN)^{1/2}) + M_5N\exp(-(\pi\eta dN)^{1/2})\Big)\end{aligned} \tag{5.9}$$

where M_1, M_2, M_4 and M_5 are constants independent on N and $\|F^{'}(z)\|_\infty \leq M_3$. Thus

$$\|\widehat{\mathbf{G}}-\widetilde{\mathbf{G}}\|_\infty \leq M_6N^{3/2}\exp(-(\pi\eta dN)^{1/2}), \tag{5.10}$$

where $M_6 = M_1 + sM_2 + \mu\Delta t M_3M_4 + \mu\Delta t M_3M_5$.
By using mean value theorem we obtain

$$F(\widehat{\mathbf{C}}) - F(\widetilde{\mathbf{C}}) = \Big(\int_0^1 \mathbf{JF}\big(\widehat{\mathbf{C}} + t(\widehat{\mathbf{C}}-\widetilde{\mathbf{C}}\big)dt\Big)\Big(\widehat{\mathbf{C}}-\widetilde{\mathbf{C}}\Big) \tag{5.11}$$

in which **JF** is Jacobin matrix of F.
According (5.6) and (5.11) we get:

$$\mathbf{E}(\widehat{\mathbf{C}}-\widetilde{\mathbf{C}}) = \widehat{\mathbf{G}}-\widetilde{\mathbf{G}}, \tag{5.12}$$

where

$$\mathbf{E} = \Big(\mathbf{A} + \mu\Delta t\int_0^1 \mathbf{JF}\big(\mathbf{C} + t(\mathbf{C}-\widetilde{\mathbf{C}}\big)dt\Big), \tag{5.13}$$

now, by taking L^2-norm from (5.12) and using (5.10) we have

$$\begin{aligned}\|\widehat{\mathbf{C}}-\widetilde{\mathbf{C}}\|_2 \leq \|\mathbf{E}^{-1}\|_2\|\widehat{\mathbf{G}}-\widetilde{\mathbf{G}}\|_2 \leq \|\mathbf{E}^{-1}\|_2\sqrt{N}\|\widehat{\mathbf{G}}-\widetilde{\mathbf{G}}\|_\infty \\ \leq M_6N^2\|\mathbf{E}^{-1}\|_2\exp(-(\pi\eta dN)^{1/2}).\end{aligned} \tag{5.14}$$

Finally, applying (5.3), (5.4) and (5.14), we can obtain

$$\|\hat{\mathbf{u}}_N - \tilde{\mathbf{u}}\|_\infty \leq K_1N^2\exp(-(\pi\eta dN)^{1/2}). \tag{5.15}$$

where $K_1 = k_1M_6\|\mathbf{E}^{-1}\|_2$. □

Theorem 5.2. If $\hat{u}(x)$ be the exact solution of (4.1) and $\hat{u}_N(x)$ be its Sinc approximation defined by (4.2), then, under the assumptions of Theorem 2.4 and 5.1 there exists a constant K ,independent of N, such that

$$\|\hat{\mathbf{u}}_N - \hat{\mathbf{u}}\|_\infty \le KN^2 \exp(-(\pi\eta dN)^{1/2}). \tag{5.16}$$

Proof . The triangular inequality implies

$$|\hat{u}_N(x) - \hat{u}(x)| \le |\hat{u}_N(x) - \tilde{u}(x)| + |\tilde{u}(x) - \hat{u}(x)|, \tag{5.17}$$

by applying Theorem 1 for second term of right hand side of (5.17) there exists a constant K_2 independent of N such that

$$\|\tilde{\mathbf{u}} - \hat{\mathbf{u}}\|_\infty \le K_2 N^{\frac{1}{2}} \exp(-(\pi\eta dN)^{1/2}). \tag{5.18}$$

Also, by using Theorem 3 for first term in the right hand side of (5.17) we can obtain

$$\|\hat{\mathbf{u}}_N - \tilde{\mathbf{u}}\|_\infty \le K_1 N^2 \exp(-(\pi\eta dN)^{1/2}). \tag{5.19}$$

Finally, by applying relations (5.17-5.19) we have

$$\|\hat{\mathbf{u}}_N - \hat{\mathbf{u}}\|_\infty \le KN^2 \exp(-(\pi\eta dN)^{1/2}). \tag{5.20}$$

where $K = \max\{K_1, K_2\}$. □

6 Numerical results

In this section, we bring two numerical examples to confirm the theoretical result in previous section. The maximum nodal errors and the convergence rate is calculated. In all of the examples, we choose $\eta = 1$ and $d = \frac{\pi}{2}$ which yield $h = \frac{\pi}{\sqrt{2N}}$, also the maximum pointwise errors are reported on uniform grids

$$U = \{z_0, z_1, \ldots, z_p\}, \quad z_r = \frac{r}{p}, \quad r = 0, 1, \ldots, p. \tag{6.1}$$

When the exact solution of problems is not available , we estimate maximum errors using the double mesh principle as

$$E^N = \max_r |\hat{u}_{2N}(z_r) - \hat{u}_N(z_r)|, \tag{6.2}$$

also, order of convergence is calculated with the following formula

$$\rho^N = \log_2\Big(\frac{E^N}{E^{2N}}\Big). \tag{6.3}$$

Example 6.1. Consider the equation (1.1) with $\gamma = 1$ and $f(u) = u^3 + u^2 + u$ for $x \in [0, 1]$ and $T = 1$.

According to the exact solution is unknown for this example, so errors are estimated as (6.2). The estimated errors E^N and rate of convergence ρ^N for this example are shown in table 1 for different values $\alpha = 0.2, 0.5, 0.8$ with $\Delta t = \frac{1}{500}$. In table 1, we compare our results with obtained results in [19], these comparisons show that the our results are better than the results in [19].

Example 6.2. We consider the equation (1.1) with $\gamma = 0$ and $f(u) = \dfrac{u^2}{2} + u$ for $x \in [-4, 4]$ and $T = 1$. In this case, the generalized Benjamin-Bona-Mahony (BBM) equation change to Benjamin-Bona-Mahony-Burger (BBM-Burger) equation.
For this example, errors E^N and rates ρ^N based on (6.2) and (6.3) are computed in Table 2 for different values of α and N with $\Delta t = 0.001$ at $T = 1$. These results illustrate efficiency and accuracy of the proposed method. Also, it can be seen that the errors decrease with increasing values of N.

For these examples, the graph of convergence curves are represented for various values of α in figure 1. This figure shows that the treatment of maximum errors is exponential with increasing N and verifies the theoretical results.

Table 1: *Comparison of estimated errors E^N and rates ρ^N for Example 1 with $p = 2N$, $\Delta t = \frac{1}{500}$ and $T = 1$.*

	$\alpha = 0.2$		$\alpha = 0.5$		$\alpha = 0.8$	
N	[19]	Our scheme	[19]	Our scheme	[19]	Our scheme
10	–	$1.84e-04$	–	$1.78e-4$	–	$1.81e-4$
	–	4.8276	–	4.8135	–	4.8105
20	$3.27e-4$	$6.48e-6$	$3.35e-4$	$6.33e-6$	$3.44e-4$	$6.45e-6$
	1.9938	5.7429	1.9923	5.7332	1.9916	5.7243
40	$8.21e-5$	$1.21e-7$	$8.42e-5$	$1.19e-7$	$8.65e-5$	$1.22e-7$
	2.0011	4.4492	2.0011	4.4782	2.0011	4.5414

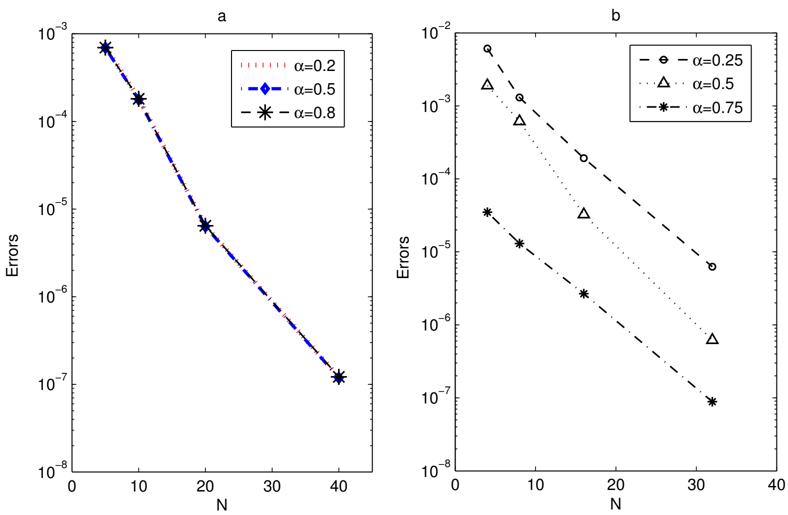

Figure 1: Convergence curves for various values of α, (a) Example 1,(b) Example 2.

Table 2: *Estimated errors E^N and rates ρ^N for Example 2 with $\Delta t = 0.001, T = 1$ and $p = 2N$.*

N	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.9$
8	0.0016	0.0013	$6.13e-4$	$1.30e-5$	$1.22e-5$
	2.7673	2.7593	4.2374	2.2782	3.5522
16	$2.35e-4$	$1.92e-4$	$3.25e-5$	$2.68e-6$	$1.04e-6$
	4.9373	4.9388	5.5766	4.9319	3.8207
32	$7.67e-6$	$6.26e-6$	$6.81e-7$	$8.87e-8$	$7.36e-8$
	4.9373	4.9388	5.5766	4.9319	3.8207

7 Conclusions

In this paper, we developed a numerical method to solve the time fractional generalized BBM equation. Firstly, the time-fractional derivative was discretized and we arrive at a semi-discrete scheme. Also, we have proven that the time-discrete scheme is unconditionally stable. Secondly, the Sinc collocation method is used to approximate the solution of the semi-discrete scheme. Besides performing some theorems, the exponential convergence rate of the Sinc method is illustrated. Finally, we demonstrate the efficiency and accuracy of the proposed method with two test problems. In order to show the efficiency and accuracy of the approach, maximum errors E^N and order of convergence ρ^N are computed and compared with some earlier works. We conclude that our scheme produces better results in comparison with other known works.

8 Open problem

In this paper, we have successfully employed our approach to solving the time fractional generalized BBM equation. Due to the accuracy and efficiency of the method, it is an open problem that the method can be easily extended for solving a broad class of fractional nonlinear partial differential equations. Also, for our future research, we can apply the Sinc quadrature rule to estimate the time-fractional derivative for these problems.

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